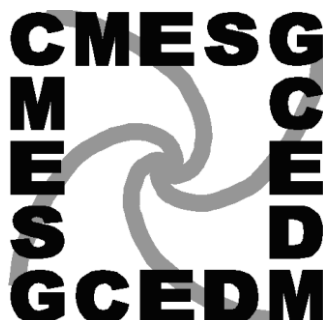CANADIAN MATHEMATICS EDUCATION
STUDY GROUP

GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE
DES MATHÉMATIQUES

PROCEEDINGS / ACTES
2017 ANNUAL MEETING /
RENCONTRE ANNUELLE 2017

McGill University
Montréal, Québec
June 2 – June 6, 2017

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PROCEEDINGS OF THE 2017 ANNUAL MEETING OF THE CANADIAN MATHEMATICS EDUCATION STUDY GROUP / ACTES DE LA RENCONTRE ANNUELLE 2017 DU GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

41st Annual Meeting
McGill University
Montréal, Québec
June 2 – June 6, 2017

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INTRODUCTION

Olive Chapman – President, CMESG/GCEDM
University of Calgary

The 41st annual meeting of the Canadian Mathematics Education Study Group/Groupe Canadien d'étude en didactique des mathématiques [CMESG/GCEDM] was another eventful learning and social gathering. With Montréal, Québec as the venue, this was our largest meeting with many newcomers that offered new opportunities and challenges in maintaining the uniqueness of the meeting as a discussion-oriented, community-based event. This was reflected, for example, in the large number of traditional poster presentations during the Gallery Walk. However, the meeting was a success in terms of the social and scientific programs that requires acknowledgement of everyone responsible for making this possible.

On behalf of the CMESG/GCEDM executive and participants of the meeting, I acknowledge the excellent work of our hosts at McGill University who offered us a social program in which we were well-fed, entertained, and accommodated, in addition to having the opportunity to enjoy Montréal. Special thanks to our colleagues and co-chairs of the local organizing committee, Annie Savard and Limin Jao, for their leadership, time commitment and thoughtful planning and hosting of the conference. The meeting was excellently organized and managed from beginning to end. Thanks to the other members of the local organizing team for their valuable contribution to the planning and smooth running of the conference. Specifically, thanks to the graduate students who did a tremendous amount of work both before and during the meeting: Alexandre Cavalcante, Dominic Manuel, Brandes Hadas, Hailey Iacono, Hannah Chestnutt, Laura Broley, Marta Kobiela, Noor Affana, Sarah Mathieu-Soucy, Scosha Merovitz, Sophie Pinard, and Vandana Chandrasekhar. Thanks also to faculty members Marta Kobiela and Dawn Wiseman. Thanks to the Department of Integrated Studies in Education and the Faculty of Education of McGill University for their support of the meeting and to the following sponsors for their contributions: Calvin Leung (Clear Educational Solutions), Centre de Recherches Mathématiques, Chuck Hughes (Crown Salts), Concordia Department of Mathematics and Statistics, Institut des Sciences Mathématiques, and McGill Bookstore.

I acknowledge the CMESG/GCEDM (2016-2017) executive for organizing another interesting scientific program with topics relevant to our membership of mathematicians, mathematics teacher educators and mathematics education researchers. On behalf of the executive, thanks to the invited session speakers/leaders for making the program possible. Thanks to the two plenary speakers: Dr. Yvan Saint-Aubin (Université de Montréal) for engaging us in his work on writing meaningful mathematics exercises for students in his university mathematics courses that stimulated rich discussions and interest regarding designing, selecting, and using mathematics exercises; and Dr. Annie Selden (New Mexico State University) for taking us on an insightful journey of her 40+ years of teaching and thinking about university mathematics students, proofs, and proving. Thanks to Joel Hillel for presenting the "Elder Talk" that took us down memory lane as he shared highlights from his long active membership in CMESG/GCEDM. Thanks to our colleagues who took on the demanding roles of leaders of the six Working Groups. We are honoured to have such colleagues who are willing to commit their time and expertise to guarantee the success of this central aspect of our meeting. Thanks to the presenters

of the three Topic Sessions, the twelve new-PhD presenters, the Math Gallery Walk presenters, the presenters and coordinator of the Panel, the Ad Hoc discussion leaders, and all the participants for making the 2017 meeting a meaningful and worthwhile experience.

This publication of the proceedings of the CMESG/GCEDM 41st Annual Meeting offers readers the opportunity to learn about some of the ongoing mathematics education research and interests of our community. It also provides a means for participants to further reflect and build on their experiences at the meeting and for others to share in and be inspired by the work of the mathematics education community in Canada.

Horaire

Vendredi 2 juin	Samedi 3 juin	Dimanche 4 juin	Lundi 5 juin	Mardi 6 juin
	8h45 – 10h15 **Groupes de travail**	8h45 – 10h15 **Groupes de travail**	8h45 – 10h15 **Groupes de travail**	8h45 – 10h15 **Panel de clôture**
	10h15 – 10h45 Pause café	10h15 – 10h45 Pause café	10h15 – 10h45 Pause café	10h15 – 10h45 Pause café
Réception des amis de FLM vendredi, 2 juin 15h30 – 16h20	10h45 – 12h15 **Groupes de travail**	10h45 – 12:15 **Groupes de travail**	10h45 – 12h15 **Groupes de travail**	10h45 – 11h45 **La parole aux anciens** 11h45 – 12h30 **Séance de clôture**
	12h30 – 13h30 Dîner	12h30 – 13h30 Dîner	12h30 – 13h30 Dîner	
	13h30 – 14h30 **Café, dessert et galerie mathématique**	13h30 – 14h00 **Séances ad hoc**	13h30 – 14h00 **Petits groupes**	
		14h10 – 15h10 **Plénière II**	14h10 – 15h10 **Discussion de la plénière II**	
14h30 – 18h45 **Inscription**	14h40 – 16h20 **N^{lles} thèses**		15h20 – 16h20 **Séances thématique**	
	16h20 – 16h50 Pause café	15h10 – 19h00 **Temps libre pour explorer la ville**	16h20 – 16h40 Pause café	
	16h45 – 17h20 **Petits groupes**		16h40 – 18h00 **Assemblée générale annuelle**	
17h00 – 18h15 **Souper**	17h30 – 18h30 **Discussion de la plénière I**			
18h15 – 19h30 **Séance d'ouverture**	18h30 – ? **Souper libre**	19h00 – ? **Souper**	18h30 – ? **Souper**	
19h30 – 20h30 **Plénière I**				
23:30 – 22:00 **Réception**				

Schedule

Friday June 2	Saturday June 3	Sunday June 4	Monday June 5	Tuesday June 6
	8:45 – 10:15 **Working Groups**	8:45 – 10:15 **Working Groups**	8:45 – 10:15 **Working Groups**	8:45 – 10:15 **Closing Panel**
Friends of FLM Reception Friday, June 2 15:30 – 16:20	10:15 – 10:45 Break	10:15 – 10:45 Break	10:15 – 10:45 Break	10:15 – 10:45 Break
	10:45 – 12:15 **Working Groups**	10:45 – 12:15 **Working Groups**	10:45 – 12:15 **Working Groups**	10:45 – 11:45 **Elder Talk**
				11:45 – 12:30 **Closing Session**
	12:30 – 13:30 Lunch	12:30 – 13:30 Lunch	12:30 – 13:30 Lunch	
	13:30 – 14:30 **Coffee, Dessert and Mathematics Gallery**	13:30 – 14:00 **Ad hoc**	13:30 – 14:00 **Small Groups**	
		14:10 – 15:10 **Plenary II**	14:10 – 15:10 **Plenary II Discussion**	
14:30 – 18:45 **Registration**	14:40 – 16:20 **New PhDs**		15:20 – 16:20 **Topic Session**	
	16:20 – 16:50 Break	15:10 – 19:00 **Free time to explore the city**	16:20 – 16:40 Break	
	16:45 – 17:20 **Small Groups**		16:40 – 18:00 **Annual General Meeting**	
17:00 – 18:15 **Dinner**	17:30 – 18:30 **Plenary I Discussion**			
18:15 – 19:30 **Opening Session**	18:30 – ? **Dinner on your own**		18:30 – ? **Dinner**	
19:30 – 20:30 **Plenary I**		19:00 – ? **Dinner**		
23:30 – 22:00 **Reception**				

Plenary Lectures

Conférences plénières

THE MOST UNGLAMOROUS JOB OF ALL: WRITING EXERCISES

Yvan Saint-Aubin
Université de Montréal

The author reviews his learning process on how to write mathematical exercises. To understand his conclusions reached by trial and error, Anna Sfard's three stages of mathematical concept development are illuminating.

INTRODUCTION

You are a teacher. You have goals and expectations for your students. You must manage wisely your time and theirs to help them reach these goals. Your time will be divided between preparing lectures, delivering them, writing exercises, outlining assignments, grading them, meeting students during office hours, preparing exams, and grading them. You have other duties as an employee in your school; but your abilities as a teacher are most challenged by the day-to-day activities in the previous list.

Several facts will influence the time you spend on each of these activities. Like a performance artist, you need to be ready to meet your students at scheduled times. These encounters provide you instant feedback, either by your students' questions or their bemused silence. If everything goes well, the periods will have helped your students and given you some encouragement to go prepare your next meeting with them. Writing and choosing exercises for students to work on at home or as part of an assignment is more delicate. There is no immediate feedback, no witnessing of mental processes, no sense of the students' psychological reactions to the challenges you posed. The grading will provide some indications of where your students stand. However, contrary to classroom interactions, trying to assess your students' progress through grading is like admiring a landscape on a foggy day: outstanding features will be caught, but fine details are likely to be missed. Preparing exams and grading them are somewhat different than preparing homework. Your goal here is to build a tool to assess which concepts your students have mastered and at what level. Besides the mathematics, your mind will be busy probing all the material, tuning the difficulty, making sure the questions represent what was taught and exercised, et cetera. In other words, your outstanding concern will be fairness. All these pedagogical activities play an important role in your teaching. How should you divide your time between them?

Let us turn the table. How much time do (or should) your students spend (or should spend) on the various activities to master what you teach? Most attend your classes and write your exams. That is granted. But how much time will they allot to the exercises you suggest to them and to your assignments? The answer may vary from one level to another of the schooling system. In primary school, there are only assignments, usually to be handed in the morning after. Starting with high school, students are expected to do some exercises on their own, whether or not these

exercises will be formally graded. At the university level, students will be required to do as many exercises as they can, without supervision or grading. The official requirement calls for one hour of work at home for every classroom hour; and my own experience, as a student and a teacher, shows me that this requirement is far from sufficient. Finally, for some of these courses, feedback will be given through exams only. If guidance is to be given, it is through a good selection of exercises.

The necessity of exercises is recognized by both teachers and students, even uninterested ones. The roles of teachers and students in this part of their pedagogical contract are somewhat disconnected. My perception of this part is that the teachers suggest the exercises, the students try to do them, but scarce communication flows one way or the other. If there is any truth in the last sentence, the situation is rather problematic. Mathematical ideas, algorithms and concepts might be introduced during classes, but, however great your teaching ability is, it is while doing exercises that your students will grow familiar with and eventually apprehend them. In other words, good understanding rarely occurs in the classroom; it occurs during independent work. As critical the job of writing and choosing exercises is, it is very difficult due to the very limited feedback on how the students work through exercises and what they eventually learn from them. Moreover praise for good teachers is usually given due to classroom performance, fairness in grading, and moral support provided to students, but rarely, if ever, does it acknowledge the quality of the exercises proposed. Writing exercises is crucial but totally unglamorous!

I am a mathematician, not a 'didacticien'. Most of what I have learned about mathematics teaching is through intuition, and trial and error. My path to writing good exercises was rather sinuous. Recently a colleague suggested that I read Anna Sfard's (1991) reflections on mathematical processes and objects. My impression of Sfard's paper is that she captures vividly the various stages of my personal acquisition of any mathematical concept. This reading brought me to put words on how I now design a series of exercises targeted to learning a given mathematical idea or concept. The next two sections will try to describe my understanding of Sfard's ideas on the fruitful tension between the *operational and structural* sides of a concept (Learning a Mathematical Concept: Algorithm vs Abstract Object) and her three stages of concept developments: *interiorization*, *condensation* and *reification* (The Three Stages in Concept Development). Only after will I give examples of exercises from one of my courses and draw the parallel with Sfard's three stages (Writing Exercises). In the conclusion, I shall explain how I design sets of exercises now (Conclusion).

LEARNING A MATHEMATICAL CONCEPT: ALGORITHM VS ABSTRACT OBJECT

My first recollection of fractions (or of rational numbers as mathematicians call them) is tied to the *règle de trois* (cross-multiplication): a mom has four apples and always treats her children equally. How many apples does she give each if she has two kids? My early understanding of this difficult concept of fraction was rooted in the process of sending the number 2 under the number 4 to make a fraction and eventually realizing that this fraction was also the integer 2. The exercise became much harder when the number of kids changed, for example, to three. The fraction would not simplify, and I had to realize that, first, I had no words for the result and second, that result needed a numerator and a denominator to be written. We practiced this *règle de trois* on all type of situations (and it brought me a lot of joy). I thus learned the rational numbers through a recipe: send an integer under another one, look for common factors, remove them if any. It is hard for me to pinpoint the moment in my understanding of fraction when this algorithm gave way to a more abstract object: that of numbers that might be integers or lie between them, and that, in a sense, superseded integers by including them in a richer family.

Many mathematical notions were historically discovered and are now learned through computations, an algorithmic procedure or an operational process. Sfard (1991) calls this aspect of this mathematical concept *operational conception*. But a different aspect of the same concept is often used.

Even though rational numbers are not integers, one can use the addition and multiplication of integers to define how to add and multiply rational numbers. To master these operations, I had to give up my understanding of fractions as a dynamical process (write *a* over *b*, look for common factors, simplify them). This dynamical process needed to be replaced by a more abstract definition, that of a pair of integers, identified respectively by the names numerator and denominator. After a transition period, the number $\frac{8}{3}$ no longer triggered the image of a mom dividing eight apples among three kids. My new conception of $\frac{8}{3}$, as an object on which new operations could be defined, made accessible to me a larger set of problems involving fractions.

The adaptability of a mathematical concept to many situations often depends on a more abstract definition, one independent or partially removed from its algorithmic incarnation. Sfard (1991) calls this new aspect of the mathematical concept *structural conception*. The definition of a rational number as a pair (*numerator*, *denominator*) provides an example of such a structural conception. The abstract definition does not replace the algorithmic one; instead the former completes the latter. Proving properties on rational numbers will often require going back and forth between the two ways of thinking about the concept.

Sfard (1991) argues that, on a student's learning path toward a given mathematical concept, its operational conception is often mastered before its structural one. Of course our understanding of *operational conceptions* of mathematical concepts should not be limited to computational algorithms whose output is a number. The mathematical concept of vector is often introduced as an arrow drawn on a plane. At first it is simply a geometrical drawing. The addition of two vectors (or two arrows) is then obtained algorithmically by a geometric construction: Move the second arrow so that its base sits at the tip of the first, then join the base of the first arrow to the tip of the (translated) second. The result of the addition is a new arrow. This, for me, is an operational conception of a vector. A student pursuing mathematics at the university level will see this introductory conception of a vector enlarged by several definitions, getting more and more abstract, each one encompassing the previous ones into a broader, more structural, conception.

Sfard's (1991) key argument is that, for a large set of mathematical concepts, the learning of an operational conception *before* a structural one will be the most efficient path for a majority of students. Her stand is not that the teaching of a concept should follow its historical development. Instead she argues that, in the process of many types of learning, some universal characteristics are independent of external stimuli, like teacher's preferences, the textbooks, the curricula, et cetera. The precedence of the operational conception over the structural one appears to be such a characteristic.

A second example may be useful to distinguish the operational and structural sides of a concept. I have chosen the concept of the derivative of a function, in part because it is taught at the end of high school or during the first of the undergraduate years (or in cégeps in the Province of Québec). This two-sided development of a mathematical concept appears to be useful at all stages of the mathematical education.

I first encountered the concept of the derivative of a function during my first year of cégep. I believe that my experience is similar to that of many student's first contact with this concept.

The textbook I learned from used the formal definition to introduce the concept. If $f : \mathbb{R} \to \mathbb{R}$ where $\mathbb{R}$ denotes the set of real numbers, the derivative of f at x is given by

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \, . \qquad (*)$$

I recall that I did not get much from this definition. I had barely any understanding at this point of what a limit is. A better understanding of this formal definition, i.e., a structural conception, would have to wait for my first real analysis course at the university level and a structural conception of the mathematical limit. Two operational conceptions were fortunately provided.
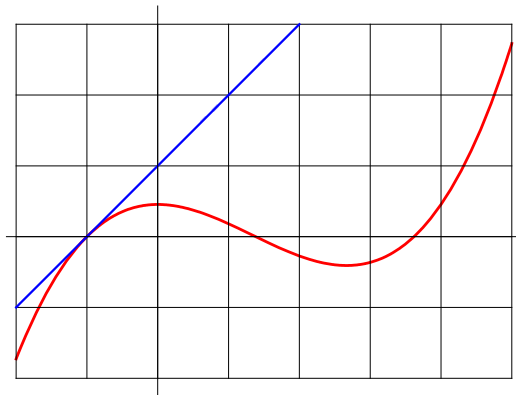
Figure 1. The graph of a function f (in red) with the tangent to the graph (in blue) drawn at the point $x = -1$. Here the derivative is 1.

The first one is the concept of derivative as calculating the slope of a tangent to a graph. Having mastered the slope of a line in high school, it was easy for me to understand the 'number' that the derivative was computing. This operational conception does not lead to a precise value of the derivative. Indeed to get a gross estimate of the derivative, one has to draw as carefully as possible the graph of the function, draw (again carefully) the tangent at the desired point, and read the slope from the graph paper. Of course, nowadays a student can use a computer to draw the graph and the tangent, but the estimate of the exact value of the slope will still be difficult to measure by visual means only. These limitations do not restrict the power of this first operational conception. It provides a (geometric) algorithm to get the number, however grossly.

The second operational conception of derivative was given to me through a series of computational rules. I still recall the beginning of the list as I still use it regularly in my work. If the reader has learned it once in his or her life, the list will be (very!) familiar.

$$\frac{d}{dx} x^n = n x^{n-1}$$

$$\frac{d}{dx} \sin ax = a \cos ax$$

$$\frac{d}{dx} \cos ax = -a \sin ax$$

$$\frac{d}{dx} e^{ax} = a e^{ax}$$

$$\frac{d}{dx} \log ax = \frac{1}{x}$$

$$\frac{d}{dx} (f + g) = \frac{df}{dx} + \frac{dg}{dx}$$

$$\frac{d}{dx} (fg) = \frac{df}{dx} g + f \frac{dg}{dx}$$

...

If I recall correctly, some of these rules were justified, either geometrically, graphically or algebraically. However none was given a proof starting from the formal definition (*). My introduction to the derivative thus stopped short of a structural conception, but I think it was a good first step: These two operational conceptions provided tools to apply the new concept to other disciplines (physics being an obvious one) and to introduce geometrically the difficult concept of the limiting process involved in the definition. The path toward a structural conception had been opened.

A lack of understanding of the formal definition rules out any formal proofs. Many scientists pursue fruitful careers, even though they have not mastered the abstract definition (*). As shall be argued in the next section, their understanding of the derivative goes beyond the purely algorithmic computation and has reached some form of structural conception. In other words, a structural conception is not a strict synonym of an abstract definition.

Mathematicians tend to conceive a mathematical concept through both its algorithmic and abstract definitions. One might think that the operative conception is only a step toward the structural conception, a step that can be forgotten once the structural stage has been reached. This is not the case. A mathematical concept is the amalgam of the two conceptions. Even professional mathematicians will use both in their discussions. To underline this point, Sfard quotes Paul R. Halmos: "To try to decide which component is more important is not much more meaningful than to debate whether for walking you need your right foot more than your left" (cited in Sfard, 1991, p. 9).

THE THREE STAGES IN CONCEPT DEVELOPMENT

Beside the two complementary operational and structural conceptions of a mathematical concept, Sfard (1991) also proposes a model of how an individual comes to develop an understanding of a new mathematical concept. She identifies three stages through this progress that she names *interiorization*, *condensation* and *reification*. This section describes these stages and gives examples of them.

SFARD'S THREE STAGES

A student's first contact with a new mathematical concept is the stage of *interiorization*. This step is mainly the acquisition of the operational processes that are attached to the concept. The mastering of the *règle de trois* (cross-multiplication) was definitely a first contact with fractions, one based on an algorithm leading to the unavoidability of some numbers of the form (*numerator / denominator*). These algorithmic processes use objects introduced previously and mastered to the point of being considered 'elementary', 'basic', 'low-level'. Of course, what is low-level will change as the student progresses through school. For example, the mathematical concept of rational numbers, that is a focus of study throughout primary school, will become a low-level tool during high school.

Through practicing the algorithmic processes, the student will develop mental representations of the new mathematical concepts. Sfard (1991) declares that interiorization has been reached whenever these mental representations can be conjured or summoned up without the student having to perform the steps of the process. I guess I interiorized fractions when $\frac{8}{3}$ stopped being related to eight apples and a family of three children and started being some object on its own.

The student then starts a step where the processes attached to the mathematical object are compressed, condensed and perceived as a whole. An example of this *condensation* occurs when the two processes of 'computing the slope of the tangent to the graph' and 'apply some

algorithmic rules to transform a function into another one' merge into two representations of the same object called the derivative of a function. Using Sfard's (1991) words, a "new concept is 'officially' born" (p. 19). In the learning of fractions, the stage of condensation might also include the mastery of addition and multiplication. These two new algorithmic processes would be interiorized by themselves and eventually be condensed into the larger concept of fraction.

The last stage in the acquisition of a new mathematical concept is the *reification*, from the Latin word *res* meaning a thing. (The same root is used in *republic*, the 'entity' of the 'people'.) Sfard (1991) describes this step as "an instantaneous quantum leap: a process that solidifies into object, into a static structure" (p. 20). It happens when various representations are unified in a purely static structure, a structure detached from any process. This structure can then be used to solve problems and new processes can be built in which it appears as a building block, a new 'low-level' object.

Sfard (1991) notes that certain minds, e.g., those of certain professional mathematicians, might be able to manipulate a new abstract object without becoming proficient in some algorithmic processes that underlie it, but she claims (and I certainly agree with her) that mastering a new concept through first interiorization, then condensation and finally reification is the prevailing path. This said, I would prefer not to present reification as an ontological quantum leap. Instead I think that reification may happen over such a long period that it is difficult to pinpoint when it has actually occurred. Merging various representations of an object requires first learning these many faces of the same mathematical concept. In my experience, their fusion or solidification into one abstract unit is often a slow process. Finally, I believe that reification of a given object might mean a different thing for different people. The derivative is, without a doubt, a mathematical concept. As such, mathematicians will want its rigorous definition given in equation (*) to be one of the representations coalescing with others into its reification. However, for engineers working in the design of airplane wings, this rigorous definition might not be necessary; it may not even be known to them! That does not mean these engineers have not reified the concept. Their use of this mathematical object in the partial differential equations modeling airflow around a wing has all the characteristics that Sfard calls for reification.

The following examples might help understand Sfard's (1991) three stages. I should warn the reader that these examples are my own and reflect my understanding of her work.

EXAMPLE 1: THE RATIONAL NUMBERS

For this first example, I chose to pursue the description of the mastery of fractions, or rational numbers, because the learning and eventual reification of this mathematical concept spans usually more than a decade of study. Here is a (partial!) list of what 'rational numbers' gather in my mind. I hope that a large part of it coincides with the reader's own list. I formulate them as actions that I can now perform. In a course syllabus, I would present them as goals to be achieved or abilities to be acquired. But below, I will use the word competencies that seems to be the preferred one nowadays.

1. I know how to answer problems similar to that of the mother giving apples to her kids with something of the form a/b.
2. I know that a/b is the same thing as $(2a)/(2b)$; it is also the same thing as $(7a)/(7b)$ and so on.
3. I know how to add a/b and c/d and express the result in the form e/f.
4. I know how to multiply a/b and c/d and express the result in the form e/f.
5. I know how to solve a real-life problem that calls for fractions (e.g., to decide how large are two pieces of the same pie put together).
6. I know that the addition of 0 with any fraction a/b gives back a/b.

7. I know that the multiplication by 1 of any fraction a/b gives back a/b.
8. I know that the addition and multiplication of two fractions are commutative and associative.
9. I can decide whether a/b is smaller, equal or larger than c/d.
10. I know how to solve algebraic equations like $3x = 2$.
11. I know that solutions of linear equations formulated with fractions are themselves fractions. For example the equations $2x/3 + 3y/2 = 1/2$ and $3x/2 - 2y/3 = 2/3$ have a solution where x and y are fractions.
12. I know how to prove that solutions of $x^2 = 2$ are not rational.
13. I know that rational numbers can be defined rigorously as equivalence classes of pairs (n, d) where n is an integer and d is a non-zero integer.
14. I know how to prove commutativity and associativity of the operations of $+$ and $\times$ defined on these equivalence classes.
15. I know that the set of rational numbers forms a field, that it is the smallest containing the integers and how to show these facts.

And the list goes on. It is organized in the order I have acquired them and contains competencies that are mastered from primary school to a bachelor degree in mathematics. If you have not done a BSc in mathematics, some might be unfamiliar. (Do not worry, keep on reading!) Before moving to the next paragraph, I invite you to stop for a while and decide which competencies are learned during the interiorization, condensation and reification of the concept of rational numbers.

I really hope you did the exercise. There are many algorithmic processes involved, some aimed at the concept of fraction (1. and 2.), at their addition and multiplication (3. to 7.), and others at the solution of algebraic equations (10. and 11.), et cetera. Is the mathematical concept of fractions dissociable of the operations of addition and multiplication on them? In other words, can one understand fractions without knowing how to add them? If yes, then the mastery of the first two competencies would complete the interiorization of the fractions. If not, then I would propose that the seven first would be necessary for this first stage. I would put the competencies 8., 9. and 10., and probably 11. at the stage of condensation; they all require an ease to think about fractions more abstractly and 'visualize' addition and multiplication without actually doing them. The last competencies, starting at 11. or 12., require conceiving the set of fractions as a whole, together with the various operations defined on them: addition, multiplication, but also taking the power of a fraction, writing and solving algebraic equations, et cetera. I would say that they are achieved while reification is happening or after.

With such a long list of competencies, one can also see the interweaving of the development on various mathematical conceptions and concepts. The 'real-life' competency 5. is almost synonymous to the procedural conception 4, but it is likely that both will require some efforts. The concepts of identity elements for $+$ and $\times$ in 6. and 7. appear intimately related to the operational conceptions 3. and 4.

It is clear that the actual moment of completion of one stage or another is difficult to decide precisely and may depend on what is included in the mathematical concept. That being said, the extremes are easily distinguished as belonging to interiorization and reification respectively: on one hand, the basic *règle de trois* and, on the other hand, the structural conception of the rational numbers as one of the sets of numbers distinguished by their properties from other sets, like the integer, the real or the complex numbers.

The rational numbers also give an example of the fact that the two conceptions of a mathematical concept described in Learning a Mathematical Concept: Algorithm vs Abstract

Object are necessary to do mathematics. The usual proof of the fact that '$\sqrt{2}$ is not a rational number' requires writing a solution in the form n/d and obtaining a contradiction. Understanding the statement itself requires the structural conception of the rational numbers, obtained through condensation and probably reification of the concept of fraction. Its proof requires the algorithmic conception (*numerator / denominator*) of fractions acquired at the stage of interiorization (competencies 1. and 2.). In other words, the two conceptions play a symbiotic role, even at the higher level of statements requiring reification.

EXAMPLE 2: MATRICES AND LINEAR TRANSFORMATIONS

This second example targets a mathematical concept that is first encountered at the end of high school (and at the cégep level in the Province of Québec). Its condensation and reification might only occur during the undergraduate years or even after. I chose it because the algorithmic steps usually presented to introduce matrices are very similar to those used to teach fractions. I propose the following list of competencies.

1. I know when two matrices can be added and how to add them when they can.
2. I know when two matrices can be multiplied and how to multiply them when they can.
3. I know that the matrix full of zeroes is the identity element for the addition.
4. I know what the identity matrix is and why it is the identity element for the multiplication.
5. I know how to recognize when a function is a linear transformation.
6. I know that only the first column of B is used to compute the first column of the product $A \cdot B$ of the two matrices A and B.
7. I know how to prove that the matrix multiplication is associative.
8. I know that every matrix corresponds to a linear transformation.
9. I know that, given bases, every linear transformation corresponds to a matrix.
10. I understand that the image of the product $A \cdot B$ of two linear maps is a subspace of the image of A.
11. I understand that isometries encountered in Euclidean geometry that leave a point fixed are linear transformations.

This list is shorter than the one presented for fractions. It covers only parts of the first courses of university-level linear algebra. Even students who master all competencies of the list might not reach the level of reification, in part because these are covered in a very short period. For me, reification is a slow process.

Matrices and linear transformations are often taught as two different concepts. The interiorization of the first is reached by doing exercises aimed at competencies 1. to 4. The interiorization of the second is done in the same way, but in relation to competency 5. The exercises are usually quite different in taste, with those for matrices involving integer- and rational-valued matrices and those for linear maps given in terms of more abstract functions. Contrary to rational numbers whose learning is spread over more than a decade, the fact that matrices and linear transformations are two faces of the same object is taught during the same course. Competencies 7. to 9. definitely require the fusion of these two representations into a unique object. The best students therefore reach condensation of the concept within a term. This concept is used in competencies 10. and 11. to state properties that mix it with other (new or old) knowledge: the concept of vector space for 10., and the concepts of isometry and scalar product for 11. For the best students, 'inserting the concept 'matrix-linear transformation' into this larger framework might represent the beginning of its reification.

WRITING EXERCISES

By 'writing exercises', I mean designing a set of exercises aimed at a specific group of students following a given course with some lectures (usually given by me) and based on a given textbook or notes that I have written.

I have learned how to write exercises the hard way: by trial and error. My main errors were to write exercises that I wanted the students to be able to do, without helping them to reach that 'right' level. Based on the framework developed in Learning a Mathematical Concept: Algorithm vs Abstract Object and The Three Stages in Concept Development, it seems obvious that the exercises should be chosen or written so that they help the students reach the interiorization of a given mathematical concept and start its condensation. Rarely will it be possible to guide students to achieve reification of the concepts that form the content of the particular course being taught, but relationships between these concepts and those mastered earlier in other courses might trigger reification of the older ones. It is absolutely necessary to provide enough algorithmic or procedural exercises for the student to grasp the technical aspects of the concepts being introduced, but these kinds of exercises are not sufficient. More exercises are needed to lead the students into some form of condensation so that the various aspects of the technical gymnastics coalesce into a well-circumscribed object. Of course these exercises leading to condensation would be useless (and cruel!) if the algorithmic aspects had not been somewhat transcended.

The course MAT 1101 Mathématiques fondamentales is a compulsory course for the students registered in the program for future teachers of high-school mathematics at Université de Montréal. (I have taught it twice by now.) It reviews and defines rigorously the common number sets (the natural numbers, the integers, the rational, real and complex numbers) and introduces the main structure of mathematics (group, ring and field). Even though most mathematical objects are familiar, the abstraction level of the course makes it hard. The following exercises are taken from notes I have written for this course. They all appear in a section on isometries of the plane and their group structure. This section thus provides an example of the group structure for transformations that form the basis of the concept of congruence of geometric objects, like triangle, polygons, angle, et cetera. The latter concepts are taught in Euclidean geometry in high schools of Québec, as I assume they are elsewhere.
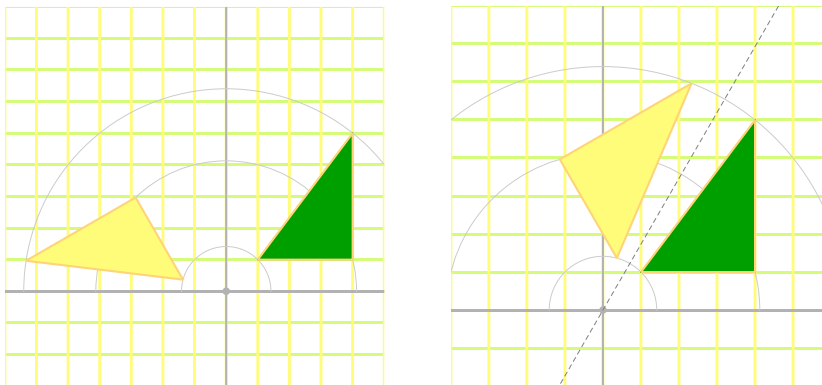
Figure 2. The isometry on the left represents a rotation of $2\pi/3$ of the green triangle onto the yellow one, that on the right is reflection through the line at angle $\pi/3$ with the horizontal axis. Both transformations preserve the distances. For example, the distances between the vertices of the green triangle coincide with those of the yellow one.

When students start MAT 1101, they have encountered isometries during their high-school years, but under another name. These isometries include the rotations and translations of the plane, which are called *rigid motions*. They also include reflections through a line of the plane. These transformations of the plane all have in common that they preserve angles and the distances between any pair of points. That is why they are called isometries. Several new properties will be introduced in the course. First, it will be shown that any transformation of the plane that preserves the distance between all pairs of points is either a rotation, a translation, a reflection, or a combination thereof. Second, while the Euclidean geometry classes will have emphasized ruler-and-compass constructions, the course will explore Cartesian-coordinate expressions for these transformations.

The exercises that are described below are a subset of those that I propose to the students after the formal discussion in the classroom. During these lectures, I have defined an isometry as a function $f: \mathbb{R}^2 \to \mathbb{R}^2$ that preserves distances and shown that they are indeed the translations, rotations and reflections. The expressions of the functions f in Cartesian coordinates are then given, and the following theorem proved:

> **Theorem**. For any isometry $f: \mathbb{R}^2 \to \mathbb{R}^2$, there exists a 2×2 matrix A and a vector $\vec{t}$
> such that (i) A is an orthogonal matrix and (ii) $f(\vec{x}) = A\vec{x} + \vec{t}$ for all $\vec{x} \in \mathbb{R}^2$.

The fact that isometries form a group has been partially proved (an exercise will complete the proof), and the relationship between isometries (rigid motions and reflections) and congruent geometric objects has been recalled: two geometric figures are congruent if there exists an isometry that brings the first onto the second. (The objects of the same shade in Figure 3 are congruent.) Sometimes I have time during the lectures to discuss some of these problems, otherwise a teaching assistant helps the students work their way through some of the exercises. The rest is the students' responsibility and could be discussed during my office hours.
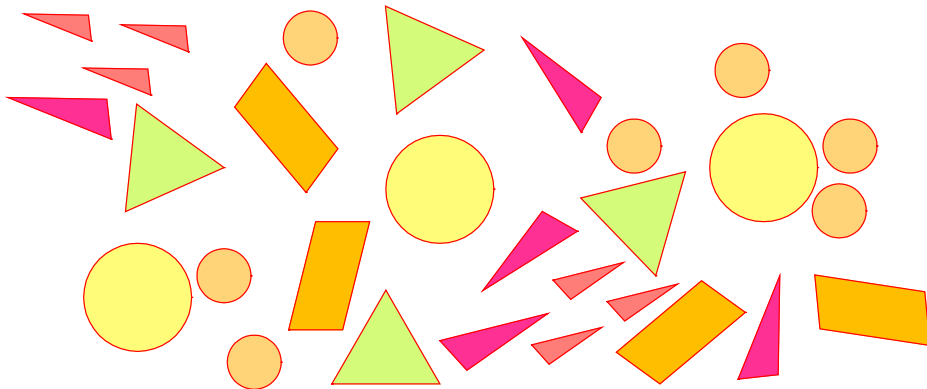
Figure 3. The objects of the same shade are congruent.

Preparing the exercises, I wanted to be sure that the Cartesian expression $A\vec{x} + \vec{t}$ was understood and what it meant for figures drawn in the plane. The first exercise below is a direct application of the formula $f(\vec{x}) = A\vec{x} + \vec{t}$ for a rotation and a reflection. The formula for the matrices A for the two cases of a rotation and a reflection was given in the notes and discussed in class.

> **Exercise 1.** Consider the square of vertices $(1,1), (1,2), (2,2), (2,1)$.
> (a) Give the vertices of its image under a rotation of $-\pi/2$. Check the computation by drawing the new vertices.
> (b) Same question for the reflection through a line at an angle $-\pi/4$ with the horizontal one.

The exercise requires putting the right value of the angle (here $-\pi/2$ or $-\pi/4$) in the formulae given in the notes and applying the isometry to the four vertices of the square, drawing the vertices, and checking that the resulting figures correspond to the rotated (or reflected) square. It is completely algorithmic and actually does in coordinates what students have done in high school either by moving the drawing over one another or by ruler and compass. It provides a new procedural representation of a rigid motion (or reflection) and is thus clearly at the level of interiorization. I thought the problem was easy, but it did give a hard time to some of the students. The new procedural element 'Cartesian form $f(\vec{x}) = A\vec{x} + \vec{t}$ of an isometry' definitely needed to be interiorized.

The following two exercises are more difficult. Both suppose some mastery of two mental representations of an isometry: as a rigid motion like in high school and as a function is Cartesian form. Exercise 2 also uses the Cartesian equation describing a line; it is recalled in the question. This exercise also proves a property of isometries (they transform lines into lines), thus adding to the (hopefully) emerging mathematical object.

> **Exercise 2.** The parametric equation of a line is of the form $d(x) = x\,\vec{a} + \vec{b}$. The line goes through $\vec{b}$ and has $\vec{a}$ as direction vector.
> (a) Obtain the image of this line by an isometry $f(\vec{x}) = A\vec{x} + \vec{t}$.
> (b) Show that this image is a line.
>
> **Exercise 3.** TRUE or FALSE. If TRUE, give a proof; if FALSE, give a counter-example.
> (a) Two segments of a line are congruent if they have the same length.
> (b) Two angles are congruent if they have the same measure.
> (c) A reflection applied to a polygon does not change its area.
> (d) Two isosceles triangles of the same area are congruent.
> (e) To be congruent is an equivalence relation between polygons.

I first comment on the 'easiness' of Exercise 3. The student's familiarity with the high school representation of rigid motions made it easy to decide between TRUE and FALSE for each statement. Only statement *(d)* caused some discussion. Some students noted that an isosceles triangle with a base and height of length 1 would have the same area as one with base 2 and height ½. Clearly these two triangles would look very different and no rigid motions would bring the first onto the second, so it was quickly agreed upon that *(d)* was FALSE. Statement *(e)* used the concept of equivalence relation that was introduced earlier in the course. Maybe it gave yet another representation of an equivalence relation and contributed to the beginning of a reification, not of the concept of isometry, but of that of equivalence relation.

Proving statements is always a challenge at this level and Exercise 2 proved to be the hurdle here. The main difficulty was to apply the isometry to the line, that is, to compute the compositions of two functions. The composition of functions is introduced in cégeps in Québec and maybe before. It is also discussed in courses taken by the students during their term before MAT 1101. But it remains difficult. In terms of Sfard's (1991) stages, students have not reified (and even condensed) the concept of composition of functions. Even proving the true statements of Exercise 3 remained difficult after the solution of the previous exercise. My teaching assistant and I knew about these difficulties, and we tried to help the students as follows.

After Exercise 1, it was natural to solve Exercise 2 by writing down this composition of functions in terms of Cartesian coordinates, spelling out the horizontal and vertical coordinates. In the short period of time allotted to the exercise period, it is unlikely that a complete interiorization is reached. Despite this, Exercise 2 was meant to lead toward condensation. Sfard (1991) identifies certain characteristics of condensation as follows: ability to think about a given

process as a whole, without the urge to go into details; and growing ease in alternating between different representations of the concept. In helping the students through the proof of Exercise 2, my teaching assistant suggested avoiding explicit Cartesian coordinates in favor of using simply $\vec{x}$ as the point upon which the isometry was applied. But the transition toward a simpler notation, and thus omitting the details of the procedural conception, was not easy for all.

I consider Exercise 4 below very difficult. It calls for many mathematical concepts that the students have mastered at this point, but have rarely used together. According to Sfard (1991), a key feature of a reified concept is that it can be used as an input into new processes. In this proof, the isometry, seen as a function on coordinates, is such an input.

> **Exercise 4.** Does every isometry have an inverse? If so, compute it. If not, give a counter-example.

The solution below identifies by numbers the many concepts that enter the proof. Despite the fact that all these are known, the proof is very difficult for my students.

> **Proof:**
> (1) The inverse g of a function f is such that $f \circ g(\vec{x}) = g \circ f(\vec{x}) = \vec{x}$ for all $\vec{x}$.
> (2) Let f and g be isometries. These are thus of the form: $f(\vec{x}) = A\vec{x} + \vec{t}$ and $g(\vec{x}) = B\vec{x} + \vec{u}$.
> (3) If g is the inverse of f, then
> $$\vec{x} = g \circ f(\vec{x}) = g(A\vec{x} + \vec{t}) = B \cdot (A\vec{x} + \vec{t}) + \vec{u} = (B \cdot A) \cdot \vec{x} + (B \cdot \vec{t} + \vec{u})$$
> and thus (4)
> $$B = A^{-1} \quad and \quad \vec{u} = -A^{-1} \cdot \vec{t}$$
> and (5) every isometry has an inverse because the matrix A is always invertible.

At step (1), the concept of the inverse of a function is used together with the composition of two functions. It is in (2) and (3) that the specific form of an isometry in Cartesian coordinates is used. Step (3) requires the computation of the composition, the associativity of matrix multiplication and regrouping terms so that they have the form of an isometry. Step (4) solves for B and $\vec{u}$, and uses the existence and unicity of solution of linear equations. Finally step (5) uses a property proved in the course that the matrix A is orthogonal, thus invertible.

This exercise is difficult, and I propose some of this type as challenges only once in a while. They are not meant to help reach any of Sfard's (1991) three stages in particular, though they might. Some students try (and enjoy) them, others stay away. After having tried them, a few may come to my office to have their solution checked or to ask for help. Discussions with these intrepid students are always inspiring. If these exercises have any use, it is in encouraging students to use any concepts and results they find appropriate for the problem at hand, and to not limit their horizon to the objects just introduced in class.

CONCLUSION

So, how do I write exercises about a given new mathematical concept? I let my mind wander and try to capture properties that I know about it, to draw parallels with lower-level objects that students master already, to indicate why the concept was introduced, how it is used, and so on. This brainstorming usually gives me a fairly long list of exercises. I then try to order them from what seems the easiest to the most difficult. Then comes the sobering question: will the students be able to do them? The answer is almost always no! The list must then be completed so that the students will be able to progress from one to the other. This part is the most difficult as it requires one to assess intuitively what difficulties the students will experience and what is a good way to lead them in the right direction though yet another exercise.

Do I use Sfard's (1991) three stages to do this job? To be honest, no. As said before, I have developed my way of preparing exercises by submitting many students to too difficult exercises and witnessing their disheartenment. It took me a while to understand their comments that "these are way too difficult!" It dawned on me that it is necessary to practice each step toward a good understanding which, in a course, often means condensation. This led me to break down the mathematical concept under study in minute pieces that could be understood easily, a step that made me start, in most of the cases, with algorithmic techniques. Did I discover and formulate Sfard's three stages? No, I did not. Is Sfard's model obvious to practicing mathematicians? Probably not. To formulate clearly the various steps of the very complex process of understanding mathematics is a major step toward better teaching. And her three stages capture very well what I go through when I learn new mathematics. Now that I know her model of the development of a mathematical concept, I can often recognize in my graded sets of exercises some that could help them reach interiorization and maybe condensation.

Since Sfard's work is already a quarter-century old (1991), there must be many refinements of her thoughts that I unfortunately ignore. The remaining lines raise questions that might have already been settled. They are aimed at a competency that is very hard to acquire, namely, the ability of writing proofs. What is the range of applicability of Sfard's model of development of mathematical concepts? Is teaching 'how to write a proof of a mathematical statement' similar to teaching a mathematical concept? Are there stages of interiorization, condensation and reification in the process of acquiring this ability? These questions are intimately related to the other plenary lecture of Annie Selden on the difficulties met by university mathematics students with proofs and proving. She and her husband have written extensively on these difficulties (as a starting point, see Selden, 2018) and have designed courses to help advanced students in the development of abilities in proving statements and recognizing when a logical reasoning is *or* is not a proof. Several universities have developed similar courses, either at the undergraduate or graduate levels. Clearly the ability of proving a statement is crucial for mathematicians, but the necessity to help students acquire this ability has only started to be recognized fully.

ACKNOWLEDGEMENT

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40+ YEARS OF TEACHING AND THINKING ABOUT UNIVERSITY MATHEMATICS STUDENTS, PROOFS, AND PROVING: AN ABBREVIATED ACADEMIC MEMOIR

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INTRODUCTION: HOW WE GOT INTO THE FIELD

I plan to take you on a journey through how my husband John and I, who have PhDs in mathematics and spent our early academic years in pure mathematics, got into research in mathematics education. Along the way, I will discuss the kinds of research we have done, some challenges we have faced, and where we are in our thinking today. We have been mainly concerned with university students' learning of mathematical ideas, concepts, and processes, especially with proof and proving.

Before we got interested in mathematics education as a research subject, we were mathematicians teaching at least some upper-division and graduate mathematics courses using the Moore Method (Mahavier, 1999). Doing so gave us a lot of exposure to students' proving difficulties in courses like abstract algebra and topology. However, in other courses, like calculus, we lectured. The problem, as we saw it, was that despite our seemingly well-constructed and thoughtful lectures, students often had misconceptions and didn't perform as we wished. We wondered why. This was the germ of our interest in mathematics education research at the university level.

I finished my PhD in topological semigroups in l974, when academic positions in U.S. mathematics departments were scarce and the number of new PhDs in mathematics was perhaps at an all-time high. In order to obtain two academic teaching positions, we accepted posts at universities in Turkey (1974-1978) and later in Nigeria (1978-1985). While teaching mathematics at the University of the Bosphorus in Turkey, we wrote a paper analyzing university students' errors in logical reasoning (Selden & Selden, 1978). In that paper, we analyzed Turkish undergraduate university students' proofs submitted in a Moore Method course in abstract algebra. When we returned to the U.S. in 1985, we took up two academic teaching positions in the Department of Mathematics at Tennessee Technological University. When we heard of the 1987 *Cornell Conference on Misconceptions and Educational Strategies in Science and Mathematics*, we decided to 'recast' our earlier paper on Turkish undergraduate university mathematics students' errors in logical reasoning in terms of misconceptions (Selden & Selden, 1987). This conference itself was an 'eye opener'. There we met Shlomo Vinner, who encouraged us to continue our research in mathematics education and to attend the conferences of the International Group for the Psychology of Mathematics Education (PME).

A little while later, in 1988, we attended the *Calculus for a New Century Symposium*, held at the National Academy of Sciences in Washington, DC. That was a 'heady' time with all sorts of people, mathematicians, administrators, engineers, and so forth, thinking about calculus reform and how to teach mathematics more actively and better. Shortly thereafter, we conducted a small study on whether students, who had completed first-semester differential calculus with a grade of C, could solve non-routine calculus problems, that is, problems they had not specifically been taught how to solve (Selden, Mason, & Selden, 1989). This led to two further studies of students' ability to solve non-routine calculus problems (Selden, Selden, & Mason, 1994; Selden, Selden, Hauk, & Mason, 2000). All three calculus studies are described below. This was our excursion into examining calculus students' ability to solve non-routine problems. After that, we switched our research to students' difficulties with proof and proving. The bulk of this paper is devoted to this work, including our "unpacking" paper (Selden & Selden, 1995), our "validation" paper (Selden & Selden, 2003), and our "affect" paper (Selden, McKee, & Selden, 2010), all leading up to a discussion of our more recent theoretical work (Selden & Selden, 2015).

OUR THREE CALCULUS STUDIES

After having attended 1987 *Cornell Misconceptions and Educational Strategies in Science and Mathematics* and the 1988 *Calculus for a New Century Symposium*, we decided to conduct a study on the ability of students who had completed differential calculus with a grade of C and were currently taking integral calculus, to solve non-routine differential calculus problems. This was partly inspired by the often expressed view of science and engineering professors that students just "can't do applications", where by applications they presumably meant the kinds of mathematical problems that come up in their courses. Our conjecture at the time was that applications, per se, were not the primary difficulty, rather we speculated that C differential calculus students weren't able to solve any non-routine problems, whether in pure mathematics or applied subjects.

This led to our developing, in conjunction with our department chair, Alice Mason, a one-hour test of five non-routine differential calculus problems which we administered to 17 volunteer students who had completed differential calculus with a grade of C (Selden et al., 1989). The way we decided that these were non-routine for our differential calculus students was to ask the teachers of all the first calculus sections for that year whether they had taught their students how to do any of the problems we developed. None had. For example, the second of the five non-routine problems was:

> Does $x^{21} + x^{19} - x^{-1} + 2 = 0$ have any roots between –1 and 0? Why or why not?

We analyzed all responses to the problems, with the major finding being, "Notably, not a single student solved an entire problem correctly and most solution attempts relied heavily on earlier, more elementary, mathematics" (Selden et al., 1989, p. 45). This led us to ask: *How would A and B differential calculus students do on the same problems? And, did they have the prerequisite knowledge?*

This led to our second calculus study (Selden et al., 1994), in which we administered the same non-routine test and a subsequent routine test of prerequisite knowledge questions to 20 volunteer A and 19 volunteer B students who had completed differential calculus with those grades and were currently enrolled in integral calculus. For example, the matching prerequisite knowledge questions for the second non-routine problem, given above, were:

> If $f(x) = x^5 + x$, where is f increasing?
>
> If $f(x) = x - 1$, find $f'(x)$.

If 5 is a root of $f(x) = 0$, at what point (if any) does the graph of $y = f(x)$ cross the x-axis?

The major finding in this study was:

Although they [the participating volunteer A and B students currently enrolled in integral calculus] *performed slightly better on our test of non-routine problems, two-thirds of the students failed to solve a single problem completely and more than 40% did not make substantial progress on a single problem. The routine test confirmed that these students possessed an adequate knowledge base of relevant calculus skills.* (Selden et al., 1994, p. 19)

Convinced that students eventually do learn differential calculus, we decided to explore the 'folk theorem' that students really learn a course in the next course that uses it. This time the same non-routine and routine tests were administered to 28 students who had gotten A, B, or C in their differential calculus courses, but who were currently taking differential equations. The major finding was:

More than half of these students [the participating volunteer students currently enrolled in differential equations] *were unable to solve even one problem and more than a third made no substantial progress toward any solution. A routine test of associated algebra and calculus skills indicated that many of the students were familiar with the key calculus concepts for solving the non-routine problems; nonetheless, students often used sophisticated algebraic methods rather than calculus in approaching the non-routine problems.* (Selden et al., 2000, p. 128)

Thus, one can have appropriate knowledge, but not think of using it. Why? We conjectured at the time that these students were accustomed to solving problems using worked examples from sections of their mathematics textbooks, and hence, had never had to think of how to start an arbitrary problem. Or, as Lithner (2008) later wrote, they were good at imitative reasoning, rather than creative reasoning.

While we conducted the third of our calculus studies not long after our second calculus study, as they say, "life happens". We got busy with teaching and other research and put aside the data for about six years until we had an opportunity to mentor a young fellow mathematician, Shandy Hauk, into mathematics education research. This delay in analyzing our data enabled us to follow-up on what happened to at least some of the participating students, and to ask the question, "*Does it matter whether students are able to solve non-routine problems?*" Perhaps surprisingly, our answer was both *yes* and *no*.

No, because the students in this study were among the most successful at the university by a variety of traditional indicators, both at the time of the study and subsequently, yet half of them could not solve a single non-routine problem. They had overall GPAs [grade point averages] of just above 3.0 [a B average] at the time of the study and almost double the graduation rate of the university as a whole. At least seven of them subsequently earned a master's degree and one a PhD in mathematics. (p. 147)

And

. . . yes, it does matter. Most mathematicians seem to regard this kind of [non-routine] problem solving as a test of deep understanding and the ability to use knowledge flexibly. In addition, most applied problems that students will encounter later will probably be at least somewhat different from the exercises found in calculus (and other mathematics) textbooks. It seems likely that much original or creative work in mathematics would require novel problem solving at least at the modest level of the problems in this study. (p. 147)

We have continued to think about the difficulty that students have in 'bringing to mind' factual knowledge that they possess when they might make good use of it during problem solving or

theorem proving. We have tried planting 'seeds' in the form of asking students to prove a useful, related technical set theory theorem early in our current 'proofs course' in order to see if they would recall that technique later when it would be useful for proving a point-set topology theorem. So far this has not proved to be the case, and we have found no useful psychology research on how one might cause *spreading activation* of a useful sort to occur when needed. *Spreading activation* is a way that cognitive psychologists explain the phenomenon that a person is able to more quickly recall information about a topic once a related concept has been introduced. As with much of the psychology research literature, phenomena are studied in order to understand how they work 'naturally', rather than how they might be harnessed to work in the service of some objective, such as solving mathematical problems or proving theorems.

A MOVE TO RESEARCH ON PROOF AND PROVING WITH OUR "UNPACKING" PAPER

After having worked on the three calculus studies, described above, and perhaps because we had taught Moore Method courses and transition-to-proof courses in a variety of ways, we had seen numerous student difficulties with proving. As a result, we switched our research interests to various aspects of proof and proving. Another reason for this may have been that, starting in 1989, we began attending the Advanced Mathematical Thinking Working Group sessions at the annual PME Conferences. That is how we became acquainted with the group's work on the book, *Advanced Mathematical Thinking* (Tall, 1991), which was already well underway when we joined the group. However, somewhat later, Tommy Dreyfus agreed to edit a special follow-up issue on advanced mathematical thinking for *Educational Studies in Mathematics*, and we were invited to submit a paper. For that special issue, we analyzed 'found data' from tests and examinations given in several of our own transition-to-proof courses (Selden & Selden, 1995).

Before writing a bit about what is in that paper, it might be interesting for those who might think that publishing research in mathematics education is easy, that for our first attempt at writing this paper, we received rather lengthy reviews with the strong suggestion that we "totally re-conceptualize" it. For us, as mathematicians, who had published a number of papers in mathematics research journals and who had directed or co-directed PhDs in mathematics, this was a rather puzzling injunction. However, we went to work for perhaps half a year, attempting to do what the reviewers had advocated. We later heard, via the grapevine, that our paper was perhaps read more in advance of publication than later. However, according to recent *Google* citations, what we refer to as our "unpacking" paper has had 269 citations to date.

Our "unpacking" paper has both theoretical and empirical parts. The empirical part came from the analysis of our students' test and examination papers. The students, who had previous experience with logical translation in their transition-to-proof courses, were asked to translate (i.e., unpack) calculus statements, not all true, written in mathematical English into equivalent logical versions, using the symbols $\forall, \exists, \vee, \wedge, \neg, \rightarrow$, and $\leftrightarrow$, and inserting all variables and quantifiers. One such mathematical statement was:

If f is defined at a, then $\lim_{x \to a} f(x)$ exists implies f is continuous at a.

As it happens, this particular calculus statement, is not true and you might want to figure out why. Correct responses were not unique. However, one sample correct unpacking, given in the paper, was:

$(\forall f \in F) (\forall a \in \mathbb{R}) \{(f \text{ is defined at } a) \rightarrow \lim_{x \to a} f(x) \text{ exists}) \rightarrow (f \text{ is continuous at } a)]\}.$

Empirical results included the following: For simplified informal calculus statements, just 8.5% of all unpacking attempts were successful. For actual statements taken directly from calculus texts, this dropped to 5%. We inferred that these students would be *unable* to reliably relate

informally stated theorems with the top-level logical structure of their proofs. Hence, these students could not be expected to construct proofs or evaluate their validity (Selden & Selden, 1995, p. 123).

In addition to the above empirical findings, we introduced, or extended, some theoretical constructs, only some of which have since been taken up by the mathematics education research community. We extended the notion of *concept image* to *statement image*, because we thought that individuals could have images of both definitions of mathematical concepts and of theorems relating mathematical concepts. We introduced preliminary versions of the notions of *proof framework* and of *proof validation*, which have recently been taken up by at least some researchers in undergraduate mathematics education (e.g., Inglis & Alcock, 2012, Weber, 2008). By a *proof framework* we meant the portion of a proof that can be written from just the logical structure of the statement of the theorem. We have since expanded upon this idea and I will discuss that later. By *proof validation* we meant reading and checking a proof for correctness. In the Appendix of our "unpacking" paper, we included a hypothetical validation of the theorem that the sum of two continuous functions is continuous. Tommy Dreyfus as editor said, at the time, that we could make the appendix into a separate paper, not for the special issue. But having had so much trouble getting our "unpacking" paper accepted, we declined.

In addition to the above theoretical distinctions, we also contrasted *formal* and *informal* mathematical statements. For example, *Differentiable functions are continuous*, is informal because a universal quantifier and a variable are omitted, and because it departs from the usual *if-then* form of the conditional. A corresponding formal version would be: *For all real-valued functions f, if f is differentiable, then f is continuous.* We see the former version as being more memorable, while the latter version facilitates students in beginning to construct a proof by writing a proof framework.

It took us awhile to get our next study conducted and published. This may have been because, in addition to teaching, we did a lot of expository writing, which unfortunately, while needed, and a service to the community, is not well rewarded in academia. In particular, at the time of the calculus reform movement, circa 1987, some individuals in the mathematics community were awarded an NSF-grant to publish a newsletter, *UME Trends: News and Reports on Undergraduate Mathematics Education*, about reform efforts.[1] Ed Dubinsky was selected as editor and he asked us to write a column which he described as like 'movie reviews' for mathematics education research papers. Thus, we developed the *Research Sampler* column, for which we wrote 26 columns and 36 news/feature articles, published in *UME Trends* and *MAA Online* from 1989 through 2001. Since newsletters are considered 'ephemeral', I do not think many of them are archived anywhere, but we have made our columns and news/feature articles available on www.academia.edu and www.researchgate.net, where some people have found and even cited them. Also, beginning in 1994 and continuing to this day, we write short abstracts of mathematics education research articles for the Media Highlights section of the *College Mathematics Journal*. Our next published research paper was concerned with the validation of proofs by undergraduates.

OUR "VALIDATION" STUDY

This was an exploratory study, published in the *Journal for Research in Mathematics Education*, of how eight mathematics and secondary education mathematics majors at the beginning of a transition-to-proof course validated (read and checked for correctness) four

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[1] In 2000, Ed Dubinsky wrote a *FOCUS* article about the founding of *UME Trends*, which is available at http://www.math.kent.edu/~edd/FocusArticle.pdf

student-generated arguments purported to be proofs of a single theorem (Selden & Selden, 2003). The theorem was:

> **Theorem**. For any positive integer n, if n^2 is a multiple of 3, then n is a multiple of 3.

We began the paper by expanding on what we meant by *validation*:

> *A validation is often much longer and more complex than the written proof and may be difficult to observe because not all of it is conscious. Moreover, even its conscious part may be conducted silently using inner speech and vision. Validation can include asking and answering questions, assenting to claims, constructing subproofs, remembering or finding and interpreting other theorems and definitions, complying with instructions (e.g., to consider or name something), and conscious (but probably nonverbal) feelings of rightness or wrongness. Proof validation can also include the production of a new text—a validator-constructed modification of the written argument—that might include additional calculations, expansions of definitions, or constructions of subproofs. Towards the end of a validation, in an effort to capture the essence of the argument in a single train-of-thought, contractions of the argument might be undertaken.* (Selden & Selden, 2003, p. 5)

There was a theoretical part to this paper[2], but I will concentrate here on the empirical findings. The participating students were told that the four 'proofs' that they were to read and evaluate had been generated by students like themselves in a previous year. The first student-generated 'proof' was the following:

> **Proof:** Assume that n^2 is an odd positive integer that is divisible by 3. That is $n^2 = (3n + 1)^2 = 9n^2 + 6n + 1 = 3n(n + 2) + 1$. Therefore, n^2 is divisible by 3. Assume that n^2 is even and a multiple of 3. That is $n^2 = (3n)^2 = 9n^2 = 3n(3n)$. Therefore, n^2 is a multiple of 3. If we factor $n^2 = 9n^2$, we get $3n(3n)$; which means that n is a multiple of 3. ∎ (Selden & Selden, 2003, p. 12)

We found at least 10 errors in the above 'proof'. Indeed, we presented it to our participants as the first argument to consider precisely because there are so many errors, with the notational ones being very easy to spot. However, all participants took a good deal of time reading and trying to make sense of the above argument, with one participant spending 15 minutes reading and rereading, before finally deciding it was a proof, except for one minor notational error.

The empirical findings on validation included the following. Participants' correct judgments on whether a given argument was, or was not, a proof went from 46% correct (i.e., essentially chance level) to 81% correct, after having considered and reconsidered the arguments. "Most of the errors detected were of a local/detailed nature rather than a global structural nature" (p. 24). To our knowledge, this was the first published paper to investigate students' proof validations, and later studies have confirmed this result. For example, the eye-tracking study by Inglis and Alcock (2012) found that "compared with mathematicians, undergraduate students spend proportionately more time focusing on 'surface features' of arguments" (p. 358).

At the end of the interview, participants were also asked debrief questions such as, (1) *When you read a proof is there anything different you do, say, than in reading a newspaper?* and (2) *Specifically, what do you do when you read a proof?* From their responses, we concluded that:

> *What students say about how they read proofs seems to be a poor indicator of whether they can actually validate proofs with reasonable reliability. They tend to "talk a good line." They say that they "check proofs step-by-step, follow arguments logically,*

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[2] For example, we took the view that the meaning of a proof resides in the proof text itself. Much like, for Martin Luther, the meaning of Scripture could be found in a deeper reading of the Bible (Selden & Selden, 2003, p. 6, Footnote 2).

> *generate examples, and make sure the ideas in a proof make sense."* (Selden & Selden, 2003, p. 27)

The "unpacking" and "validation" papers, described above, while introducing some theoretical concepts such as proof frameworks and proof validation, were largely empirical. After that, we became more theoretical in our research and writings.

MOVING TOWARD A THEORETICAL PERSPECTIVE ON PROOF AND PROVING

The next research paper, which we sometimes refer to as our "affect" paper, appeared in a special issue of the *International Journal of Mathematical Education in Science and Technology* (*iJMEST*). It considered the role of consciousness in the proving process and introduced the ideas of *<situation, action>* pairs, *behavioral schemas*, and *non-emotional cognitive feelings* (Selden et al., 2010). We wrote:

> *We see (at least the conscious part of) cognition in general, and the proving process in particular, as a sequence of mental and physical actions, such as writing or thinking a line in a proof, drawing or visualizing a diagram, reflecting on the results of earlier actions, or trying to remember an example. Many such actions appear to be guided by small 'habits of mind' that often link a particular recognized situation to a particular action. Such <situation, action> pairs, or habits of mind, can reduce the burden on working memory.*

> *As a person gains experience, much of proof construction appears to be separable into sequences of small parts, consisting of recognizing a situation and taking a mental or physical action. Actions which once may have required a conscious warrant become automatically linked to triggering situations. From a third-person, or outside, perspective these regularly linked <situation, action> pairs might be regarded as small 'habits of mind' [14]. On the other hand, taking a first-person, inside, or psychological perspective, they are lasting mental structures that we have called behavioral schemas.* (Selden et al. 2010, p. 204)

An example of a *non-emotional cognitive feeling* would be a feeling of being on the right track. A *behavioral schema* that might be invoked during proving begins with a situation. For example, one might be starting to prove a statement having a conclusion of the form '*p* or *q*'. This would be the situation. Having encountered this situation many times before, one might readily write into the proof "*Assume not p*" and proceed to attempt to prove *q* or vice versa. While this action can be warranted by logic ('if not *p* then *q*', is equivalent to, '*p* or *q*'), there would no longer be a need to do so.

In that "affect" paper, we presented a six-point theoretical sketch of the genesis and enactment of behavioral schemas, which I paraphrase below:

1. Within very broad contextual considerations, behavioral schemas are immediately available. They do not normally have to be recalled, that is, searched for and brought to mind.
2. Simple behavioral schemas operate outside of consciousness. One is not aware of doing anything immediately prior to the resulting action—one just does it.
3. Behavioral schemas produce immediate action, which may lead to subsequent action. One becomes conscious of the resulting action as it occurs or immediately afterwards.
4. One cannot 'chain together' behavioral schemas in a way that functions entirely outside of consciousness and produces consciousness of only the final action.
5. An action due to a behavioral schema depends on conscious input, at least in large part.
6. Behavioral schemas are acquired through practice. To acquire a beneficial schema a person should actually carry out the appropriate action correctly a number of times—not

just understand its appropriateness. Changing a detrimental behavioral schema requires similar, perhaps longer, practice.

(Selden et al., 2010, pp. 205-206).

Behavioral schemas can be detrimental, as well as beneficial. An example of a detrimental behavioral schema that students often learn tacitly is that $\sqrt{a^2 + b^2}$ equals $a + b$. Showing a counter example is ineffective—almost any two numbers substituted for a and b show this is false. However, students persist. We would not refer to this as a misconception, but rather characterize it as a 'misaction'. Perhaps the most interesting detrimental schema reported in our "affect" paper was the case of a mathematics graduate student, Sofia, in our 'proofs course' who had developed an *unreflective guess* schema, which we also thought of as *grasping at straws*. Enacting this schema often prevented Sofia from making progress on constructing proofs. Since one such unreflective guess often led to another for Sofia, we wanted to rid her of this 'bad habit', so we suggested substitute actions such as: draw a figure, look for inferences from the hypotheses, reflect on everything done so far, or even do something else for a while (p. 212). By the end of the semester, we felt Sofia had made much progress. To substantiate this, we included in the paper the following:

> *As the course ended, our intervention of directing Sofia to do something else, whether it be draw a diagram or review her notes, was beginning to show promise. For example, on the in-class final examination Sofia proved that if f, g, and h are functions from a set to itself, f is one-to-one, and f ∘g = f ∘h, then g = h. Also on the take-home final, except for a small omission, she proved that the set of points on which two continuous functions between Hausdorff spaces agree is closed. This shows Sofia was able to complete the problem-centred parts of at least a few proofs by the end of the course, and suggests her 'unreflective guess' behavioural schema was weakened.*
> (Selden et al., 2010, p. 212)

AN EXCURSION INTO RESEARCH ON READING

We next investigated how precalculus and calculus students read their textbooks (Shepherd, Selden, & Selden, 2012). This study was conducted with a mathematician colleague, Mary Shepherd, who wished to be mentored into the field. The students in this study had high ACT[3] mathematics and reading scores and did much of what good readers do. The selected participants were invited to read passages, a little ahead of where they currently were reading for the course, but which their teacher judged were accessible to them. The research question we asked was: Could they work straightforward tasks associated with the reading soon after reading passages explaining, or illustrating, how the tasks should be carried out, and with those passages still available to them?

Our empirical findings included: Only three of the eleven volunteer students could independently work at least half of the tasks. Why? Their difficulties seemed to arise from: (1) insufficient sensitivity to, or inappropriate response to, confusion or error; (2) inadequate or incorrect prior knowledge; and (3) insufficient attention to details, often due to mind wandering (Shepherd et al., 2012, p. 238).

We wondered: Why are mathematicians good readers of mathematical texts? While we did not go on to investigate this, our colleague, Mary Shepherd did. She explored how mathematicians read mathematical material unfamiliar to them (Shepherd & van de Sande, 2014). Being a

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[3] Most students in the US are required to have at least minimum scores (set by each university) on a national reading comprehension test and a national mathematics test, either the ACT or the SAT, as well as other qualifying materials in order to be admitted to the university. The ACT tests are provided by American College Testing, Inc.

mathematician herself, she selected the beginning section of a volume on differential geometry, a subject her participants were unfamiliar with, but was in her area of expertise. Perhaps the most interesting finding of this study, was that the mathematicians, but not the graduate student participants, often engaged in what she called *reading-the-meaning*. For example, when coming to the definition of a metric space, the mathematicians quickly noted this without reading the words or symbols—it was as if they were seeing a familiar icon. In addition, the authors proposed a framework for reading mathematical exposition from novice to intermediate to expert (Shepherd & van de Sande, 2014, p. 85).

Of interest, to young researchers perhaps, is that it took 10 years from my first answering Mary Shepherd's RUME email listserv request for help until the publication of our joint paper. Not knowing the reading comprehension research literature, Mary and I delved into it for about two years and corresponded via email about it. Then we, with John, met to design the study and Mary collected the data. Then, for awhile, I got sick and the study was set aside. After we finally submitted our manuscript, the journal editor took about a year to get back to us and rejected it. Then the three of us took what we thought were the legitimate criticisms of the reviewers, rewrote the paper, and submitted it to *Mathematical Thinking and Learning*. This time we got back a "*revise and resubmit*", rewrote the paper again, and eventually it was published. (Shepherd et al., 2012).

BACK TO PROOF AND PROVING: OUR CURRENT PERSPECTIVE ON PROOF CONSTRUCTION

Much of our current theoretical perspective was detailed in our *RUME Proceedings* paper (Selden & Selden, 2015), which was given an honorable mention by the Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education (SIGMAA on RUME). It was an attempt to 'weave together' all of our previous work on proof, together with more recent research from 10+ years of teaching our inquiry-based 'proofs course'. That course is taught mainly to beginning mathematics graduate students who feel they need help with proof construction. It is taught entirely from our own notes, with students constructing original (to them) proofs and receiving, sometimes extensive, critiques in class. Topics very briefly covered include sets, functions, real analysis, abstract algebra (in the form of semigroups), and if time permits, some point-set topology. A major aim of the course is to facilitate students' learning through experiences constructing as many different kinds of proofs as possible. Another important aim of the course is to have students learn to write proofs acceptable to their other professors. (For a more detailed description of the course, see Selden, et al., 2010, p. 207.)

We video and analyze all classes and many planning sessions. As a consequence, we have developed a theoretical perspective of the proving process that includes: (a) mathematical aspects and (b) psychological aspects. First, I will discuss some of the mathematical aspects of our theoretical perspective including the genre of proofs and the structure of proof texts.

MATHEMATICAL ASPECTS: THE GENRE OF PROOFS

Our thinking and research on this topic actually began around Summer 1999 when we attended the Institute for Advanced Study/Park City Mathematics Institute (PCMI), which is designed for mathematics researchers, post-secondary students, and mathematics educators at the secondary and post-secondary levels. We had previously observed in our teaching that students sometimes find the manner in which proofs are written perplexing. That is, it is often at variance with other genres of writing, and we had identified some significant features that generally occur in proofs.

While at PCMI, we interviewed volunteer mathematicians about what they thought about some of our conjectured features of proofs, while they were looking at one of their own published mathematics papers. These features are indicated below.

1. Proofs are not reports of the proving process.
2. Proofs contain little redundancy.
3. Symbols are (generally) introduced in one-to-one correspondence with mathematical objects.
4. Proofs contain only minimal explanations of inferences, that is, warrants are often left implicit.
5. Proofs contain only very short overviews or advance organizers.
6. Entire definitions, available outside the proof, are not quoted in proofs.
7. Proofs are *logically concrete* in the sense that quantifiers, especially universal quantifiers, are avoided where possible.

(Selden & Selden, 2013b)

None of the above features is very surprising for mathematicians, especially the first one. However, we know of no other study on this topic. But as has so often happened with us, life and other academic duties interfered, and the data were put aside and not actually published until the book originating from the Symposium in Honor of Ted Eisenberg's retirement was published (Selden & Selden, 2013b).

STRUCTURES OF PROOFS

Some of our more recent thinking on this was incipient in our earlier papers (e.g., Selden & Selden, 1995, 2003). A proof text can be divided into a *formal-rhetorical* part and a *problem-centered* part. The *formal-rhetorical* part is the part that depends only on unpacking the logical structure of the statement of the theorem, associated definitions, and earlier results. In general, this part does not depend on a deep understanding of the concepts or genuine problem solving in the sense of Schoenfeld (1985, p. 74). We call the remaining part of a proof the *problem-centered* part. It does depend on problem solving, intuition, heuristics, and understanding the concepts involved (Selden & Selden, 2011).

A feature that can help write the formal-rhetorical part of a proof is what we have called a *proof framework*, an idea we introduced earlier in our "unpacking" paper. However, we have since expanded on this idea to include several different kinds, and in most cases, both a *first-* and a *second-level* framework. We have detailed this in a recently published *PRIMUS* paper (Selden, Selden, & Benkhalti, 2017). Briefly, given a theorem of the form "For all real numbers x, if $P(x)$ then $Q(x)$", a *first-level proof framework* would be "Let x be a real number. Suppose $P(x)$. … Therefore, $Q(x)$", with the remainder of the proof ultimately replacing the ellipsis. *A second-level framework* can often be obtained by 'unpacking' the meaning of $Q(x)$ and putting the second-level framework between the lines already written for the first-level framework. Thus, the proof would 'grow' from both ends toward the middle, instead of being written from the top down.

To write a second-level framework, one often needs to convert formal mathematical definitions and previously proved results into their *operable interpretations*—something that we initially found surprising. For example: Given a function $f: X \rightarrow Y$ and $A \subseteq Y$, one defines $f^{-1}(A) = \{x \in X \mid f(x) \in A\}$. An operable interpretation would say, "If you have $b \in f^{-1}(A)$, then you can write $f(b) \in A$ and vice versa". One might think, as we previously often did, that translation into an operable form would be unnecessary or easy, especially because the symbols in $\{x \in X \mid f(x) \in A\}$ can be translated into words in a one-to-one way, but for some students it requires both help and practice.

THE NEED FOR PREVIOUS RESULTS—PROOFS OF TYPES 0, 1, 2, 3

In order to enhance the possibility of student successes in our inquiry-based 'proofs course', we have classified theorems of increasing difficulty in our course notes which consist of statements of theorems, definitions, and questions. While we had been thinking about proof difficulty for some time, I believe we first discussed some of these proof types in an invited article for a special issue of *The Mathematics Enthusiast* (Selden & Selden, 2013a). Our current classification into proof types is as follows:

> *Type 0* often follows immediately from definitions.
>
> *Type 1* may need a result in the notes.
>
> *Type 2* needs a lemma, not in the notes, but relatively easily to discern, formulate, and prove.
>
> *Type 3* should have at least one of discern, formulate, or prove be difficult.

Here is an example of a Type 3 proof of a theorem from our course notes: *A commutative semigroup S with no proper ideals is a group*, when one is provided only the definitions of *semigroup* and *ideal*. One first needs to observe that, for $a \in S$, aS is an ideal, so $aS = S$. This, in turn, implies that equations of the form $ax = b$ are solvable for any $a \in S$ and any $b \in S$. We could have formulated these two facts as lemmas in advance of the theorem statement, but we chose not to. If one formulates and proves these two lemmas, then, using some clever instantiations of the equation $ax = b$, one can obtain an identity and inverses, and conclude S is a group. At last count, in our 'proofs course', only two of 74 students, after much hard work, have been able to prove this theorem on their own. (Additional details about this proof and about one mathematician's proving attempt can be found in John Selden's Topic Session paper in this *Proceedings*.)

One of the reasons for classifying proofs into types according to difficulty is to be able to estimate, in advance, which students to call on to present their proof attempts at the board. We feel that if a proof is too easy for a student or if a proof is too hard for a student, then probably nothing will be learned by that student or the class. Worst of all, being called upon to present a proof attempt that one does not think is worth discussing may lower that student's sense of self-efficacy, a topic addressed below.

THE NEED FOR UNGUIDED EXPLORATION

In constructing some proofs, one may reach a point where there is no 'natural' way forward. One has come to an impasse, that is, colloquially, one is 'stuck'. In what we call *unguided exploration*, one may need to find, or define, an object and prove something about it, with no idea of its usefulness, that is, one may need to 'explore' the situation to get an insight.

For example, in proving the above Type 3 semigroup theorem, this kind of exploration, followed by a helpful insight, can happen at least twice. The first helpful insight comes when one notes that aS is an ideal, and hence, $aS = S$. The next helpful insight comes when one sees that the set equation, $aS = S$, implies that element equations of the form $ax = b$ are solvable for any $a \in S$ and any $b \in S$. We feel that many Type 3 proofs may require considerable perseverance and self-efficacy. We try to engender this in students by arranging for early proving successes, followed by assigning proofs of increasing difficulty. Discussing associated heuristics well before a target theorem arises may also be useful.

THE NEED TO UNPACK THE LOGICAL STRUCTURE OF A THEOREM STATEMENT

We made the distinction between informal and formal mathematical statements in our "unpacking" paper, mentioned briefly above. *Informally stated theorems* are commonplace in

everyday mathematics. They are not ambiguous or ill-formed because widely understood, but rarely articulated, conventions permit their precise interpretation by mathematicians and less reliably by students. Such informally stated theorems can be memorable and perhaps easily brought to mind, but may be difficult to unpack and prove, especially for beginners.

We have found that proof frameworks are relatively easy to write for theorems stated in the customary *if-then* format. Thus, early on in our notes, we write theorem statements in this format. However, we know that students must eventually be able to unpack informally stated theorems into their *if-then* format in order to decipher what the theorem is stating, what the hypotheses are, and what the conclusions are in order to begin the process of proof construction by writing a proof framework. Next, I will discuss some psychological aspects of the proving process.

PSYCHOLOGICAL ASPECTS OF THE PROVING PROCESS

As mentioned above, we view proof construction as a sequence of actions which can be physical (e.g., writing a line of the proof or drawing a sketch) or mental (e.g., changing one's focus from the hypothesis to the conclusion, trying to recall a theorem, or bringing up a feeling). The sequence of actions that eventually leads to a proof is usually considerably longer than the final proof and is often not constructed from the top down. Somewhat surprising to us, we once had a mathematics education graduate student from the School of Education, who did not know that proofs were not constructed from the top-down. Her recollection from a prior real analysis course was that the professor always wrote proofs from the top-down in lectures, so she had just assumed that was the way proofs are constructed.

Some of what I write next may seem similar to what I wrote above. That is, no doubt, because theoretical perspectives develop slowly over time and our current perspective grew out of our earlier theoretical observations and empirical studies.

SITUATION-ACTION LINKS, AUTOMATICITY, AND BEHAVIORAL SCHEMAS

If, during several proof constructions in the past, similar situations have corresponded to similar reasoning, leading to similar actions, then a *link* may be learned between them, so that another similar situation evokes the corresponding action in future proof constructions without the need for the earlier intermediate reasoning. Using such *situation-action links*, or *<situation, action>* pairs as we called them earlier, strengthens them, and after sufficient experience/practice, they can become overlearned and automated, and hence, become *behavioral schemas*. These are the same behavioral schemas that I described above and whose six properties I mentioned.

There are cognitive advantages to invoking automaticity appropriately during proof construction. So we aim to help our students convert System 2 (S2) cognition into System 1 (S1) cognition where appropriate. *S2 cognition* is slow, conscious, effortful, evolutionarily recent, and calls on considerable working memory. In contrast, *S1 cognition* is fast, unconscious, automatic, effortless, evolutionarily ancient, and places little burden on working memory (Stanovich & West, 2000). Converting S2 cognition into S1 cognition conserves working memory, a precious resource.

In discussing automaticity, we are largely depending on the work of social psychologists (e.g., Bargh & Chartrand, 2000). In general, according to Bargh (1994), an individual executing an automated action tends to: (1) Be unaware of any needed mental process; (2) Be unaware of intentionally initiating the action; (3) Execute the action while putting little load on working memory; and (4) Find it difficult to stop or alter the action. However, not necessarily all four tendencies occur in every situation. We feel that the first three of these tendencies, appropriately

harnessed during proof construction, would help conserve students' working memory for the truly hard parts of proofs.

We view behavioral schemas as belonging to a person's knowledge base. They can be considered as partly conceptual knowledge (recognizing and interpreting the situation) and partly procedural knowledge (doing the action), and as related to Mason and Spence's (1999) idea of "knowing-to-act in the moment". We aim to encourage and develop beneficial behavioral schemas for proving in our students and discourage, and hopefully extinguish, detrimental behavioral schemas for proving, such as Sofia's detrimental *unreflective guess* schema, described above.

As a result of enacting beneficial proving behavioral schemas advantageously, students might simply not have to think quite so deeply about certain portions of the proving process, and might have more working memory available for the harder parts of a proof. However, helping students develop beneficial behavioral schemas is no easy task, because the process of learning a behavioral schema can often be implicit, although the situation and the action are at least, in part, conscious. That is, an individual can acquire a behavioral schema without being aware that it is happening. Indeed, such unintentional, or implicit, learning happens frequently (e.g., Cleeremans, 1993).

NONEMOTIONAL COGNITIVE FEELINGS IN PROOF CONSTRUCTION

We are particularly interested in the kinds of affect that might occur during proof construction, and have considered feelings, especially *nonemotional cognitive feelings*, as mentioned above. Often the terms *feelings* and *emotions* are used more or less interchangeably, perhaps because both appear to be conscious reports of unconscious mental states, and each can, but need not, engender the other. However, we follow Damasio (2003) in separating feelings from emotions because emotions are expressed by observable physical characteristics, such as temperature, facial expression, blood pressure, pulse rate, perspiration, and so forth, while feelings are not.

Here are some examples of the kinds of nonemotional cognitive feelings we are interested in: (1) a feeling of knowing that one has seen a theorem useful for constructing a proof, but which one is not able to bring to mind at the moment; (2) a feeling of familiarity; and (3) a feeling of rightness. Such nonemotional cognitive feelings can guide cognitive actions. For example, these can influence whether one continues a search for a solution or a proof or aborts it.

Feelings seem to be summative in nature and pervade an individual's whole field of consciousness at any particular moment. For example, one can have a feeling of unease in the midst of concentrating on developing a proof or solving a problem. Finally, we conjecture that feelings may eventually be found to play a larger role in proof construction than indicated above, because they can provide a direct link between the conscious mind and the structures and possible actions of the nonconscious mind, which can process many streams of information in parallel.

THE ROLE OF SELF-EFFICACY IN PROOF CONSTRUCTION

In order to prove harder theorems—ones with a substantial problem-centered part—students need to persist in their efforts, and such persistence can be facilitated by a sense of *self-efficacy* (Selden & Selden, 2014). According to Bandura (1995), *self-efficacy* is "refers to beliefs in one's own capabilities to organize and execute the courses of action required to manage prospective situations" (p. 2). Of developing a sense of self-efficacy, Bandura (1994) stated that, "The most effective way of creating a strong sense of self-efficacy is through *mastery experiences*" (p. 71, emphasis in original), that performing a task successfully strengthens one's sense of self-efficacy. Also, according to Bandura (1994), "Seeing people similar to oneself

succeed by sustained effort raises observers' beliefs that they too possess the capabilities to master comparable activities to succeed" (p. 72). Bandura's ideas 'ring true' with our past experiences as mathematicians teaching courses by the Modified Moore Method. Seeing a professor succeed at proving a theorem does little to promote students' sense of self-efficacy.

Some of the ways we attempt to develop students' self-efficacy are by: (1) letting them know early on that the *raison d'être* of the course is to help them with their proof writing, (2) arranging for early proving successes, and (3) slowly increasing the difficulty of the proofs in the course notes.

THE DEVELOPMENT AND USE OF LOCAL MEMORY

In constructing a proof of some complexity, often much more relevant information is brought to mind than can be held in one's working memory. When such information is lost from consciousness, it may remain partially activated and easily accessed. We refer to such partially activated information as *local memory*. While we have experienced such easily accessed information during our own mathematical research, as well as its loss when too many days or weeks have passed by, we know of little research about its development, maintenance, or uses even in the psychology literature. Nonetheless, it seems to us that conscious thought can sometimes influence the activation of such information, that is, help bring something helpful to mind. We have observed of ourselves, when attempting an intricate complex proof, that a considerable amount of information is generated, but cannot all be kept in mind; however, it is easily recalled. We speculate that many mathematicians experience this when conducting their own research. We feel local memory is an area worthy of future research.

What good is a theoretical perspective if one cannot use it for anything? Below, I discuss how we use our perspective to construct proofs and analyze students' proof attempts. I also indicate how we use our perspective in designing our 'proofs course'.

USING OUR PERSPECTIVE TO CONSTRUCT AND ANALYZE PROOFS

First, I show how one can construct a sample correct proof of a theorem using a proof framework and operable interpretations of definitions. Then I use our perspective to analyze an incorrect student proof attempt of the same theorem.

CONSTRUCTION OF A SAMPLE CORRECT PROOF

The theorem I consider comes from the semigroup portion of our 'proofs course'. I begin with the statement of the theorem and its proof, showing both the first- and second-level proof frameworks. The second-level proof framework, lines [3] and [4] below, comes from unpacking the meaning of *commutative*.

> **Theorem.** Let S be a semigroup with an identity element e. If, for all s in S, $ss = e$, then S is commutative.
>
> **Proof:**
>
> [1] Let S be a semigroup with identity e. Suppose for all $s \in S$, $ss = e$.
> [3] Let a, b be elements in S.
>
> …
>
> [4] Thus $ba = ab$.
> [2] Therefore, S is commutative. QED.

I now continue with the rest of the proof, filling in the ellipsis above. This may involve a bit of 'messing around' with equations, that is, it may require quite a bit of exploration. A prover of

this semigroup theorem can write many unhelpful equations before coming upon useful equations that result in a proof. After selecting parts of his/her exploration and rearranging them, the remaining part could have been written as lines [5], [6], and [7] below.

> **Theorem.** Let S be a semigroup with an identity element e. If, for all s in S, $ss = e$, then S is commutative.

> **Proof:**

> [1] Let S be a semigroup with identity e. Suppose for all $s \in S$, $ss = e$.
> [3] Let a, b be elements in S.
> [5] Now, $abab = e$, so $(abab)b = eb = b$
> [6] But $(abab)b = aba(bb) = abae = aba$.
> [7] So, $b = aba$, so $ba = (aba)a = ab(aa) = abe = ab$.
> [4] Thus $ba = ab$.
> [2] Therefore, S is commutative. QED.

We are not claiming that teaching students to write proof frameworks is a panacea. Rather, being able to write a complete proof framework exposes the 'real problem' to be solved during the proof construction process. We are also not claiming that mathematicians write proofs this way—only that they will accept the results of proofs written this way.

A STUDENT'S INCORRECT PROOF ATTEMPT OF THE SAME THEOREM

First, I will present what the student wrote and submitted as a proof of the theorem. Then I will analyze it using our theoretical perspective, looking for beneficial actions *not taken* and detrimental actions *taken*. Recall that the theorem is:

> **Theorem.** Let S be a semigroup with an identity element e. If, for all s in S, $ss = e$, then S is commutative.

The student's incorrect proof attempt, including scratch work, is given below:

Let S be a semigroup with an identity element, e. Let $s \in S$ such that $ss = e$. Because e is an identity element, $es = se = s$. Now, $s = se = s(ss)$. Since S is a semigroup, $(ss)s = es = s$. Thus $es = se$. Therefore, S is commutative. QED.	Scratch Work 7.1: A semigroup is called commutative or Abelian if, for each a and $b \in$ S, $ab = ba$. 7.5: An element e of a semigroup S is called an identity element of S if, for all $s \in S$, $es = se = s$.

ANALYSIS OF THE STUDENT'S INCORRECT PROOF ATTEMPT

First, we look at beneficial actions *not taken*. The second sentence should have been "Suppose for <u>all</u> $s \in S$, $ss = e$". With this change to include "all $s \in S$", the first-level framework would have been correct. In addition, the student did not produce a second-level framework by introducing arbitrary a and b at the top, followed by "Then $ab = ba$" right above the conclusion.

Had the student written the correct second sentence and taken the above actions, the situation would have been appropriate for exploring and manipulating an object such as *abab*. We think that such exploration calls for some self-efficacy, but can lead to a correct proof.

Next, I consider what the student wrote 'in the middle', and analyze it line-by-line. It was:

> Because e is an identity element, $es = se = s$.
> Now, $s = se = s(ss)$.
> Since S is a semigroup, $(ss)s = es = s$.
> Thus $es = se$.

The first line of 'the middle' above violates the mathematical norm of not including definitions that can easily be found outside the proof, in the proof. Also, it does not move the proof forward. The next three lines are not wrong, but also do not move the proof forward because to prove commutativity, one needs two arbitrary elements. We consider these actions detrimental because they can convince the student that he/she has accomplished something when that is not the case. This completes the analysis of the incorrect student proof attempt.

HOW WE USE OUR PERSPECTIVE IN DESIGNING OUR 'PROOFS COURSE'

I only mention a couple of things we do in designing and teaching the course. We want students to have early successes, so they gain a sense of self-efficacy. Therefore, we try to have relatively easy theorems at the beginning of our notes and gradually increase their difficulty. At the beginning of the course, we have students practice writing proof frameworks (without necessarily having to write complete proofs). There are several kinds of proof frameworks, but I have only demonstrated one kind. For example, other proof frameworks can involve proofs by cases or proofs by parts. There is yet another proof framework for proofs by contradiction.

As for operable interpretations of definitions which are needed to write second-level proof frameworks, we have used handouts (with definitions on the left side of the paper and operable interpretations on the right side). The idea of doing this was to have students make flash cards using these handouts and practice these operable interpretations. Unfortunately, while these handouts were somewhat helpful, we found some students, who despite having these handouts available on our examinations, still did not use them appropriately in writing their proofs.

TEACHING AND FUTURE RESEARCH CONSIDERATIONS

We believe this perspective on proving, using situation-action links and behavioral schemas, together with information from psychology, is mostly new to the field. Thus, it is likely to lead to additional insights and teaching interventions. This brings up the question of priorities. For example: Which proving actions of the kinds discussed above are most useful for mid-level university mathematics students to automate when they are learning to construct proofs?

Since every proof can be constructed using a proof framework, we consider constructing proof frameworks as a reasonable place to start. Doing so will get students started on proofs, as well as know where they are headed. This is preferable to staring at a blank piece of paper and not knowing what to do. Furthermore, we are not claiming that mathematicians write proofs this way—only that professors will accept the results of writing proofs this way.

For students to have early successes and build self-efficacy, one can begin with more formally stated *if-then* theorems and later go to more informally stated theorems, which are harder to unpack. Furthermore, we have observed that some students do not write a second-level proof framework, perhaps because they have difficulty unpacking the meaning of the conclusion into an operable form. What are some effective ways of getting students to 'unpack' and use mathematical definitions in their proofs?

SUMMARY AND CONCLUSION

In this paper, I first briefly mentioned: (1) how we got into mathematics education research; (2) our early calculus studies; (3) our "unpacking" paper; (4) our "validation" paper; (5) our "affect" paper; and (6) our "reading" paper. Then, I discussed at greater length our current theoretical perspective on proof construction, how we use it to analyze student proof attempts,

and how we use it in designing our 'proofs course', as well as mentioning a couple of research questions. We would be pleased if others considered some of these ideas.

ACKNOWLEDGEMENTS

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Elder Talk

La parole aux anciens

'ELDER TALK'—A REVISIONIST VERSION

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When I gave my 'elder talk' at McGill this past spring I intended to cover the first ten years of CMESG by recounting some personal experiences and anecdotes. I mistakenly assumed that the 45 minutes allotted for my talk were to be measured in 'elder minutes' (1 'elder minute' = 5 minutes) but I was disabused of this notion by the Chair who told me that time was up when I was still somewhere in the middle of Year 2.

I hope that during my talk I at least managed to convey the tremendous influence two people had on my 'conversation' from mathematics to mathematics education. One was David Wheeler who joined the Math Department at Concordia in 1975. The other was Tom Kieren who, by sheer luck, happened to be on sabbatical in the Math Education Department of the University of Georgia at the same time as I was. I could not have had better mentors and I am including a photo of Tom, David, and I taken at some CMESG meeting.

Figure 1. Photo of Tom Kieren, David Wheeler and Joel Hillel (left to right) at CMESG.

In what follows, I will recount few of the stories that I did not get to tell during my talk.

1978-1981: THE PROBLEM SOLVING YEARS

Problem Solving was the dominate theme of math education in the 1970's and David Wheeler and I embarked on a 3-year research project on problem solving which was funded by Quebec's FCAR grant (Leslie Lee joined the team fairly early on). It was the first venture into math education research for me and, surprisingly, also for David, notwithstanding all the incredible contributions that he had already made to mathematics education. While most research projects on problem solving at the time focused on the heuristic enterprise (i.e., the possibility and effectiveness of teaching heuristics to enhance problem solving), we decided to focus our research on examining the mental operations used by 'naïve' students during the solving process. We deliberately chose to use 'non-standard' mathematics problems (e.g., cryptarithm-type problems: "Can you cut a square into 7 square pieces?") so as not to make our subjects think that they are being tested on school math knowledge. We adopted the 'thinking aloud' methodology that was becoming more acceptable as a research technique once Krutetskii's (1976) book on mathematical abilities became better known.

These were days before elaborate research protocols on the use of 'human subjects' were necessary and getting participants for the study was a simple matter of walking to the closest high school and asking the math teacher to solicit volunteers ("Johnny, why don't you go and work with this math professor from Concordia", "yes, sir"). Our subjects were mostly junior high school students and over the course of the research we ended with nearly 50 transcribed interviews.

The 'thinking aloud' methodology had its limits. Most solvers, even those who were good verbalizers, became silent during critical junctures of the solution process. But, with some probing along the way, post-solution interviews, and having a record of the solvers' written work, one could construct a pretty reliable description. (An aside: after spending lots of time talking about the 'thinking aloud' technique and how we were to conduct ourselves as interviewers by minimizing our interventions, I remember when we listened to the tape of the first session conducted by David. Within a millisecond of the student getting stuck, David was already jumping in and offering suggestions. Old teaching habits die hard, but we did improve our interviewing technique as we progressed.)

There was the familiar tension of what to make out of all the data we accumulated. Part of the challenge was to see what remained after taking out the idiosyncratic features of each of the protocols. What we ended up with was a much better understanding of the kinds of challenges that our experimental problems created for 'naïve' solvers. This led to a much sharper 'task analysis' whereby each problem was analyzed in terms of a set of 'demands' (mathematical, structural—including goal definition, operations, and relations, psychological). The solvers' behviours were subsequently analyzed in terms of their attempts to deal with these demands.

One of the most striking aspect of the analysis was the observation that the process of 'understanding the problem' was often continuing till nearly the very end of the solution process. Certainly, Pólya's (1945) normative 4-stages description of the problem solving process ('understand the problem', 'devise a plan', 'carry out the plan', 'look back') was a far cry from the behavior of our novice solvers. Neither was the Information Processing model of problem solving with its goal-oriented approach at all evident with our solvers and with the type of problems we used.

When it came time to disseminate our results we, not unlike our experimental subjects, were 'naïve' researchers. Rather than publish our result in a journal we deliberately opted to produce a comprehensive research report which we thought would be a better reflection of the overall project. The report covered all aspects of the project from the planning stage to the actual

research (which included the modifications as we went along, particularly with the choice and wording of the experimental problems) and our method of analyzing the protocols. Furthermore, we also felt it important to offer alternative analyses of the same protocols. Other math educators (David Tall, Alan Bell, Lesley Lee, Stanley Erlwanger, and Allister McIntosh, all of whom happened to be around during at various stages of the research) were given a go at our protocols and came up with somewhat different analyses of the solvers' behaviours, and these were included in the report. We also attached a compendium of all the transcribed protocol that we envisaged would serve as an 'open source' for other researchers in the field. It turned out, in retrospect, that we grossly overestimated researchers' patience with reading lengthy research reports (a bit like reading theses) and, with the exception of Nicolas Balacheff, who read the report and made lots of astute comments, the research remained hidden in the literature on problem solving. We did, eventually, give a couple of presentations at PME meetings in Grenoble in 1981 and Shoresh in 1983, and by the time we started another research project, we were both wiser and more strategic.

1980 MEETING, LAVAL UNIVERSITY: MUCH ADO ABOUT NOTHING

Our first time meeting in Laval had Caleb Gattegno as one of the invited plenary speakers. Gattegno was, undoubtedly, one of the most influential math educator of the previous century with non-traditional views about teaching and learning ("only awareness is educable"; "subordinate teaching to learning" [Gattegno, 1981, p.5]). His friendship with David Wheeler went back a good 30 years. Around the time of the founding of the Association of Mathematics Teachers in England and before joining the department at Concordia, David was involved with Gattegno's 'Educational Solutions' enterprise in New York. There were also other members of the Study Group who have worked with, and attended seminars, by Gattegno, and they often spoke of him with reverence usually reserved for a guru. David had tried to get Gattegno as a plenary speaker since the start of CMESG and was delighted that Gattegno finally accepted.

Gattegno was the main proponent of the use of the Cuisenaire Rods, as well as other manipulatives, such as the Geoboard in the teaching of mathematics to children. He also espoused the use of the available technologies (TV initially and calculators later) in teaching and was starting to appreciate the potential for 'micro computers'. In his CMESG lecture he talked about entering the 'Era of Nothing' and then added that "working on computers and TV makes me more aware every day that we must all become experts (among other things) in working on 'nothings'" (Gattegno, 1981, p. 7).

Gattegno had always argued that real learning does not require a lot of expenditure of energy but many in the audience were a bit mystified (and possibly skeptical) about the notion of 'nothings'. In the post lecture discussions, Gattegno was pressed to clarify his ideas. Somehow, and for the first and only time that I can remember, the mood of the session turned combative. He became impatient with the questions and basically seemed to imply that 'if you don't understand what I am saying then it's your own problem', leaving some members of the audience somewhat angry and frustrated.

David Wheeler was clearly distraught at the turn of events. He tried to put a diplomatic spin on what we subsequently referred to as 'the Gattegno Affair' by writing in his introductory comments of the Proceedings, that "Gattegno baffled, intrigued and stimulated […]. The participants were unable to discuss this matter—'nothings'—with the sensitivity and persistence that Gattegno demanded" (Wheeler, 1981, p. 2). But privately, he knew that his friend somehow blew it, and he lamented to me, "Oh, but he could have been so good."

1986 MEETING, MEMORIAL UNIVERSITY: LAKATOS, SIGNAL HILL, AND DYNAMIC GEOMETRY

The meeting at Memorial was our smallest ever with under 30 participants. The meeting coincided with the annual meeting of the Canadian Math Society (CMS), so for the first time we had scheduled an event in common, namely, inviting Alan Schoenfeld (1987) as a plenary speaker to talk about his work on problem solving.

I do not remember how the idea of doing a live reading of Lakatos' (1976) *Proofs and Refutations* came about. There was a bit of free time on a Saturday night, after supper and before a scheduled reception, during which the CMS was holding its AGM. A table was set just outside the hall where the AGM was held and several members of CMESG took on the roles of main characters in Lakatos' masterpiece. The idea of the reading was, in part, an attempt to capture the attention and interest of the 'math types' when they came out of their AGM. As it turned out, it did not quite work that way—most of the mathematicians leaving the meeting walked right by, oblivious to the reading and probably already looking ahead to the reception. Few had a momentary 'what's that' look as they went by us but then shrugged their shoulders and continued on. But, there was a silver lining to the endeavor as three mathematicians were curious enough to stay and listen and were obviously captivated by the dialogue. At the end of the reading they wanted to know what the book was and who was its author. Possibly, an eye-opening experience for them as it was for me when David Wheeler, during his first year at Concordia, handed me the book and said, "I think you might enjoy reading this."

The Sunday afternoon break in the CMESG meetings was not nearly as organized an event as it is nowadays, and we were often left to fend for ourselves. Someone found out that on that particular Sunday afternoon was the annual run up Signal Hill and 10 of us decided to take part in the event.

Figure 2. Signal Hill runners.

By the time the run was over and we got our ribbons for participating, we realized that we would be too late for dinner so we stopped at a Fish and Chips joint for a quick bite. It turned out that 'quick' was not a word in Newfoundland's vocabulary and by the time we got our portions, we were just a few minutes away from the start of Schoenfeld's plenary. We rushed back to the venue by cabs and entered the lecture hall as discretely as possible (where Schoenfeld was getting the longest imaginable introduction in order to delay the start of the talk) still holding out newspaper-wrapped fish and chips. Alas, discretion was futile as the aroma of our yet uneaten fish and chips got every head in the hall turning toward us in unison—all we could was apologize profusely (and try to eat quietly as the lecture went on!).

I participated in the working group on Microcomputers in Teacher Education which examined some recently developed educational material available on discs. At the end of the second day we were given a look at Judah Schwartz' 'The Geometric Supposer', which was the first available dynamic geometry package (yes, Cabri came afterward), and it generated a lot of excitement among most of the participants. One member of the working group was a mathematician from Memorial who was very quiet throughout the sessions and did not seem much interested in the going-ons (I wondered if he was cajoled by Ed Williams, the organizer of the meeting, to attend). In the beginning of the third day, he surprised us by having thought about and written down a collection of delightful geometric problems, adding that "these are the kind of problems I would give my students to investigate with the Supposer". He obviously got a lot more of the WG than we imagined, reminding us of some research findings that pointed out that sometimes it is those quiet students in the classroom who actually gain the most.

FINAL COMMENT

I could go on and talk about the Friday evening of our 1988 Winnipeg meeting where a dozen of us were walking through cow pastures (part of the Agricultural School) in total darkness, gingerly avoiding cow patties, with our elegantly dressed plenary speaker Christine Keitel in tow, as we desperately tried to find our way back from a pub to the residence building (it was a lot simpler to get to the pub from the residence when it was still daylight). But I better leave something for the next talk.

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Working Groups

Groupes de travail

TEACHING FIRST YEAR MATHEMATICS COURSES IN TRANSITION FROM SECONDARY TO TERTIARY

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INTRODUCTION

The transition (here on in referred to as 'the Transition') from high school to post-secondary—including university, Cégep, community and technical colleges—is well documented as being difficult for students. In addition to jumping out of one academic culture and into another, with the numerous social changes brought, students often experience subject matter that appears and feels disjointed from what they had previously encountered. This being especially pronounced in mathematics is what brought this working group together.

In this working group, we engaged in the following activities and discussions:

- a discussion of the nature of the Transition;
- a discussion of 'what mathematics?' Is the mathematics experienced in high school fundamentally different than that in post-secondary? And why mathematics is unique in the Transition;

- differences in instruction and assessment between secondary and post-secondary and how we teachers, instructors, and professors feel both prepared and unprepared for our teaching, and how our students appear unprepared; and
- suggestions for ways to ease the Transition from secondary to post-secondary learning in mathematics.

We solicited feedback from our participants to narrow the focus of the group but remained largely within the areas identified above. We acknowledge that the Transition is complex, with many factors extraneous to but heavily influencing mathematics, and is not easily summarized in a few brief bullet points. Our discussions were very productive—bordering on heated—and we summarize the flurry of activity that was our working group the best we can below.

TRANSITION

We began by identifying issues in the Transition. These are manifold and come from a variety of perspectives. Students experience a shift in social norms, academic norms of knowing and doing and learning, personal responsibilities and motivations, among a variety of others. These shifts are often not continuous and can be jarring. The participants recognized that, though perhaps more dramatic than others, the Transition is not the only transition faced by students in their academic careers. Depending on the jurisdiction, students undergo a transition from elementary to junior high to high school, or from high school to Cégep to university. On a smaller scale, students may experience a transition between semesters or years or instructors. So, there are discontinuities built into our educational systems—we were left wondering if there is a better way. Can these inherent transitions be made to be smoother?

It became clear to the group that to address this question, we must first attempt to better understand the students. To that end, we asked the following two sets of questions:

1. Who are the students? Who do they become?

The students' worlds, once small, begin to enlarge in university. Many arrive with aspirations of being a medical doctor, a lawyer, or an engineer, but begin to realise the vast options available to them. Entire disciplines lay hidden—math often included—and are gradually 'discovered' by the students. Gradually, students choose paths, not all of which lead to an M.D., J.D., or B.Eng.

One goal stated by our participants as educators is to ignite the curiosity in their students necessary for them to expand their horizons. The careers we train them for likely do not yet exist and so all of us, students and educators alike, must come to terms with and navigate uncertainty.

2. Who were the students? Who are they now?

Students enter university from a variety of backgrounds—socio-economic, linguistic, academic, regional, national, personal identity—and the confluence of individuals to one institution has a homogenising effect: these once disparate people now have a shared identity. This may not make the Transition any easier—students must now confront their new-found adulthood, with accompanying responsibilities and decisions and commitments. The sudden autonomy, especially as it pertains to academics, can cause stress for the students, since now the link between their actions and outcomes becomes more apparent. Academically, there is seldom someone in place to monitor the students' progress and they are left to manage their time and effort devoted to studies. Grades are often lower than those received in high school, partly because of the high entrance cut-offs of Canadian universities, and students often must come to terms with being a little fish in a large pond.

In brief, there's much to get accustomed to.

MODELS OF THE TRANSITION

This working group, of course, is not unique in its desire to better understand the Transition. Numerous models have been proposed to explain and describe aspects of the Transition. We chose two such models developed, fortuitously, by members of our working group.

In the first, Clark and Lovric (2008) conceptualize the Transition as a 'rite of passage'. The authors draw on the field of anthropology, specifically on the work of Arnold van Gennep (1960), to identify features of a rite of passage. These are

- *Life crisis*: an individual's customary routines are perturbed,
- *Phase of separation*: the individual leaves the established community,
- *Liminal phase*: an activity that facilitates the changes required of the individual, and
- *(Re)joining a community*: changes complete, the individual (re)joins a community.

These features are salient in the Transition—high school, due to its duration and dominance in all aspects of a student's life, is the customary routine; post-secondary perturbs this, often violently. This is accompanied by the phase of separation, which is often a literal, physical separation from the established community. The liminal phase involves the new forms of education and assessment and the social freedom of choice and movement. Ultimately, the student joins the post-secondary community, which does not always happen, and becomes accustomed to new ways of knowing and working.

Seeing these features in the transition from high school to university mathematics is perhaps more of a challenge. Other authors do identify that post-secondary mathematics, especially university-level mathematics, is fundamentally different than that of the previously encountered mathematics that is often rooted in physical experience:

> *Advanced mathematics, by its very nature, includes concepts which are subtly at variance with naïve experience. Such ideas require an immense personal reconstruction to build the cognitive apparatus to handle them effectively. It involves a struggle [...] and a direct confrontation with inevitable conflicts, which require resolution and reconstruction.* (Tall, 1991; p. 252)

So the transition between mathematical ways of being and knowing is a rite of passage, less obvious than the jarring social transitions in the Transition, but just as jarring.

The second model is from Corriveau and Bednarz (2017) and lends a complementary perspective to that of Clark and Lovric (2008)—the unit of focus is the teachers/instructors, rather than the students. These authors view the Transition as a change of cultures, where culture in this context is the teachers'/instructors' *Ways of Doing Mathematics* (WMDs), in particular in relation to mathematical symbols and symbolism. High school teachers' WMDs are described in Corriveau and Bednarz (2017) as

- *Progressive*—the symbolism is gradually implemented.
- *Transparent*—the accessibility to the student of the symbolism is valued.
- *Chosen*—the choice of symbolism is maintained by the teacher in relation to their class and might not follow convention.

In contrast, the Cégep instructors' WDMs are:

- *Explicated*—giving conventional symbolism meaning to the students.
- *Determined & exterior*—adopting conventional symbolism.

- *Compact symbolism*—the symbolism is dense with meaning.

With these distinctions in mind, the students do experience a movement between cultures of symbolism in mathematical practice.

In our working group, we presented participants with two passages from the articles we considered and afforded time for them to engage with how they might identify the ideas present in their own students, institutions, and mathematics. These are

> *Rite of passage cannot be successful without a proper social context and without the involvement of (relevant) communities. The primary purpose of ceremonies that accompany a rite of passage is to help the person pass from one well-defined, established and accepted position in life to another, which is equally well-defined, established and accepted. The success of the rite depends, in large measure, on the 'logistical' assistance that the parties interested are able to offer to the individual undergoing the rite of passage.* (Clark & Lovric, 2008, p. 26)

> *These findings […] shed light on aspects of mathematics different from those that have, until now, been explored in research on transitions from secondary to postsecondary instructional settings. Indeed, such research has directed little attention to the theme of symbolization. And when it has done so, this theme emerges only indirectly, through an analysis of the tasks proposed at each teaching level in a way serving to highlight the additional demands confronting postsecondary students (e.g., formalization, breaking up of the meaning of symbols). By grappling with the informal mode of culture encountered at each level, the present research offers another perspective. It brings to light the complexity of the transition from one mathematical culture to another, in particular by showing up the "different rules of the mathematical game" (Drouhard, 2006).* (Corriveau & Bednarz, 2017, p. 15)

We left the group to meditate on these quotes with the hopes that they will facilitate further exploration. We acknowledged there is a great milieu in the Transition, which we necessarily relegate to the background to discuss the object of our focus: the mathematics.

QU'EST-CE QUE LES MATHÉMATIQUES ?

Pour établir notre objectif global rapidement, notre groupe a commencé par discuter des mathématiques prises dans la Transition. Afin de prendre du recul, nous nous sommes concentrés sur les thèmes suivants :

- Quelles sont les mathématiques enseignées / réalisées / apprises / évaluées ?
- Quelles mathématiques dans (la dernière année de) l'école secondaire sont différentes de celles enseignées à (la première année d') université ?
- Qu'est-ce que ces mathématiques entravent/avancent/permettent ?

Fait intéressant, mais pas surprenant, il était difficile pour le groupe de se concentrer sur les mathématiques. D'autres aspects de la vie des enseignants et des étudiants ont continué à « interrompre » la conversation : la gestion de classe, le contexte éducatif, la diversité, les contraintes institutionnelles telles que la taille des classes et le temps passé en classe. Cependant, les questions que nous avons proposé de discuter ont été reconnues comme pertinentes, mais difficiles à aborder.

De nombreux exemples de différences apparemment concrètes dans l'enseignement des mathématiques au secondaire et à l'université ont rapidement été présentés : différences de terminologie, accent mis sur les concepts versus les algorithmes, vrai versus correct—et la notion de vérité en mathématiques, exactitude versus approximation, l'approche unique versus les approches multiples—la vision utilitaire des mathématiques versus la quête de la

connaissance, la différence dans les manuels scolaires et comment les élèves et les enseignants devraient les utiliser, différences dans les évaluations et dans l'accent mis sur les compétences métacognitives. Reconnaitre que ces différences pourraient bien être des mythes ou des moyens simples d'ignorer un problème complexe a également été reconnue. L'objectif de présenter ces différences, ou mythes, était de remplacer le narratif blâmant par une conversation dans le but de mieux comprendre les préoccupations dans la transition et la façon de les aborder. Cet exercice a permis au groupe d'envisager de passer d'un mode de pensée « préparatoire » et centré sur la « remédiation », vers différents modèles pour concevoir l'enseignement des mathématiques élémentaires et avancées.

Tout en abordant les questions proposées, les participants et les co-leaders ont eu l'occasion de discuter des enjeux, à différents moments, dans des groupes homogènes et dans des groupes hétérogènes. Il a été reconnu que d'offrir ces possibilités aiderait peut-être à résoudre de nombreuses difficultés didactiques que rencontrent les enseignants des écoles secondaires, des collèges/cégeps et de l'université. D'une part, les participants ont souligné le manque d'occasions (institutionnalisée) pour discuter avec leurs pairs, par exemple, des mathématiques enseignées / réalisées / apprises / évaluées, et de ce que cela entrave / avance / permet. D'un autre côté, les participants des trois niveaux se sont plaints du manque d'espace (institutionnalisé) pour discuter avec des collègues d'autres ordres. Peu importe le niveau, les enseignants ont souligné qu'ils ne savaient pas « ce qui se passe » aux deux autres niveaux et comment ils devraient se connecter, se référer, ou pas du tout. Nous ne savons pas ce que les étudiants sont en train d'apprendre, ce qu'ils sont censés savoir, ce qu'ils savent réellement, et nous ne savons pas comment le découvrir—peut-être qu'il n'existe aucun mécanisme (institutionnel) conçu pour nous.

Le groupe s'est engagé à expliquer explicitement les raisons pour lesquelles les enseignants d'un niveau peuvent être si peu familiers avec les autres niveaux. Ce n'est pas seulement que le temps s'est écoulé et que les « choses » ont changé depuis « *que je suis allé au secondaire/cégep/ à l'université* ». La réalité est que les enseignants n'ont pas toujours fait partie du même système que leurs élèves font ou vont faire partie. La mobilité nationale et internationale des étudiants *et* des enseignants se traduit souvent par le fait que les enseignants n'ont aucune idée de ce que leurs élèves ont été enseignés et de ce qu'ils ont appris, ainsi que de ce qui les attend dans leurs futures études. Ce n'est pas clair pour nous, participants à ce groupe de travail, que ce soit mauvais en soi—mais au moins, il semble que le déséquilibre entre les attentes des étudiants et des enseignants et les hypothèses entre les niveaux jouent un rôle dans les difficultés des étudiants et des enseignants dans l'apprentissage et l'enseignement des mathématiques.

Certains professeurs universitaires de mathématiques du groupe ont été capables d'articuler une définition des mathématiques : la mathématique est l'étude abstraite des structures. Ils réfèrent aux mathématiques comme une activité ludique—en tant que mathématiciens professionnels, nous jouons avec des problèmes. Bien que certains (ou tous!) les professeurs d'université du groupe aient développé des approches d'enseignement qui se concentrent sur l'étude abstraite des structures et mettent en évidence le caractère ludique de l'activité, il est assez incertain que c'est l'approche générale de l'enseignement des mathématiques universitaires, au moins dans les premières années universitaires. La recherche en didactique des mathématiques en lien avec l'enseignement du calcul et de l'algèbre linéaire semble suggérer que ce n'est pas le cas.

Cela a conduit à discuter des objectifs éducatifs de l'enseignement des mathématiques dans les différents niveaux. En lien au contenu, par exemple, il y a les attentes des professeurs d'université envers les étudiants, que ceux-ci arrivent avec un bagage de connaissances spécifiques (ce qu'est une équation, qu'est-ce qu'une variable, priorité des opérations de base), mais surtout avec des compétences métacognitives : persévérance, créativité, pensée critique et

théorique. Peut-être que de mieux définir et articuler les objectifs éducatifs (pas nécessairement, ou pas seulement, dans les documents ministériels et curriculaires, mais dans le langage et les cultures institutionnelles des enseignants et des étudiants qui doivent atteindre ces objectifs) aidera à établir des ponts entre les ordres.

WHAT'S UNIQUE ABOUT MATHEMATICS IN THE TRANSITION?

Mathematics often acts as a 'gatekeeper' course, both at the secondary and tertiary levels. This leads to 'service' courses in first-year university where one or both of calculus or linear algebra are required courses for degree/diploma programs but are 'terminal' in that they do not lead to further courses in mathematics. Many people, including teachers, have very emotional reactions to mathematics. It is a subject area where it is socially acceptable for people to say, "I'm not good at math" (search online for t-shirts that say "allergic to algebra" [girls only], a pink t-shirt that says "I'm too pretty to do math", "I suck at math" and MATH "Mental Abuse To Humans"). This is not a sentiment that is socially accepted in reading, writing or other disciplines.

In mathematics, we often work on big ideas but work on small, condensed parts while other areas take a statement or idea and expand upon it. In addition, there is an abstract nature to mathematics that is not rooted in the 'real world'. The study of mathematics can be viewed as cumulative in nature, particularly with the current focus in secondary school that leads to calculus. For example, a thematic approach can be taken in studying history or literature where the order of material is flexible. It is possible to study the times or works of the renaissance and then subsequently study Ancient Rome, but it is more difficult to sequence the teaching and learning of content in mathematics. Mathematics also uses symbols and notation that can make it inaccessible to 'outsiders'. There is a rigor and precise use of language and symbols. While it is easier to identify achievement in other domains such as music, members of the working group wondered what achievement in mathematics looks like.

INSTRUCTION & ASSESSMENT

Two of the most noticeable classroom aspects of the Transition for students are in the areas of instruction and assessment. Learning mathematics in post-secondary school is noticeably different than high school: classes are often larger, students can be anonymous, the pace may be quicker, and responsibilities have shifted more to the student. Most prominently, post-secondary courses are often taught differently than in high school. Lectures remain the primary mode of instruction in post-secondary, in contrast to, at least nominally, more interactive, hands-on teaching in high school. This is partly due to the perceived infeasibility of hands-on teaching for large classes, but also reflects a difference in instructor mindset: university instructors are often trepidatious about interactivity during class time, yet many secondary school teachers are encouraged toward more interactive and responsive teaching. Assessment is also very different between post-secondary and high school settings. Post-secondary assessment is almost exclusively summative, and often high-stakes. While assessment in high school can also be high stake and summative (e.g., final exams), there is a much greater focus on the learning function of assessment where students receive personalized feedback which affords opportunity for student growth. There is also more opportunity in high school classes for conversations and observations to be used as part of informal or formal assessment. These marked differences in instruction and assessment were apparent to the participants, and we considered each, in turn.

INSTRUCTION

To initiate a deeper discussion on instruction, the participants, in heterogeneous groups, created lessons on the topic of solving systems of linear equations. This topic was chosen because it is common in high school and university.

There was a wide variety in approaches. Some groups wrote prompts to have students consider possible linear systems with a particular solution as shown in Figures 1 and 2. Some groups included a 'real world' context for solving equations, such as in Figure 3, and some included investigation of the number of solutions, such as in Figure 4, and a discussion of operations that preserve the solutions, as in Figure 5.
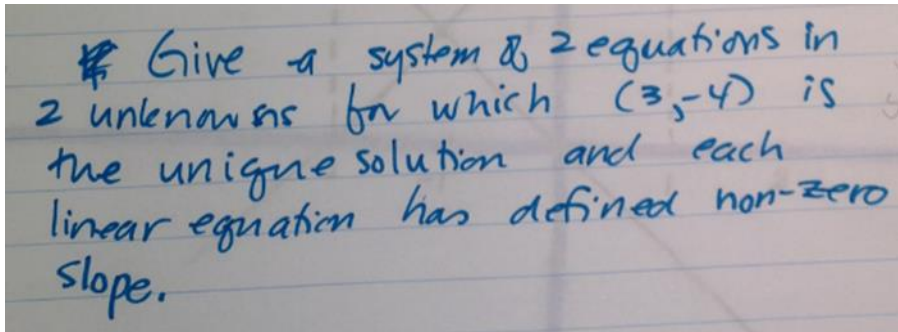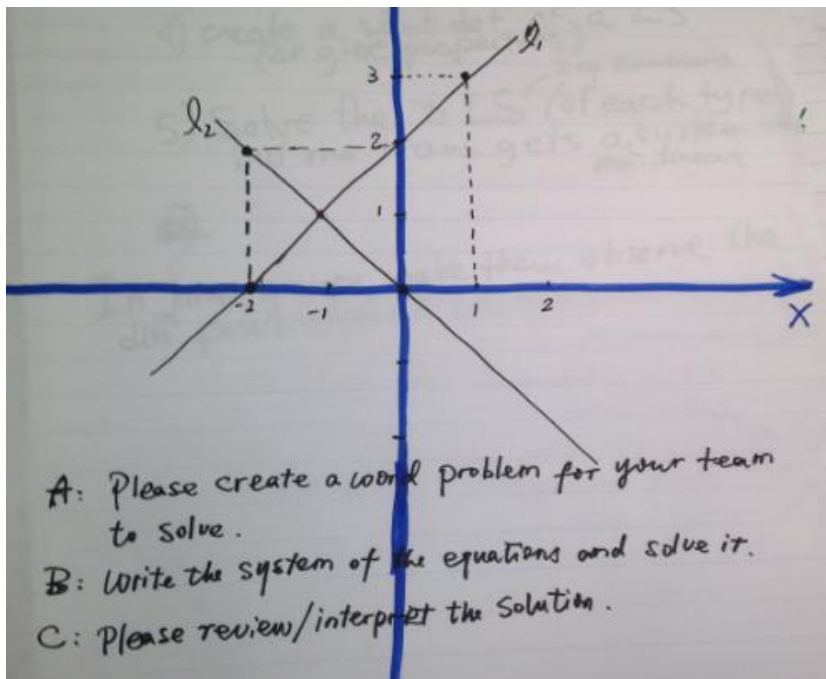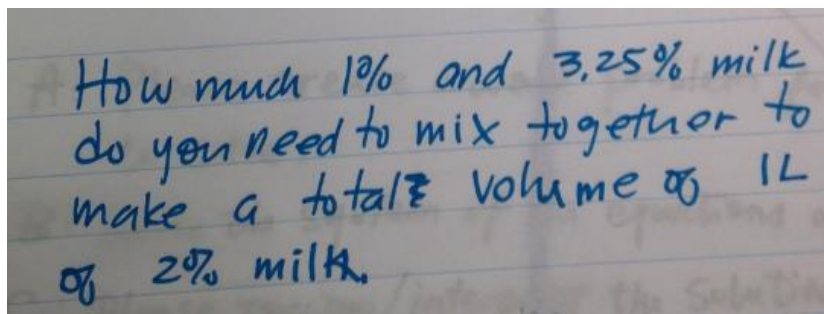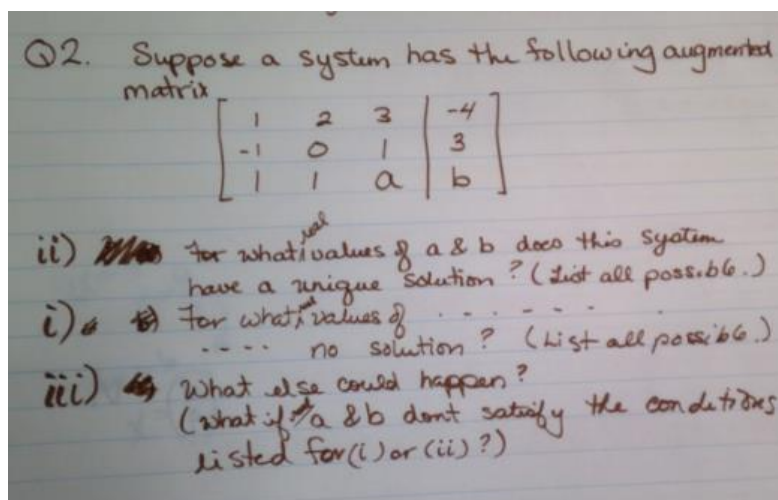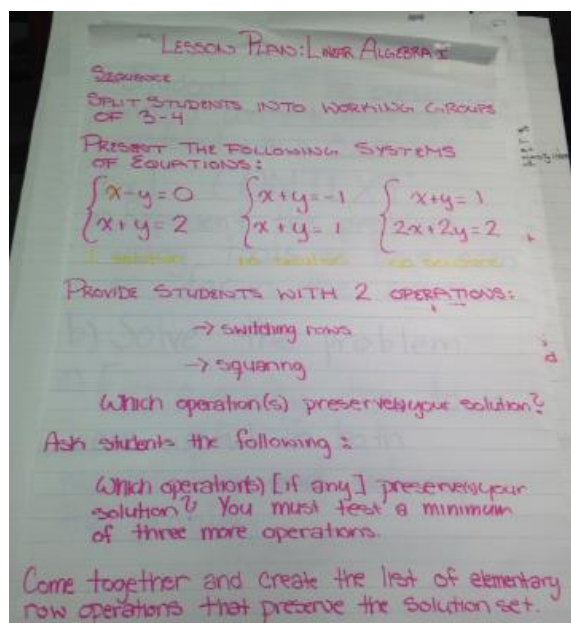

Figure 1.

Figure 2.

Figure 3.

Figure 4.

Figure 5.

What struck us as facilitators was that there were few original ideas brought forth by the participants during this session. This was surprising as our participants were all, individually, creative educators and often voiced that creativity in this and other sessions. We speculate that our groups were educational microcosms—that each individual representing a level of education in a group desired something creative and profound for their students, and through interacting with others from different levels of education, compromised on their ideas, producing an educational experience that satisfied each level but that none strongly desired.

Having planned and discussed lessons, we moved on to viewing two lessons, one from middle school, and another from university, that could be taken as archetypal: the first being taken from the TIMSS study (TIMSS, 1999) and the second from a recorded lecture at the Massachusetts Institute of Technology (MIT, 2005). There was consensus within the working group that both of these lessons were likely typical samples, but there was widespread discomfort with both lesson samples.

The middle-school lesson centred on the topic of exponential relations and exponent laws, but the teacher mainly lectured. The class was peculiar—with an awkward delivery of materials, where students were requested to take out a binder and put a sheet in, but then disregard that and put it away. What seemed to be a lesson intended to have a demonstration with manipulatives quickly degraded to a lecture. The teacher did not appear to take previous learning into account and afforded few opportunities for student involvement.

The university lecture was just that, a *lecture* in the most traditional sense of the word, and was similarly devoid of consideration of students' prior learning. It concentrated on presentation/delivery of ideas and problems, rather than consideration of the audience (in-person or on-line).

Both archetypal lessons seemed to perturb our participants. Neither went over well, and the participants generally felt that they served as archetypes of how *not* to teach. We nevertheless see value in studying these examples—a good example of how not to do something can be on par with an exemplar of how to do that thing.

ASSESSMENT

The next activity was intended to initiate a deeper discussion about assessment practices in high school and university. Participants, in their same group as in the instruction session, created assessments corresponding to their instructional lessons.

The final portion of this session involved discussing authentic assessment samples from high school and university. These are in Figures 6 and 7.

An interesting distinction emerged: post-secondary instructors, in particular the university professors, found the high school problem too ill-posed—the question did not appear to invite a response by its open ended-ness; whereas the high school teachers found that the university problem lacked richness—that it was far too routine to be invoke meaningful mathematical thoughts. Both teacher and professor retorted—the high school question did indeed make sense to the teachers, who found it to be in line with what they would likely present their students, and the university professors produced some insightful, and non-routine solutions to the university question. It seemed, then, that assessment is deeply intertwined with institutional and course-level cultures; not readily understood without the anchoring context.

Un rectangle est limité par la courbe $y = \frac{10(x-2)}{(x-3)}$ et les axes x et y. Utilisez les concepts appris en class pour explorer l'aire du rectangle. Voici un rappel des concepts: des limites, la continuité, la différentiabilité, la taux de variation instantanée, la première et la deuxième derivée, les tests de la permière et de la deuxième derivé, croquis de courbes, optimisation.

A rectangle is bound by the curve $y = \frac{10(x-2)}{(x-3)}$ and the x and y axes. Use the concepts learned in class to explore the area of the rectangle. Here's a reminder of the concepts: limits, continuity, differentiability, instantaneous rate of change, first and second derivative, first and second derivative tests, curve sketching, optimization.

Figure 6. High school example.

Let R be the region below the x-axis, to the left of the y-axis, and above the curve $y^2 + 4x^2 = 4$. Find the area of R.

Soit R la région sous l'axe x, à gauche de l'axe y, et sur la courbe $y^2 + 4x^2 = 4$. Trouvez l'aire de la région R.
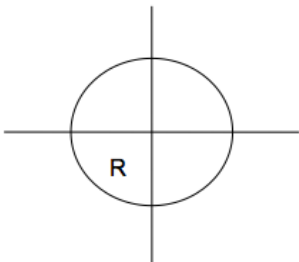
Figure 7. University example.

HOW ARE WE PREPARED/UNPREPARED FOR OUR TEACHING?

Having discussed the Transition itself, and issues related to mathematics in the Transition, including instruction and assessment, we turned our attention to how we educators are (un)prepared for our teaching. We summarize these discussions below, based on educational level.

HIGH SCHOOL TEACHERS

The high school teachers in our working group felt very prepared for their teaching with good content knowledge as well as instructional activities from their teacher-preparation courses. There was consensus that there are disconnects between the preparatory experience in the Faculty of Education, which was largely theory-based, and the practical reality of teaching. Teachers felt that generalist courses for secondary teachers are not very useful for future math teachers, because of the dominance of social sciences and languages content in the examples and discussion. The assignments in the general courses on topics such as special education and assessment were perceived as being completed for their own sake, not for learning. While it was recognized that formal lesson planning could build an understanding of how to structure math classes, there was a perceived over-emphasis on formal lesson planning during teacher-preparation. Learning about planning, assessment, and other key topics is largely left to content courses or the practicum for future secondary mathematics teachers. In addition, there was concern that many of the courses in a teacher-preparation program are mostly taught by professors who have never been a high school teacher, or have little teaching experience at the high school level.

Learning in the practicum setting was viewed as much richer, but it was pointed out that associate teachers are critical to 'making or breaking' the whole teacher education experience. Several participants had experiences where associate teachers undermined the messages received from the university faculty in the education program.

One particular area that the high school teachers in this working group felt unprepared for was assessment and evaluation. There was consensus that teaching means nothing unless learning is occurring, and that teachers sometimes assume students know things they do not. Teachers wanted a deeper understanding of how to use assessment in planning and to inform pivots/changes on the fly in the middle of lessons. Most teachers had the majority of their learning about assessment through reading and lectures, which was largely devoid of the context of high school mathematics.

Another area where high school teachers felt unprepared was in helping students prepare for programs at community colleges since very few teachers have attended college as a student.

CÉGEP/COLLEGE INSTRUCTORS/PROFESSORS

The Cégep and college instructors/professors in our working group felt very prepared for the mathematics content areas for teaching, with the exception of statistics. However, they felt they lacked preparation in didactics/pedagogy. In particular, they stated they felt they lacked a diversity of techniques in teaching particular topics, and stated it was hard to tell what 'works' in teaching at the Cégep and college levels. In particular, Cégep and college teachers/professors felt they lacked social and emotional connections with students due to class size.

UNIVERSITY PROFESSORS/INSTRUCTORS

University professors in our working group felt very confident and prepared in their knowledge of content and research, but felt they do not know their students. This is largely due to institutional constraints where class size is a major challenge to getting to know students. Professors worry about how to create a learning environment, for several hundred students, that allows students to do their best. In addition, professors do not feel prepared for the assessment and evaluation piece—what is a good test, how to assess conceptual understanding, how to use assessment to improve learning?

The class size issue has impacts on assessment, because large class sizes mean it is very hard for university professors to use formative assessment to inform teaching and improve learning. Large class sizes require teaching assistants to grade assignments and exams, which removes professors from a key step in informing instruction. There was consensus among university professors that showing caring about learning is very important to students. Professors also noted there is increasing diversity in incoming students, and that while there is an assumption that students have strong technology and computer skills, many university students do not know how to use technology to look up basic mathematics concepts. They also noted they are dealing with more mental health issues that have a big impact on attendance and evaluations.

On the topic of transitions, university professors find dialogue with high school teachers to be very productive, particularly when relationships are established to allow annual collaborations. Mentoring between university students and younger undergraduate students, as well as even younger high school students, is seen as helpful.

WHAT IS IT THAT STUDENTS DON'T KNOW OR CAN'T DO THAT YOU WANT/NEED THEM TO BE ABLE TO DO?

As the working group progressed, we turned our attention to prospective thinking about the Transition. Groups were prompted to think about what they desired of their students to know about and do with mathematics. Two themes emerged in the ensuing discussions: mathematical content and dispositions towards mathematics. However, these themes were interpreted differently by the various types of participants. Below is a summary of the major points brought up.

HIGH SCHOOL TEACHERS

The primary content areas that high school teachers brought up as problematic for students were in the areas of orders of operations, fractions and algebra. Other areas included factoring polynomials and finding roots. A particularly rich area of discussion was in how estimation has been largely ignored in curriculum, but there has been a recent resurgence through the social media presence of two California educators: Andrew Stadel (*Estimation 180*) and Dan Meyer (*Three-act math*).

In terms of mathematical dispositions, a wide range were mentioned. Of note are:

- *Perseverance*. Students tend to not persist through problems. The high school teachers voiced a desire for students to keep trying and not give up too easily. Teachers ought to facilitate practice in perseverance, and perhaps model it in their own instruction. Rather than giving the impression that math 'problems' are short and quick, teachers ought to design tasks that afford perseverance. Participants mentioned that the students demonstrate perseverance in other areas—music, art, sport—and so it is certainly possible to stimulate that perseverance in mathematics as well.

- *Independence*. High school math instills in students a dependence on the teacher or other external-to-the-student ways of validating and knowing mathematics. Students are told when they are right or wrong, and often have difficulty in determining that themselves. This situation changes very suddenly in university, where students experience a greater autonomy. And this is a source of discomfort for strong and weak students alike. The major challenge is to shift students' motivations from external to internal—doing so is the first step to independence. An internalized motivation enables a student to, for example, check their solutions/answers, not rely too heavily on a calculator, and to know when they are right or wrong.

- *Societal*. It remains a societal trope to claim one's incompetence in mathematics, despite that being a perennial complaint of the mathematically inclined. This remains surprising, since other disciplines do not experience the same level of ignorance. What are we doing to change this?

CÉGEP/COLLEGE INSTRUCTORS/PROFESSORS

Similar to the high school teachers, Cégep/college instructors voice concerns about students' competencies with particular mathematical content. In particular, syntax was a major concern—students need a greater fluency with mathematical syntax. On a more procedural note, students' work did not always follow the order of operations convention. This, of course, causes major errors in college-level mathematics.

In terms of students' dispositions toward mathematics, Cégep and college instructors tended to focus on the nature of students' knowledge. They find that students are typically disposed

toward memorization of mathematics and have the notion that there is only one way to solve any given mathematical problem. These fixed mindsets should be transformed early on; being able to work flexibly, with multiple solution approaches, and to verify one's mathematical work are valuable college-going skills.

UNIVERSITY PROFESSORS/INSTRUCTORS

University professors highlighted functional thinking as a core mathematical competency students tend not to exhibit. In addition, and similar to the other two groups of educators, professors highlighted basic algebraic skills as a point of concern. At this stage we ask, if all three levels of educators value the same foundational set of algebraic and arithmetic skills and conceptions, how is it not being developed in the students?

Content aside, professors voiced a desire for many of the same dispositions that the other groups mentioned: internal motivation, persistence, for example. Two points of divergence are worthy of note. First, professors desired the non-mathematical attributes of general academic and personal maturity. They find often that students are still too reliant on an instructor to guide them and felt they ought to be much better prepared to engage with materials related to a course independently. Professors felt they were on the receiving end of a buildup of a culture of poor communication etiquette—they desired more formality from their students. The second major point was a desire for the students to 'play' with mathematics. Far too much mathematics encountered prior to university stifled the creativity and free inquiry of the students. This contrasts dramatically with the work of mathematicians. Mathematics is a source of enjoyment and fun for mathematicians and the professors voiced a strong desire for students to experience similar joy with mathematics.

EASING THE TRANSITION

The final session in our working group centred on creatively addressing the Transition. We presented each cluster of participants a blank sheet of paper and the instructions:

> *Activity: Let's Dream! Create a plan/experience that will address transitional issues from secondary to tertiary math.*

We emphasized that the groups should not feel too constrained by current power/institutional/cultural structures and imagine experiences that might better transition students into post secondary. Here is what they came up with:
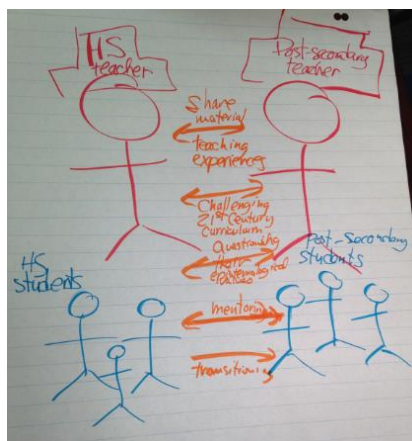
Figure 8.

Group 1: Dialogue between high school teachers/students and university or college professors/students. Teachers and professors share materials, teaching experiences and questioning epistemological issues Students are involved in mentoring activities. Development of a math knowledge network similar to the Fields Institute in Ontario.

Group 2: Week-long summer 'head start' camp like is offered at Queen's University. High school final exams could also serve as a diagnostic for the transition into first-year math. Work in small groups and meet teachers, professors and undergraduate students.

Group 3: Dialogue between university/college professors and high school teachers; many high school teachers do not love math

Group 4: One week of college/university immersion at the start of a term where there is new independence/responsibility

Group 5: Create Cégeps in all provinces!

CONCLUSION

As is a typical outcome for a working group at CMESG, we parted with more questions than answers. But there was also the distinct feeling of having networked ideas in ways not commonly encountered. We facilitators physically structured the groups present in each session to maximize diversity of institutions—high schools, Cégeps, technical and community colleges, university students and professors, and government representatives were all present and engaged with each others' perspectives. Perhaps this is what is needed, above all, in the Transition.

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L'ANXIÉTÉ MATHÉMATIQUE CHEZ LES FUTURS ENSEIGNANTS DU PRIMAIRE : À LA RECHERCHE DE NOUVELLES RÉPONSES À DES ENJEUX QUI PERDURENT

ELEMENTARY PRESERVICE TEACHERS AND MATHEMATICS ANXIETY: SEARCHING FOR NEW RESPONSES TO ENDURING ISSUES

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INTRODUCTION

This CMESG Working Group (WG) attempted to bring fresh thinking to the enduring issue of elementary preservice teachers' mathematics anxiety. While this is a topic that has resonated with teacher educators for many years, a recent upsurge of interest in the emotional component of mathematics learning at the Kindergarten to Grade 12 levels (e.g., Andersson, Valero, & Meaney, 2015; Brown, Brown, & Bibby, 2008; DiMartino & Zan, 2010; Lange & Meaney, 2011; Takeuchi, Towers, & Plosz, 2016; Towers, Hall, Rapke, Martin, & Andrews, 2016; Towers, Takeuchi, Hall, & Martin, 2015) is prompting renewed interest in examining post-secondary students' (and particularly elementary preservice teachers') emotional relationships with mathematics. We felt that it was time to ask again: What motivates, drives, and/or creates elementary preservice teachers' mathematics (teaching) anxiety? And, have these drivers changed recently? Are we as teacher educators relying on (old) assumptions about the nature of our current students' anxieties? Drawing on published and emerging research (e.g., Goulding, Hatch, & Rodd, 2003; Hobden & Mitchell, 2011; Takeuchi, Czuy, & Towers, 2016; Towers, Takeuchi, Hall, & Martin, 2017), we examined the kinds of K-12 and post-secondary

experiences that (may) have led the current generation of math-anxious preservice teachers to their relationships with mathematics, and then moved towards explorations of new responses to this enduring problem.

The three days of the Working Group were designed around a theatre metaphor: day 1 of 'actors and agents', working to understand the primary actor—the preservice teacher, and the other actor—the mathematics educator; day 2 of 'lines and scripts', exploring the prior experiences, knowledge, and beliefs these two main actors bring with them; and day 3 'the stage', exploring possibilities to interrupt the usual cycle of anxiety. An underlying premise of this Working Group was to provide many opportunities for participants to play, talk, and work together, a kind of modeling of the mathematics learning experiences we felt would act as interruptions to the cycle of anxiety and help preservice teachers in their professional learning. To that end, we designed homework for each day, included graphic, text, and auditory artifacts from preservice teachers expressing their mathematics and mathematics teaching anxiety as motivators and prompts for Working Group discussion, and provided opportunities for participants to create and express themselves in diverse ways. We worked in French and English and encouraged participants to speak whichever language they felt more comfortable with. Since we had resources in both French and English, we also encouraged the participants to familiarise themselves with the other language. To demonstrate our commitment to this principle, Manon frequently spoke in English, and Jamie and Jo spoke in French.

Through the different activities, much of our intention was to present opportunities for critical reflection-on-action (Schon, 1983); we did not try to direct the discussion towards 'an answer', but left discussions and conversations open-ended, allowing participants to continually reflect, review, and revise their conceptions and thinking. We take this approach with this presentation of the work of our Working Group, offering our thinking on preservice teacher mathematics anxiety for you to interpret for yourself.

DAY 1 'ACTORS AND AGENTS'

The first day of the Working Group started immediately with vignettes of preservice elementary teachers talking about their mathematics anxiety. These vignettes were drawn from Towers' recent study of students' experiences learning mathematics (e.g., Takeuchi, Czuy, & Towers, 2016). On purpose we did not start with introductions; we felt the nature of the interactive and collaborative activities would naturally motivate people to talk to one another, and we could effect introductions as a natural part of the first day 1 activity. We wanted the participants to express themselves through the characters they created from the vignettes. Initially, participants began by talking about preservice teachers' mathematics anxiety, but they soon began talking about themselves as fellow actors, with the preservice teachers helping them deal with their mathematics anxiety. Participants were disclosing, in essence providing introductions of, who they were as participants in the Working Group as they explained how they came to conclusions for the features of their character.

The following are the five characters that came to exist as individual and collective preservice teacher entities in our Working Group. Participants naturally referred to these characters throughout the three days we worked together. In each case, the image is a photograph of the small group's preservice teacher character, and the text in italics is a snapshot of the words used to introduce the character to the rest of the Working Group.

1. *Jane Doe, une étudiante en éducation au primaire qui se sent dépassée par la situation. Ses cours de maths se ressemblent tous et sont axés sur des exercices. Elle doit se concentrer. Elle doit être rapide. Elle ne sait pas à quoi tout cela va lui servir. Malgré la pression de réussir qui est grande (si elle ne réussit pas ses cours de maths, elle ne pourra pas être enseignante), elle n'ose pas demander de l'aide aux autres, car elle a peur d'être jugée.*

2. *Emotions, and gender were considered important—therefore the character is a girl named Rebecca; and a cognitive gap and/or awareness of the cognitive gap; grade 8 as an important transition; procedures done without meaning; not grounded in 'math' class.*
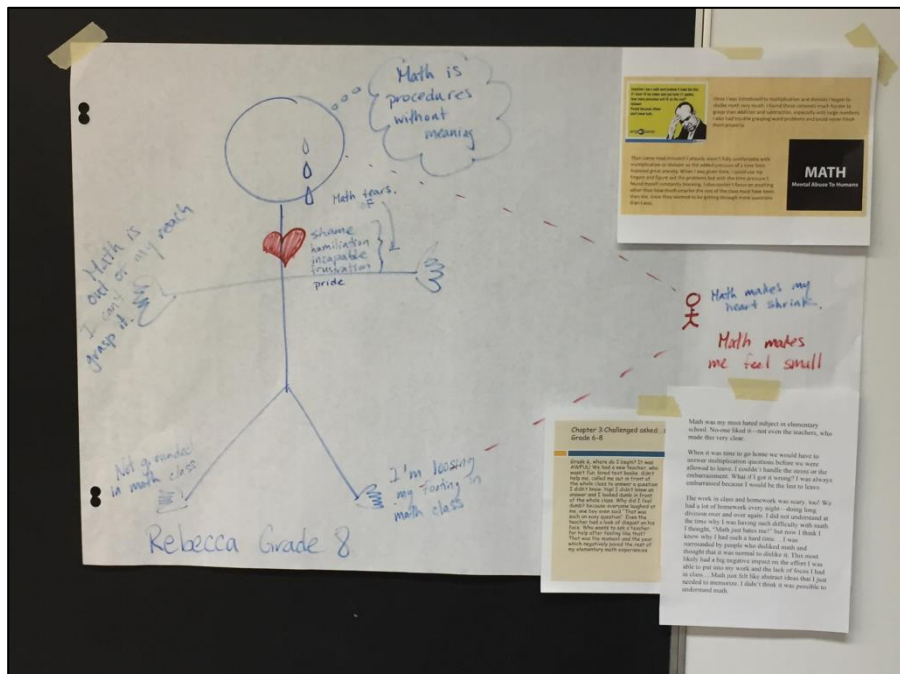
3. *Kevin, a boy, but gender was discussed; a bit of a timeline, at the beginning everything went well; evolution to becoming a preservice teacher; anxieties and frustrations; multiplication and division, glaze of the eyes; not enjoying the weekly Mad Minutes; a teacher helped show what math could be; education is an 'at school' thing, and not a 'home' thing. The name is important—Kevin is a boy who comes from a background that is far from education, then there is embarrassment because of an inability that is not in his control, a progression of learning for Kevin, school took the math away from him—the math that he knew naturally at the beginning*
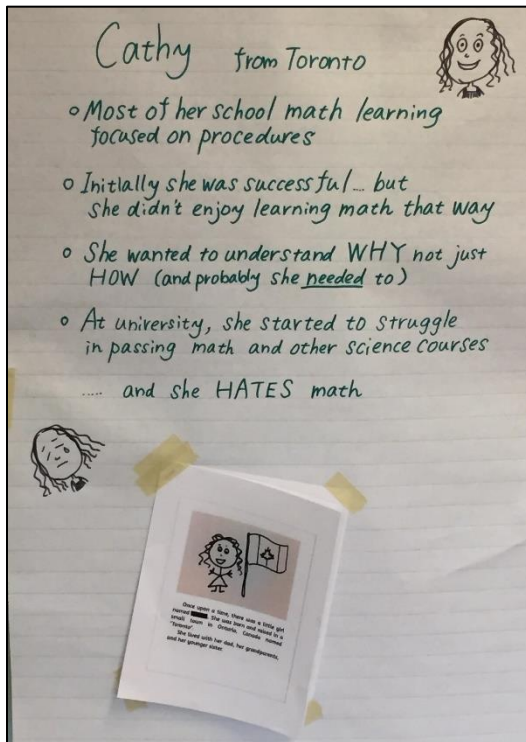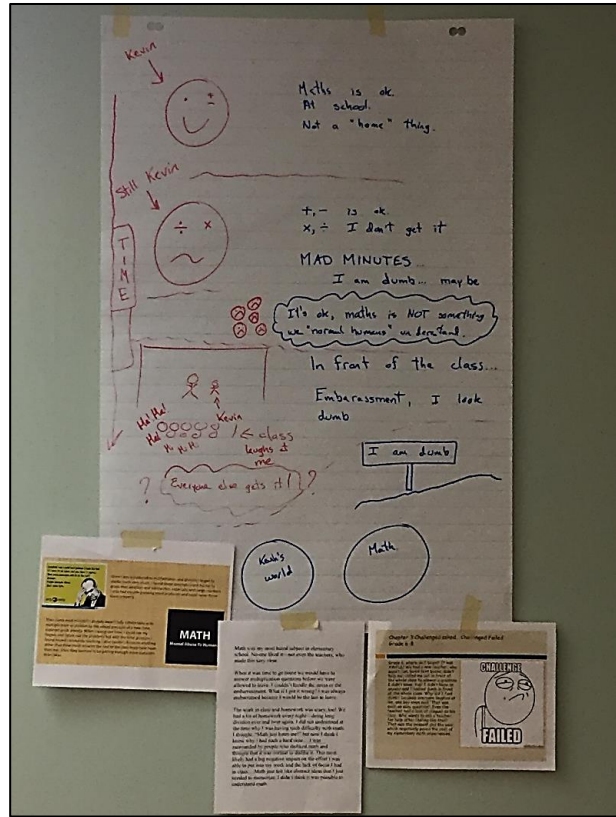
4. *Cathy, from Toronto, dropped out of the program; had success initially with a program exclusively about procedures; developed a need to understand why and not just how; discussion of conceptual vs procedural—what does the learner need?; the power of the language that the teacher uses—the language of identity laid onto the student by the teacher. Large-scale spatial awareness then helps understand the small procedural steps—such as directions in a city. Sometimes the student needs to BE an advocate for their own learning, and that is a personal kind of thing*
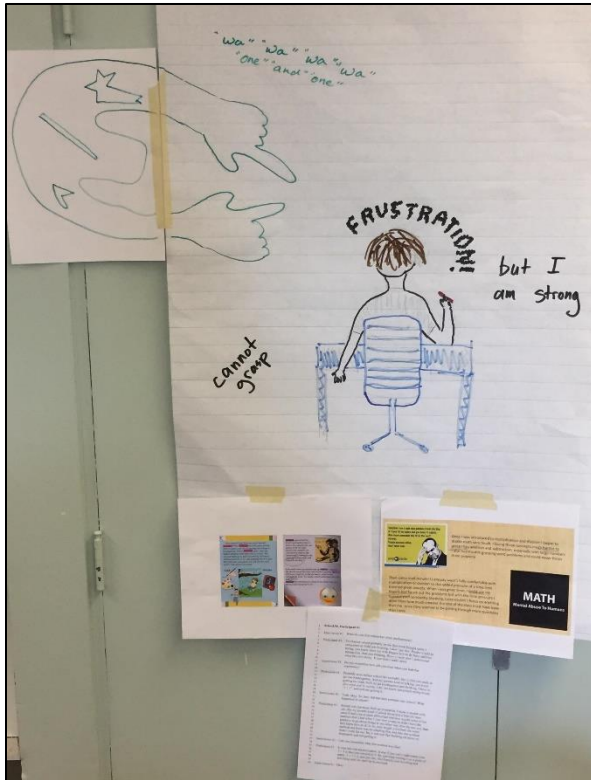
5. *No name; right from being a 4 year old not understanding 1+1 is 2, it became an identification; frustration, but not defeated because she felt confident in other areas; language about not grasping. "I am …. This is not my experience with math but I have seen it"* [a participant's comment about themselves]. *Green colour is important, there is something happening with the fingers, like adding, but there is no knowledge about what is going on behind i.e., in the thinking, therefore the green is that it is alien.*

After the mid-morning break, the focus turned to the mathematics educator. A small group placemat activity was performed with the participant mathematics educators. Three placemat spaces, representing a) *goals* (the horizon learning goals of what they hoped to accomplish with elementary school preservice teachers); b) *intentions* (what they intended to do to achieve those goals); and c) *anxieties* (relating to their own professional practice) spread from a central oval. The placemat was turned twice during individual brainstorming while participants wrote in the placemat space in front of them. Then they had a small group discussion about what was noticed in the placemat spaces. As enduring issues emerged from the discussion, those enduring issues were written in the centre oval of the placemat.

The following are the six placemats that resulted from this activity.

Buts/Goals (first diagram)

Center (oval):
- Building and expanding our tools to address students' anxiety & trauma for every learner
- Creating sustainable environments for ongoing math learning for pre-service & practicing teachers

Buts/Goals:
- increase their confidence and competence with mathematics and mathematics teaching
- Build their confidence to teach math and how to bring in the joy of the discipline
- Let students know that ... learn our style; I learn ... they can learn in my appreciation
- they are not alone. I must learn how the role(?)

Intentions:
- Find other ways ... experience math
- Teach in a different ... different ways to experience math
- many opportunities to experience math in a supported, collaborative environment
- "normalize" the anxiety of being wrong or embarrassed so students take risks and can learn as much as possible
- Situating math in a broader context re-imagining the math pedagogy they've experienced

Anxiet<ies/és>:
- Will I be able to reach every learner to help them feel comfortable exploring math?
- Will I be able to address students' difficulties? anxieties? Do I have enough tools in my tool box?
- What tools am I prepared to use with...?
- "Opening Pandora's Box"
- Will I further reinforce negative experiences that they came with? - trauma
- Balancing Contrast between my "approach to math ed. and their D'xastam(?) teachers' approaches

Buts/Goals (second diagram)

Center (oval):
1. "Accompagnement" des étudiants anxieux: comment? jusqu'où?
2. Un stress qui engendre du stress... Comment rester zen?
3. Amener ailleurs ceux qui ont eu une mauvaise expérience des maths. (et m les autres)
4. Rechercher l'équilibre.

Buts/Goals:
- Donner du sens aux motifs, voir les liens, pour la mathématique
- Les élèves, suivre leur raisonnement à bien écouter les étudiants
- Les préparer à ... du programme
- Connaître le programme
- connaître quelques difficultés d'opération chez l'élève
- Ne reproduire l'enseignement qu'ils ont reçu

Intentions (comment):
- Les étudiants font les exercices du manuel de primaire ++
- Les placer en situation d'élève (résolution de problèmes, jeux de rôles,...)
- Journée de résolution de problèmes
- "On écrit tout!"

Anxiet<ies/és>:
- Seront-ils prêts?
- Certains vont vivre du stress...
- Certains vont échouer. Quoi faire? Que dire?
- Comment anticiper les inquiétudes des étudiants?
- Comment accompagner l'étudiant·e anxieuse

As the small groups described their placemats and their thinking, it was interesting to note that the groups talked about their preservice teacher character, but often also, with pieces of an imagined elementary school student. An overriding theme was that of mistakes, "*Acknowledging the mistakes we are going to make, means we can move ahead*" (a participant). A small tangential discussion concerned the mathematics itself; that is, the mathematics of the curriculum. Is it different, does it change? Or is it the way mathematics is being taught that is changing? "*There is a tension that exists no matter what we say we are doing*" (a participant). The notion of preservice teacher anxiety and its effect on mathematics educators was an important discussion point. "*We may not be able to remove anxiety, but we could support them* [preservice teachers] *to cope with it; but stressed students bring stress to me; therefore, how can I stay ZEN?*" (a participant). The mathematics educators in the Working Group were self-reflective, "*I think it is interesting that you asked us to* complete *the anxiety section* [of the placemat] *because it* [mathematics educator anxiety] *is not something I have thought about before*" (a participant). And the preservice teachers in the Working Group expressed a sense of relief in their reflection, "*The fact that you* [mathematics educators] *have anxiety, it makes us preservice teachers feel better because we have anxiety. It makes us feel more confident and capable because we are still struggling. Thank you!*" (a participant).

HOMEWORK!

Between the end of the first day, and the beginning of the second day of the Working Group, participants were asked to take and/or select one picture from anywhere in the world that they felt had something mathematical about it or something that could be considered and discussed that is mathematical. We introduced them to an online mapping tool that allows images to be uploaded and linked to a specific geographic location. Participants were asked to turn on their geo-coordinates and location services on their photographic equipment before taking their picture or to simply link their picture to somewhere in the world while uploading it to the online map. The pictures were to be posted online in the mapping tool or could be brought to the Working Group the next day and we would post the image for them.

DAY 2 'LINES AND SCRIPTS'

La deuxième journée s'est articulée autour de deux activités, soit 1) une exploitation des géotechnologies (activité liée au devoir de la première journée) et 2) une réflexion sur la vision que les participantes et participants ont des mathématiques et sur l'effet que cette façon de voir les choses peut avoir ou non sur l'anxiété mathématique que vivent les futures enseignantes et futurs enseignants. Dans un premier temps, les participantes et participants ont travaillé à partir des images placées sur la carte en ligne (https://gcedm-2017.maps.arcgis.com/apps/StoryMapCrowdsource/index.html?appid=2dd100d066984cbe81 1e027c9e769900). Voici quelques-unes de ces images :

Les participantes et participants ont d'abord travaillé en petits groupes et se sont inspirés d'une image (la leur ou une image choisie parmi celles des autres) pour rédiger un problème. Le fait que l'objectif et le niveau ciblé (p. ex. : 1re année universitaire vs. 4e année universitaire) changent grandement le problème rédigé est un des constats qui a émergé de cette tâche. Les problèmes proposés furent riches et variés. Alors que certains ont tout simplement demandé une description de l'image, d'autres ont rédigé des problèmes plus traditionnellement mathématiques (p. ex. : À ton avis, combien de fenêtres y a-t-il à la Place Ville Marie?). Les participantes et participants ont ensuite précisé la solution que leur personnage (développé lors de la première journée) proposerait à ce problème. Enfin, ils ont identifié la façon dont leur personnage réagirait devant ce problème (actions) et la façon dont il se sentirait (émotions).

Une discussion s'ensuit sur les différents éléments abordés lors de cette activité. Les participantes et participants ont entre autres partagé ce qui s'était passé lors de la rédaction des problèmes (réels) et des solutions (réelles—la leur; et fictives—celle du personnage). L'un des premiers éléments notés fut la richesse des échanges autour des différentes images. La plupart des gens avaient un ou des problèmes en tête lorsqu'ils ont proposé une image. Or, la plupart du temps, leurs collègues ont vu d'autres mathématiques dans cette même image. En serait-il de même avec les étudiantes et étudiants? Que se passerait-il si nous leur demandions tout simplement, comme l'a proposé l'un des participants, de décrire l'image? Seraient-ils en mesure, malgré leur anxiété, de nous faire voir les choses autrement? D'autre part, à travers les discussions, les participantes et participants se sont rendu compte qu'une telle activité pouvait permettre l'exploitation à la fois des mathématiques et de la didactique des mathématiques.

Les images ont mené plusieurs personnes à rédiger des problèmes ouverts. Il semble toutefois que ce type de problème puisse à la fois nuire et avantager les étudiantes et étudiants. En effet, alors que certains affirment que de tels problèmes sont plus accessibles et moins intimidants, d'autres disent que plus la tâche est ouverte, plus les étudiantes et étudiants sont anxieux. Il semble donc indispensable de présenter des contraintes claires, afin d'encadrer, du moins en partie, le problème proposé. D'autre part, il appert de se pencher sur l'évaluation de tels problèmes. Un problème ouvert peut-il être utilisé pour évaluer les étudiantes et étudiants de façon sommative? Si oui, comment évaluer la solution proposée à un tel problème?

Après avoir « joué » avec les images pendant un certain temps et discuté de leur expérience, les participantes et les participants ont été invités à se questionner sur leur propre façon de voir les mathématiques. Le contexte suivant leur a été présenté comme mise en situation : « *Vous êtes dans votre classe. Un nouveau chargé de cours va vous remplacer dans votre cours avec les futures enseignantes et futurs enseignants du primaire pendant que vous êtes en congé sabbatique. Sans perdre de vue la continuité du programme que vous avez développé, qu'allez-vous leur dire?* ». Afin d'aider les participantes et les participants dans leurs réflexions, les questions suivantes leur ont aussi été présentées :

- Comment conceptualisons-nous les mathématiques dans les activités que nous réalisons avec les futures enseignantes et futurs enseignants du primaire?
- Quels effets les décisions des formatrices et formateurs peuvent-elles avoir sur l'anxiété vécue par les futures enseignantes et futurs enseignants du primaire?
- Est-ce que cela va atténuer leurs craintes ou les empirer?
- Que peut-on dire à propos de la façon dont les futures enseignantes et futurs enseignants se sentent prêts ou non à enseigner les mathématiques?

DAY 3 'THE STAGE'

The first task of the third day was to attempt to connect some of the pictures in the online map in some coherent manner, such as a math trail. Participants worked flexibly, individually, in pairs, and in small groups.

A second task involved a paper version of a 'map' task. *Find a minimum number of ice cream shops that can exist on the vertices of the map so that everyone can reach an ice cream shop. People only live at vertices. People are only willing to walk one edge to get to a shop.* Various versions of the map, often increasing in complexity, were explored by the participants. The depth and breadth of mathematics, appropriate for the elementary school level, as well as branching into other levels of learning, spontaneously and naturally pervaded the conversations. "*I made a cube, and a 3-D vertex, but I was not able to convince my team-mates*" (a participant). Participants were not *trying* to be mathematical, they were not *trying* to be problem solvers; they acted naturally and showed themselves to be mathematical problem solvers. Ultimately, knowledge of the 'greedy algorithm' was acquired, and implemented into their thinking.

Next, drawing again on data from Towers' research study (e.g., Towers et al., 2017), drawings of students' expressions of mathematics anxiety were explored. Participants were asked to reflect on the drawings submitted by Kindergarten to Grade 12 students and think about how they would help their own preservice teachers to help such students.

Interesting moments of reflection and thought popped up regularly for the group. The discussion was perhaps chaotic, but focused through the lens of the participants' own realizations and appreciations of their roles and responsibilities with respect to mathematics anxiety and mathematics teaching anxiety. In the following comments, participants reflected on the mathematics task we had done as a group (the ice-cream shop task)—a graph theory task—and the kind of mathematics that was referred to by the students in Towers' research study, which was predominantly related to number and operations.

"To enable them [preservice teachers] *to think about how they are thinking about a problem."*
  "Preservice teachers' experiences are often disconnected from exercise to exercise."
    "How not to see every problem as an entirely new problem."
      "It [about anxieties] *was an analogy for me about how anxiety was generated."*
"We couldn't grasp the first task before the next map was given to us, and then on to another map…"
"I was starting to feel quite dumb and lost because another group clearly got it, and I didn't get it yet."
  "I was frustrated, I could not grasp…, I know how to do this but it wasn't coming."
    "There was some positive and some negative experiences; I was feeling motivated to keep the solution to myself."
"It's not the personality type that you are, it's the history of experiences you have had that lead one to think of themselves in a negative way."
  "The anxiety in the characters is mostly about number and calculation, the pictures are most about geometry."
    "Even if we had good experiences with this graph theory, etc., how much does it address the anxiety with number; e.g., how is this different than learning to play the piano as an analogy."
      "Pictures [the presented research data] were predominantly about number and operation; there is potential that using tasks such as these students may see themselves as being able to do math in one way, and then try to do the other math."
"It may not be the task at hand that causes anxiety, it might be the environment and or the people themselves that bring the anxiety to the front. It's not the math, it's the environment; the preservice teacher classroom may show more anger and frustration, etc., with such tasks; timing of bringing tasks to students' attention can be important."
  "It depends upon the person—I didn't want any help or hints, I wanted to do it myself."

"This might not be the students we are trying to address in this WG, because they are likely less anxious."

"Therefore we also need to be careful not to frustrate those that are not anxious as we work to alleviate anxiety in others; this task was a nice one because it was easy to start."

"This may suggest the teacher has an integral role to give help where needed."

"There is also the situation of the math anxious will wait for the other one to solve it. Therefore engagement is low as they wait; these kinds of tasks are not about 'fractions'."

"I have learned to be very explicit about my own practice when I teach preservice teachers."

"Especially when tasks that are not directly connected to the curriculum."

"Whose, our focus? Just those with anxiety, those who like to work alone and therefore mostly just watch the other work through the task."

"Preservice teachers must also be able to use tasks that are not evaluated because of the curriculum."

"Now, on day three we are trying to determine what an anxious student looks like…!"

An 'aha' moment occurred for many of the participants as they thought about their work as preservice educators of preservice teachers. Preservice teacher learning may not just be about helping preservice teachers appreciate elementary students' mathematics anxiety, preservice learning may also be about our (the participants') consideration that preservice teachers may be carrying their own mathematics anxiety. Perhaps a valuable lens is the perspective that they are teaching a preservice teacher course as mathematics educators (who carry aspects and levels of their own anxiety) helping preservice teachers (who carry their own mathematics anxiety) to help students (who also have mathematics anxiety) learn mathematics. The discussion then revolved around the question, "How do we break this cycle?"

How do we break this cycle? Perhaps a moment of sense making… *"one thing I have is power as a university prof—not only about the task, the task may be more minor than how we respond to their work on the task." "Be explicit about their actions—that they are acting mathematically—giving them permission, moving slowly, to stop and make sure they are making sense, and understand." "Invite them to stop the class."*

A potential new response to this issue of reducing mathematics anxiety and mathematics teaching anxiety is to look closely at the work we do in teacher education that has the deliberate aim of working on preservice teachers' anxieties. We know that our CMESG/GCEDM community houses a wealth of expertise in developing tasks for elementary preservice teacher education that aim to challenge problematic perceptions of mathematics, expand those perceptions, and address students' mathematics anxieties. Part of the intent of this Working Group was to gather, co-develop, play with, and, for our community (and perhaps wider distribution), develop a collection of 'new' designed tasks. Although we did not achieve that last, lofty goal, we have a stronger understanding of the complexity of the underlying phenomena.

We are aware, though, that exposure to pedagogically rich mathematical tasks is not a panacea for preservice teachers' mathematics anxieties. And so we will proceed with caution, inviting Working Group participants to continue to ask: Is it possible to turn all elementary preservice teachers into mathematicians or some sense of mathematician, or even lovers of math, and indeed, should that be our goal?

We invited Working Group participants to be sensitive to the changing, and ever more political, educational landscape into which our new teachers step. For example, Ontario has recently moved to emphasize financial literacy in the mathematics curriculum; New Brunswick's recently revised high school mathematics curriculum calls upon learners to choose between three pathways that offer differing levels of applicability to everyday life and careers; and

concerns about students' competence in numeracy (rather than mathematics) are emerging in the social discourse in Alberta. In thinking about new responses to the issue of mathematics anxiety among elementary preservice teachers, we explored the questions: What effects might such moves have on preservice teachers' math anxieties? Will this ease their fears or add to them? Will preservice teachers feel more, or less, prepared to teach mathematics through such curricula? What are the implications for the work of teacher educators? How can we help new teachers feel they can be, and, ultimately, be, more engaged in the broader conversation about the role of mathematics in society?

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SOCIAL MEDIA AND MATHEMATICS EDUCATION

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INTRODUCTION

Social media has enabled a dramatic shift in how individuals can participate in society. Social media tools allow for participating in co-creating and curating ideas, experiences and constructs that are publicly accessible and frequently remixable (e.g., Cardone & MTBoS, 2015; Carpenter & Krutka, 2014; Larsen, 2016; Larsen & Liljedahl, 2017). Mathematicians, mathematics educators, students, and the general public are influencing and are being influenced by publications about the teaching and learning of mathematics posted online on various forms of social media (Freiman, 2008). This participatory culture moves quicker than traditional forms of scholarship and dissemination; however, the implications of social media for the field of mathematics education are yet to be explored. Given the above, the key purpose of this working group was to explore the possibilities that social media presents the mathematics education community.

In this endeavour, we were guided by the following questions:

- What is the nature of social media in the context of mathematics education?
- What might be implications of social media use on the teaching and learning of mathematics?
- What are considerations regarding social media for mathematics education researchers?

Since this is a relatively new topic that has not been adequately explored particularly in the context of mathematics education, we began the first day of the working group by experiencing, exploring, and defining social media.

EXPERIENCING SOCIAL MEDIA

Our first prompt for exploring social media was a YouTube video shared by Viktor of a train plowing through a large pile of snow, which happened to be filmed close to Viktor's residence (video found here: https://www.youtube.com/watch?v=Yja2VmZOfdA). This video, which lasts three minutes, captures a train steadily approaching a railway crossing that is blocked by a pile of snow from road ploughing. The video has over 10 million views, 24 thousand likes, 600 dislikes, and 1500 comments. Group members were asked to comment on the video anonymously. Comments were posted on pieces of paper around the room. The following are some of these responses.

• Nothing can stop this train!	• I thought the train would be pushed off the track because of so much snow.
• Wow! Amazing! Wish I could have seen this in person	• So much snow in Canada.
• Woo-woo! Here comes the choo-choo!	• Spectacular!
• Canadian weather sucks!	• Wow! Almost!
• Beautiful!	• Impressionnant ! Surprenant toute cette neige qui s'envole
• Woowoochoochoo!!	

Group members were then asked to comment again, but this time, to include their name and the professional affiliation they were representing at the conference. The following are some of the professionally affiliated responses, which were also posted around the room.

• Belle explosion de neige. Très joli !—*Dimitri, Cégep St-Laurent*	• Spectacular!—*Olive Chapman, UC, CMESG president*
• Cette vidéo pourrait-elle être utilisée comme tâche mathématique ? Quelle serait la question ? #UtiliserLesMathsAuQuotidien—*Mathieu Thibault, UQAM*	• How often does this happen? Woohoochoochoo—*Steven Khan, Brock U.*
• You have managed to catch a really close encounter with the train. I can use the video in my class talking about perspective.—*Jennifer Holm, University of Alberta*	• I wonder what algorithms can be used here to find out what's happened here.—*Osnat Fellus, University of Ottawa*
• Wow! I'm glad that the train didn't have an accident. Watching this was mesmerizing.—*Caitlin Furlong, Université de Moncton*	• Need to learn more about it.—*Calvin, University of Alberta/University of Dodana, Tanzania*
• Look how hard Canadians push through.—*Krista, U of C*	• How might I use this video in my math classes?—*Jill Lazarus, University of Ottawa*
• Anybody else wondering how much (m^3) snow there is actually flying to the sides?—*Christina Krause, Univ. of Duisburg-Essen*	• Does it happen all the time?—*Xiong Wang, University of Alberta*

Discussion ensued regarding these comments as well as the comments found on the video online. We found that the nature of writing and the level of language changed between when comments were anonymous and when comments had a professional affiliation attached. There was a higher register of language and attention to grammar when professional affiliation was included. There was also more hesitation when comments were being written with the professional affiliation. 'Saving face' and maintaining responsibility to the profession became a concern when commenting was professionally affiliated and public. Because professional affiliation seemed to make a difference in how we responded publicly, the notion of 'multiple

selves' and identity arose. That is, how do we form a digital identity through our participation in social media and who do we perceive ourselves to be?

Further, the relevancy of the video prompt to one's interest, context, and past interactions with social media seemed to play a role in how each of us responded. For instance, those who were familiar with the 'notice wonder' routine, an approach where mathematics teachers use a visual prompt to question students about what they notice and wonder (NCTM, 2016), saw this post as such. An example of a 'notice wonder' prompt found on Twitter is shown in Figure 1. Regardless of whether the video was treated as a 'notice wonder' prompt or not, many in the group indicated a search for deeper meaning and revealed an interest in how this video could be used in teaching mathematics within the professionally affiliated comments. Making a relevant contribution became important when identity was attached.

Figure 1. 'Notice wonder' prompt with soft drink boxes posted on Twitter.

EXPLORING AND DEFINING SOCIAL MEDIA

Since working group members had varying degrees of experience with using social media, as a group, we listed out social media platforms we are aware of that currently exist, which included Twitter, Facebook, Pinterest, YouTube, Vimeo, Instagram, WhatsApp, Tumbler, SnapChat, blogs, Reddit, Academia, ResearchGate, LinkedIn, and discussion forums (e.g., Math Forum). From this, we worked towards a shared understanding of what counts as social media for the purposes of our working group. For example, "social media" (2017) is defined to be "forms of electronic communication (such as websites for social networking and microblogging) through which users create online communities to share information, ideas, personal messages, and other content (such as videos,)", but this definition may be interpreted in various ways.

Interestingly, Francophone participants in the working group noted that social media is seen in two ways in the French language as a distinction between *social networks*, which are more professional in nature, and *social media*, which are more general in nature (Theirs, 2013). This distinction does not exist in the English language, which has interesting implications on how we perceive what counts as social media.

We eventually established a working definition of social media for the purposes of our working group, which was that '*social media are tools that can curate a variety of media and allow for interaction among users using the media*'. We intentionally chose to include both openly public

and varyingly closed forms of social media, because even when media is closed off from a global public audience, it is still social in nature as it allows for the curation of media that is made available to members who are privy to accessing it. It is understood, also, that curation includes interaction, where media is the mediating tool with which the interaction occurs.

EXPLORING MATHEMATICS EDUCATION CONTENT ON VARIOUS PLATFORMS

After identifying what social media is and how it is manifested in the daily world, we pursued identifying mathematics education content on a variety of platforms. Groups were formed to explore what are considered five prominent social media platforms: Twitter, Facebook, Pinterest, YouTube, and blogs. The findings from each group are summarized below.

TWITTER

Twitter was found to be a tool designated primarily for public broadcasting. Twitter affords knowledge mobilization as things are shared (Tweeted), re-shared (Retweeted), and commented on. Twitter can even be used as a planning tool for teaching mathematics because it gives one the ability to find resources, videos, pictures, and research links, that is, to curate material. Twitter does, however, contain substantial content and therefore runs the risk of being perceived as disorganized. Certain members of the group acknowledged that the vast amount of content to follow was a barrier to effectively using the platform in a professional setting. To combat this issue, organization, developed through hashtags, and knowing the hashtags, became the focus of discussion regarding manners of finding desired resources.

FACEBOOK

Facebook has several special interest 'groups' related to mathematics education. 'Groups' are pages on Facebook created and managed by certain members, and can be set as either private, requiring approval to join, or openly accessible to all. Members of a group can post to the group page, and each post may elicit comments. There are professional groups and pages for teachers, students, classes, and academics. Some of these include *Math Lovers, 'MathFacts', Groupe d'Action pour les Mathématiques en Acadie, Numberphile, AERA Division K: Teaching and Teacher Education, Mathematics Education Researchers*, and *CMESG 2017*. There are a lot of curricular resources shared on group pages, and there are opportunities to receive feedback on ideas from peers.

PINTEREST

Pinterest is a tool for curating external links on various 'boards' where hyperlinked content is collected and featured. Any user can create a 'board' and can select what they include on it. Pinterest has both 'users' and 'boards' that claim to be dedicated to 'mathematics'. Users could be high school students or future teachers. The mathematics education content on Pinterest seems to be primarily visual in nature. There may be concerns with copyright and maintaining focus within searches. Some members of the group noted that many early career math teachers used Pinterest to obtain lesson plans et cetera more than any of the other platforms. Possible reasons for this phenomenon were discussed.

YOUTUBE

YouTube is a video sharing platform that houses many math related videos. Through an algorithm, YouTube also suggests videos to watch based on what one has viewed. Users can choose to subscribe and receive updates when new related content is released. There are a variety of channels dedicated to mathematics (e.g., *Numberphile*). Content is also shared widely and feeds many other social media platforms, such as Facebook, Twitter, Google+, and

Pinterest, which contributes to the tendency for certain videos to go 'viral' (e.g., "Can you order 43 nuggets?") There are also comment panes under each video. Comments can be made either on the YouTube platform or on other platforms on which a video is shared, which brought us back to our initial discussion of the train.

BLOGS

Blogs present material in more depth. As such, they are more time consuming to read. Blogs may also be challenging to find as they are written by many people on a variety of platforms without established search terms; however, other platforms such as Twitter, Facebook, and Pinterest offer spaces for sharing blogs. Mathematics related blogs may be found either by categories via https://mathblogging.org/, by bloggers via http://mathtwitterblogosphere.weebly.com/list-of-tweeps-and-blogs-by-category.html, by a Google search, or by personal connections. Blogs are relatively easy to create, and users have freedom over their design. Various people in mathematics education use blogs to write about their ideas and to share their work. This includes mathematics teachers, education leaders, and academics. As such, they are a rich source of discourse about mathematics education.

The discussion about the various platforms left the group with several questions. In particular, how do the structural features of a social media platform affect its utility and affordances? How does one maintain a focus when using social media? How can learning be different through social media and what enhancements does it provide? What is the potential impact of social media on learning?

MATHEMATICAL AND PEDAGOGICAL EXAMPLES

After exploring the broad range of social media options available, the group zoned in on specific examples of mathematical and pedagogical ideas found on social media.

NOTICE AND WONDER

Figure 2. 'Notice wonder' prompt with T-shirts posted on Twitter.

Since some group members had used a 'notice wonder' approach to the train video, we engaged in a 'notice wonder' example. As noted earlier, a 'notice wonder' is an instructional routine used in classrooms that aims to promote student thinking about what they notice and what they wonder and is typically prompted by an image or situation. The goal of this strategy is to encourage a variety of responses in a non-threatening manner (NCTM, 2016). Teachers often create 'notice wonder' prompts and share them within social media circles. As such, we briefly engaged in a 'notice wonder' example prompted by an image of piled T-shirts shared by Alex

Overwijk as shown in Figure 2. Some members of the group questioned what mathematics could arise from this, and others wondered about how long a teacher would spend investigating the questions that students generated based on the prompt.

OPEN MIDDLE

Another task type prevalent on social media is an 'open middle'. 'Open middle' problems have a closed beginning and a closed end in that they start with the same problem and end with the same solution, but they emphasize an 'open middle' in that "there are multiple ways to approach and ultimately solve the problem" (Johnson, Kaplinsky, Anderson, Luevanos, & Miller, n.d.). The way these problems are curated is that users often discuss open middle examples on Twitter using the hashtag #openmiddle, and eventually submit examples to the openmiddle webpage (http://www.openmiddle.com/) where they are uploaded and stored. As a group, we explored the 'open middle' problem in Figure 3.

Figure 3. 'Open middle' number sentence problem.

To mimic the online space in our physical setting, we explored the task in randomized groups on whiteboards around the room (see Figure 4). This approach was developed by Liljedahl (2016) to promote mobility of knowledge, an increase in random interactions, and a reduction in social barriers.
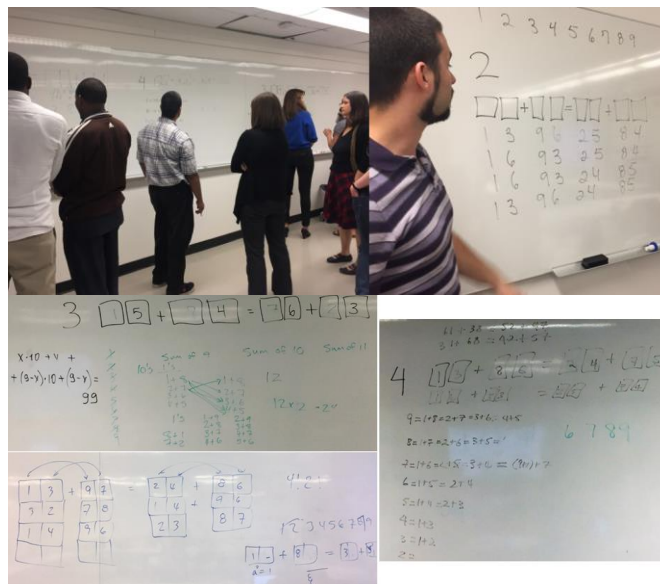

Figure 4. The group working on the 'open middle' number sentence problem.

From working on the problem and reflecting on our problem-solving process, we noticed that we quickly moved to pursuing what all the possible ways could be to satisfy the number sentence rather than just finding one way to arrange the digits as the problem requested. We also noticed that what was left on the boards could not always speak for itself in terms of revealing the reasoning of the group.

After working on the problem, we explored the responses that were made to the Twitter post made by Graham Fletcher (https://twitter.com/gfletchy/status/852494584538181634) where this problem was initially introduced. The problem was accompanied by Fletcher's (2017) statement: "Almost ready to submit this new 2nd-grade task to @openmiddle. I'll take any feedback and/or possible solutions you want to throw my way." This post garnered 43 retweets, 109 likes, and 33 response threads. We quickly noticed that many of the same approaches we had taken were made in the responses. Table 1 summarizes the group's observations and questions in relation to the treatment of this problem on social media.

What we noticed	What we wondered
• Some things are opaque and other things are transparent in such a setting. A lot is hidden. The degree to which it is clear exactly what lies behind a post varies. Some responses included media such as video or photos, while others did not include. In all cases, there is something that is not completely known, and therefore makes it 'opaque'. • One of the responders showed all the scribbles they made in working through the problem, and stated that it's a sign of a good problem if there are lots of scribbles. The fact that someone would post about scribbles is intriguing. • The asynchronicity seems to allow for a particular kind of exchange to take place. • There was communication through images happening in the threads. • Many things we had thought of were also in the thread (e.g. add 0, use playing cards, ask what are all the possible ways, etc.). Social media validated our thinking process. • This is an incredibly rich data set (comments, videos, pictures, etc.). In the mathematics research world, one video such as the one posted within this thread, would be analyzed and written about in many articles. This is only one post within one thread. • Social media is a treasure trove of data that is waiting to be analyzed!	• What grade levels could this task be appropriate for? How does this task get at the curriculum? • When a teacher sees this task, will they view it based on their own predisposition and their assumptions on student ability? • Who is being judged or evaluated? • Do respondents think about how they will be judged based on what they post? • Who are respondents and what are their digital identities? • Who can participate in the conversation? • Is participating in this way a self-fulfilling prophecy or echo chamber? How do we find people who will disagree? Do we only look for things that are relatable for us? • How are failures shared online? • What can be transparent online? • What classroom environment would this sort of task would be used in and how would it be used? • How does the visual presentation within posts change the nature of interactions? • What are the affordances of social media? • How does a community form online and what are the effects of being part of an online community such as the one that seems to be prevalent in this example?

Table 1. Summary of observation and questions to Open Middle problem.

RANKING CONVINCING ARGUMENTS IN A FORUM

We further explored the context of a forum. Viktor shared about his experiences of introducing a Discussion Forum within the CAMI virtual learning community (Freiman & Lirette-Pitre, 2009). The Forum was used by LeBlanc (2012) to conduct her research about proof evaluation by the grade 8 students using an online asynchronous discussion. Manon LeBlanc shared a set

of proofs (Figure 5) where the students were presented with five different proofs showing that the sum of two even numbers is even. The goal of the activity was to rank the proofs from most convincing to least convincing, which is what we did as a group as well.

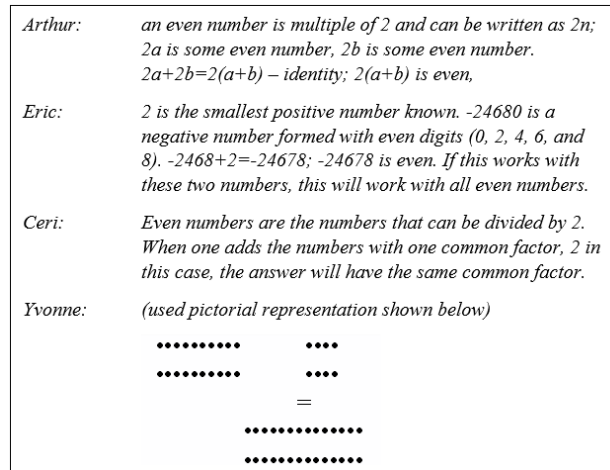|	
Arthur:	*an even number is multiple of 2 and can be written as 2n; 2a is some even number, 2b is some even number. 2a+2b=2(a+b) – identity; 2(a+b) is even,*
Eric:	*2 is the smallest positive number known. -24680 is a negative number formed with even digits (0, 2, 4, 6, and 8). -2468+2=-24678; -24678 is even. If this works with these two numbers, this will work with all even numbers.*
Ceri:	*Even numbers are the numbers that can be divided by 2. When one adds the numbers with one common factor, 2 in this case, the answer will have the same common factor.*
Yvonne:	*(used pictorial representation shown below)*

Figure 5. Five proofs for showing the sum of two even numbers is even (LeBlanc, 2012).

In our working group, we broke into small groups and worked on two tasks related to Figure 5. *Task 1* was to rank the proofs shown in Figure 5, to discuss these choices with group members, and then to think about how it can be done online. *Task 2* was to use Google Docs to debate about the problems.

We found that, among our members, everyone interpreted *Task 1* differently, which led to a rich discussion going beyond the original task of ranking proofs towards a deeper analysis of the task itself and possible affordances of social media (in our case, the discussion forum) to enrich mathematical communication and reasoning opportunities in mathematics classroom and beyond. As an extension of the activity, we also suggested the use of Google Docs by our members to collaborate on the task (*Task 2*). Google Docs is a word processor that allows for multiple users to collaborate on a given document in real-time. The group discussed how mathematical reasoning can be fostered in collaborative digital spaces.

SOCIAL MEDIA USER EXPERIENCES

As the group became interested in more nuanced aspects of using social media for mathematics education, it became important to hear various perspectives on social media use within mathematics education. Fortunately two prominent social media for mathematics education users, participating in other working groups at CMESG, were willing (we would be remiss not to mention they were invited via Twitter) to come and share their experiences with our working group. We thank Alex Overwijk and Nat Banting, as well as the leaders of their working groups for letting them join us to present.

ALEX OVERWIJK

Alex is a mathematics teacher in Ottawa. He noted that the Internet has changed the way he teaches because it offers opportunities for collaboration. Alex was also prompted to discuss his social media 'stardom' for becoming known for being the world's best circle drawer. He had been drawing 'perfect' circles in class for 10 years before one of his students videotaped him and it was posted to YouTube in 2007 (https://www.youtube.com/watch?v=eAhfZUZiwSE).

He noted that he had 2.5 million hits in two weeks, and about six years after the video went viral, he was asked to draw a perfect circle on the Today Show on NBC (https://www.today.com/video/teacher-who-can-draw-perfect-circle-does-it-live-44508739725). To this day, he is still asked to do this because, well, he is known for it. This speaks to the far-reaching effect social media can have and the identity it can contribute to. Participants of the working group were challenged by Egan to draw perfect circles. Congratulations to Mathieu Thibault who emerged from the group to take on Alex, but Mathieu, of course, lost closely to the World Champion in the final round. (see Figure 6).
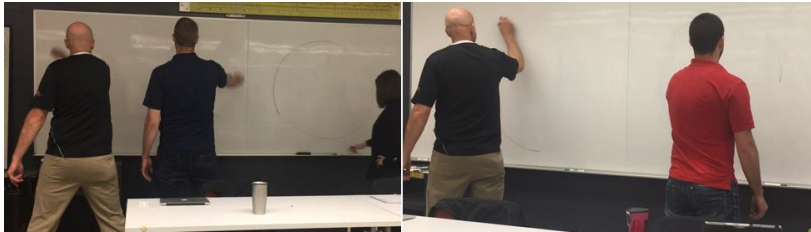

Figure 6. Drawing perfect circles.

NAT BANTING

Nat is a mathematics teacher in Saskatoon as well as a mathematics education researcher who has just recently completed his Master's degree at the University of Alberta. He spoke about his slow immersion into social media as a way to reflect on his own teaching, to organize his ideas, to feel accountable to his ideas, and to collaborate with other teachers. Nat spoke about how he uses Twitter and blogs, and his experience initiating the 'fraction talks' resource (see Figure 7).

Figure 7. Nat Banting sharing about using Twitter and blogs.

Nat started a Twitter account during his first year of teaching, and although he had many colleagues, he did not have many opportunities for collaboration. Through the Twitter network, he can connect with teachers around the world. Nat believes he has a lot of control in what he sees because he only subscribes, or 'follows', people who post interesting or useful things. This is how he builds his own 'sphere of influence'. For Nat, the Twitter mathematics teacher network (commonly referred to as the Math Twitter Blogosphere, or MTBoS) has been most powerful in terms of opportunities for interaction with other mathematics teachers. He noted that most of the interactivity with others happens in the Twitter space because that is where the network is. Nat also maintains a blog space called *Musing Mathematically* (found via natbanting.com/blog/) in which he writes about ideas prompted by his experiences as a mathematics teacher through his views as a researcher in mathematics education. The blog space does not get as much interactivity, but he can share his posts on Twitter to promote discussion.

The network allowed Nat to develop a space in which a 'fraction talks' resource developed. Nat was initially inspired by Ilana Horn, who shared about using a partitioned square to talk about fractions during one of her keynotes. He used this idea with students, and soon found his students wanted to create their own examples, which became more and more complex. When he shared about these examples on Twitter, other mathematics teachers from around the world began to send him their students' examples. To make it easier to talk about, Nat developed a fraction talks website (www.fractiontalks.com), and it reached half a million views in about eight months. There is also an associated hashtag used on Twitter (#fractiontalks) and a Twitter handle for it (@FractionTalks). Nat became known for this 'thing' and he is associated with this resource frequently, but he emphasizes that he really only provided the space for the network to work on creating the resource.

SOCIAL MEDIA AND ACADEMIA

The conversations prompted by Alex and Nat shifted our conversations towards the relationship between the mathematics teacher network on social media and the field of mathematics education. Although the two groups have different currencies to measure success and validity (e.g., a retweet versus a reference), both are outlets for scholarship around mathematics education. Parallels between the nature of the mathematics teacher network on social media (i.e., MTBoS) and CMESG were made as they are both grassroots movements where mathematics educators connect to interact about ideas in a generative way. However, there is also a bleeding between the network on Twitter and academia. For instance, MTBoS recently celebrated the first dissertation about the MTBoS, written by Christopher Parrish (2016). There are also nationally funded research projects that are originating out of the work that MTBoS members have developed (e.g., Illana Horn and Melissa Gresalfi's National Science Foundation's project titled *Playful Mathematics Learning: An Exploration of Design, Learning, and Engagement*). There is this idea that one thing can become another thing, and it feeds back onto itself—a sort of snowballing.

Pivotally, Steven Khan asked Nat and Alex about "*what the MTBoS community wants from researchers*". Nat and Alex agreed that it is about contributing and interacting. It is important that researchers not just 'study' MTBoS members, but rather, that they become involved in the ways they would around their interest areas. Nat suggested that researchers should find people who post about things they are studying and interact with them. Alex added that sometimes he posts on his blog, and a researcher will comment with a reference to an article that supports what he is doing, and he finds this very helpful. The group discussed how social media seems to be a great platform for connecting the academic and teaching worlds in an accessible manner, and it behooves mathematics education researchers not to get involved in the interactions about mathematics teaching online.

SOCIAL MEDIA AND MATHEMATICS EDUCATION

As a group, we engaged in a summarizing activity to bring out the most important things we learned as a group from the three working group meetings. Some themes arising from our discussions included identity, community, research, accessibility, rich data source, and opportunities for collaboration. More broadly, though, our key learnings can be summarized within three areas: the nature of social media, implications for teaching and learning mathematics, and implications for mathematics education researchers. These are outlined in what follows.

THE NATURE OF SOCIAL MEDIA

'Social media' consist of tools that curate a variety of media and allow for interaction among users using the media platform. The structure of various media platforms may affect how users interact. For instance, Twitter is less structured, and blogs are more structured, but Twitter allows for more interactivity. Importantly, social media are not only used for capturing and sharing moments but also for collaborating about ideas. Therefore, social media allow for curating, contributing, and collaborating. Social media can also support building networks and communities. Various roles can be taken on such as lurker, leader, critical challenger or devil's advocate within these communities. Further, anonymity seems to play an important role in how content is shared on social media. For instance, having a professional affiliation attached to one's digital identity may make one more thoughtful about making public comments. There is also a difference between a real person and what one appears as online. Although personal stories are not easily seen online, having a backstory about someone can make them more relatable and easier to understand. At times, it is possible for social media to be 'dark' in that it can disturb in-person relationships as digital identities conflict with real-life identities. However, social media becomes what you make of it. Some may view it as an echo chamber, a fire hose, or as a place for snowballing of ideas, but this is highly dependent on how it is used. For instance, hashtags are an important framework for organization and can prevent social media from being overwhelming. Also, finding and pursuing connections with others who share common purposes, even if they have diverse ideas, makes it a more personally useful space.

IMPLICATIONS FOR TEACHING AND LEARNING MATHEMATICS

The virtual mathematics teacher community (e.g., MTBoS) can be very supportive and generative in terms of building ideas for teaching mathematics. Methods of teaching are accumulated on social media and found within the posts made by mathematics educators. In-person conferences (e.g., *Twitter Math Camp*) have even developed out of social media connections between mathematics educators in relation to conversations about mathematics teaching. Inspiring stories of lived experiences pertaining to mathematics teaching are shared on social media, and authors are available to connect about these experiences, which makes a rich opportunity for networking and dialogue. Given that 'stardom' can be reached from simple things such as drawing a perfect circle, there is a powerful opportunity for influence within the realms of social media.

IMPLICATIONS FOR MATHEMATICS EDUCATION RESEARCHERS

Social media provide a rich data gold mine for research and teaching and also allow for putting research in the hands of teachers. Namely, it is a rich place for dialogue between mathematics educators on social media (i.e., Twitter) and mathematics education researchers. Although both groups are invested in success around the teaching and learning of mathematics, they have different outlets for scholarship. Social media allow for dialogue; whereas, research papers are static. Social media are also more current and allow for real-time and asynchronous discussions. For instance, teachers post problems and their colleagues respond to them with ideas, often sharing video of students working through the problems. This sort of interaction offers similar opportunities to that found in 'lesson study' approaches. It also constitutes a rich data source that offers insight into the state of pedagogy in mathematics education. We cannot stop the social media train (*Woo-woo! Choo-choo!*), so how will we utilize it to its fullest potential?

OPPORTUNITIES FOR FUTURE EXPLORATION

It is no surprise that with such a new and ever-growing topic, we are left with many unanswered questions. We have only scratched the surface of possibilities that may result in further exploration of the nexus between social media and mathematics education. We are left with the

following questions: How can teachers' and researchers' use of social media help students learn mathematics? How do we use social media with students in classrooms? What enhancements does it provide? And what is the potential impact of social media on learning? How do the structural features of a social media platform affect its utility and affordances? Who is using social media for mathematics education? How can we research the social media phenomenon? How do we build bridges between MTBoS and mathematics education researchers?

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QUANTITATIVE REASONING IN THE EARLY YEARS

LE RAISONNEMENT QUANTITATIF DANS LES PREMIÈRES ANNÉES DU PARCOURS SCOLAIRE

Doris Jeannotte, *Université du Québec à Montréal*
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INTRODUCTION

In the closing panel of 2016 CMESG, Anna Sierpinska questioned whether the kinds of tasks provided to children in elementary school enhanced their quantitative reasoning. In this working group, we worked on extending Anna's wondering by exploring the development of quantitative reasoning in the early years (pre-K-4) from different perspectives. We drew on recent literature, a range of mathematical tasks, and teachers' practice and pupils' responses as part of this exploration.

In particular, we explored the links between quantitative reasoning and early algebraic thinking through processes such as generalizing, pattern noticing, conjecturing and justifying. How are these reasoning processes utilized in tasks involving mathematical structures (e.g., even and odd numbers, commutativity, regularities in multiples, skip counting, adding decimal numbers, etc.)?

Through the three days we worked together, we addressed the following key questions:

- What is quantitative reasoning and what are the different forms of quantitative reasoning?

- Is early development of quantitative reasoning important and if so, why?
- What types of mathematical tasks emphasize quantitative reasoning processes?
- How might we enhance quantitative reasoning through teaching?

DAY ONE

During day one, we considered the following questions:

- What is quantitative reasoning and what are the different forms of quantitative reasoning?
- What types of mathematical tasks emphasize quantitative reasoning processes?

The goal was to explore the discourse around quantitative reasoning and so, to start to conceptualize what quantitative reasoning is. As researchers and educators, we already have some ideas about what quantitative reasoning is without necessarily referring to an already built theory. By exploring one task and two transcripts of children mathematical activities, we shared what we considered as part of quantitative reasoning in the early years.

THE TASK
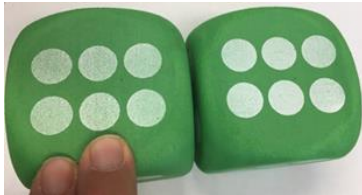
We explored a task not designed to develop quantitative reasoning. It was first used in the project MathéRéaliser (Corriveau & Jeannotte) to explore the use of manipulatives in grade 1. We chose it because, it was suitable for first-grade pupils and the task itself is based on relation between quantities.
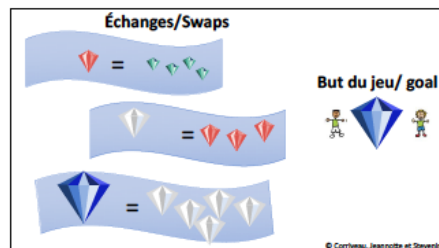
Figure 1. Rules of the Diamonds' game.

WHAT EMERGED FROM THE DISCUSSION

Even if the task is based on relation between quantities, is it possible to find quantitative reasoning in students' mathematical activities?

1. Counting vs Quantitative Reasoning
   One important element that emerged from our conversation is the difference between counting and quantitative reasoning. As the task is suitable for young children, it is also solvable only by counting. It was clear that counting a quantity was not quantitative reasoning even in early grades.
2. The Idea of Relation and Working on Relationship
   Thompson (1993) links quantitative reasoning and working on relationship. One form of reasoning that grade 1 students can do is to use the ratio between the different types of diamonds to go faster. For example, we could observe children explaining to the class that if they got 12 on the dice, they would know they can take a white diamond

directly because: "four, four, four (pointing at each group of on the dice) make three reds" (see Figure 1). But, even if this reasoning implies working on relationships, we were not convinced it was enough to talk about quantitative reasoning.

3. Question we could ask children to help the emergence of quantitative reasoning
   To enhance quantitative reasoning, we tried to think about what kinds of questions we could ask children when using this game. One that appears interesting is to stop the game before there is a winner and then ask each team who is closest to obtaining the blue diamond. In doing so, we help students focus on the difference between what they have and the target. They can also compare those differences with the relation more or less.

Pupils making their own rules would also help focus on relationship and how this can affect the game. For example, is the strategy of picking one white when we roll 12 on the dice always good regardless the relation in play? Why?

THE FIRST TRANSCRIPT

The first transcript was from Tian and Huang (2009). The study of Tian and Huang aimed at documenting spatial and quantitative reasoning abilities of young children. During interviews, 3 to 5 years old children had to answer the question how many doors, flower pots, and houses in Figure 2.

Figure 2. Spatial and quantitative reasoning (Tian & Huang, 2009).

What seemed implicit is that pupils who use quantitative reasoning would rely on a pattern (every row is the same) to answer the question versus counting what was visible. The idea of pattern, visualization, comparing and of concrete situations emerge from the discussion.

THE SECOND TRANSCRIPT

The second transcript was from Radford (2011). The task explored by grade 2 children in the transcript is a patterning activity (see Figure 3).
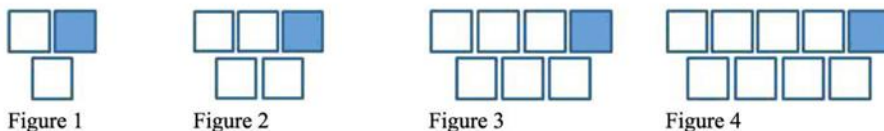

Figure 3. The first four figures of a sequence given to the students in a Grade 2 class (Radford, 2011).

In the transcript, we could read how students were reasoning to find a relation which would enable them to find the number of squares in any figure. In addition to the relationship, generalization is needed to solve this task, as it is designed to enhance algebraic reasoning. Two

questions emerged from this exploration: (1) *What are the differences between algebraic reasoning and quantitative reasoning in the early grades?* (2) *If algebraic reasoning requires analytic reasoning, is there another way of considering the unknown imply in the quantitative reasoning?*

At the end of day one, we finished with a vocabulary list that we felt was linked to the meaning of quantitative reasoning in some way.

• Concrete	• Imagining	• Relation
• In context	• Hidden parts make	• Quantities
• Structure	multiplication emerge	• Visualization
• Conversion	• Relationships	• Patterns
• Comparing	• Counting	• Patterning
• Spatial	• Representation	• Interpretation
• Imaging	• Flexibility	

Table 1. Vocabulary related to quantitative reasoning.

DAY TWO

Our focus for Day Two was to address the following questions:

- What is quantitative reasoning and what are the different forms of quantitative reasoning?
- What types of mathematical tasks emphasize quantitative reasoning processes?

Much of our discussion during Day One was wrestling with the question "What is quantitative reasoning?" Through our activities of the previous day, we found we were employing multiple forms of thinking under the general umbrella of mathematical reasoning to describe and justify perspectives. We debated whether it was worthwhile making distinctions amongst the multiple forms of mathematical reasoning that we brought forward.

What is *mathematical*
algebraic
quantitative
additive
multiplicative
arithmetic
logical
spatial
structural
proportional
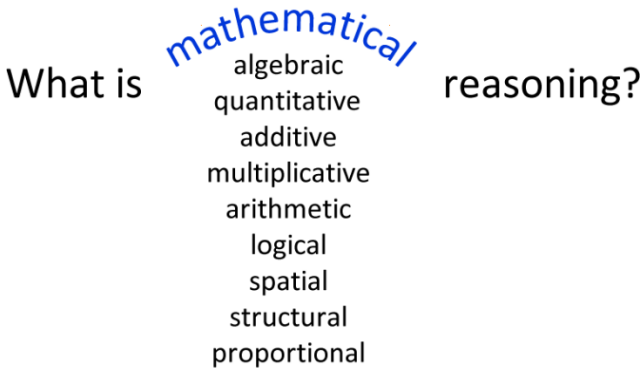reasoning?

Figure 4.

To explore the distinction between quantitative reasoning and other forms of reasoning, we turned to literature to help define quantitative reasoning. According to Smith and Thompson (2007), quantities "are attributes of objects or phenomena that are measurable; it is our *capacity* to measure them—whether we have carried out those measurements or not—that makes them

quantities" (p. 101). Understanding the nature of a quantity fed into the definition of quantitative reasoning as defined by Thompson (1993):

Quantitative reasoning is the analysis of a situation into a quantitative structure—a network of quantities and quantitative relationships (Thompson, 1989). A prominent characteristic of reasoning quantitatively is that numbers and numeric relationships are of secondary importance, and do not enter into the primary analysis of a situation. What is important is relationships among quantities. In that regard, quantitative reasoning bears a strong resemblance to the kind of reasoning customarily emphasized in algebra instruction. (p. 165)

In order to understand how to operationalize the definition of quantitative reasoning, we have engaged in a task deemed by Smith and Thompson (2007) to elicit such reasoning:

I walk from home to school in 30 minutes, and my brother takes 40 minutes. My brother left 6 minutes before I did. In how many minutes will I overtake him? (Krutetskii, 1976, p. 160).

Working group members were asked to determine a solution to the problem. They were discouraged from using algebraic solutions, at least initially. One approach was to develop a table to record fractional distances traveled at specific time intervals. The relationship of time, distance and speed were identified for "me" and "my brother" somewhat independently and the recursive relationship was applied until an equal value (or distance) was obtained.
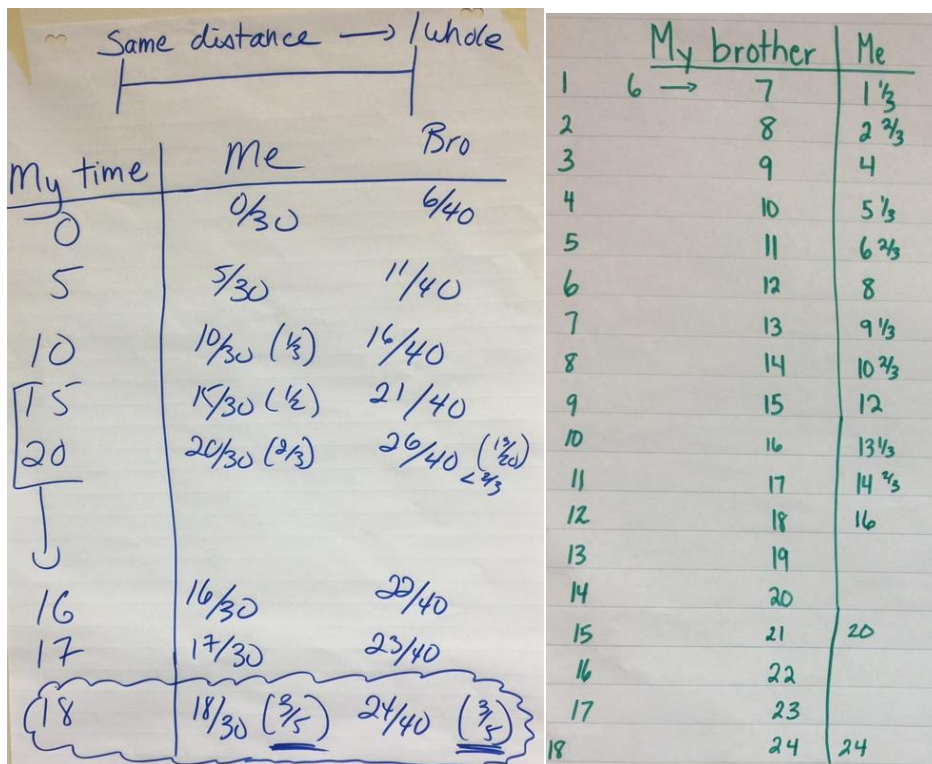
Figure 5.

Other groups attempted to establish a relationship between "me" and "my brother's" speed either arithmetically or algebraically.

Figure 6.

Several groups expanded upon the four-thirds speed relationship to help diagram the context to find the point where the person overtook the brother at 18 minutes.
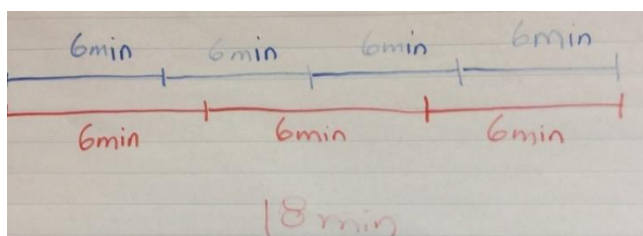
Figure 7.
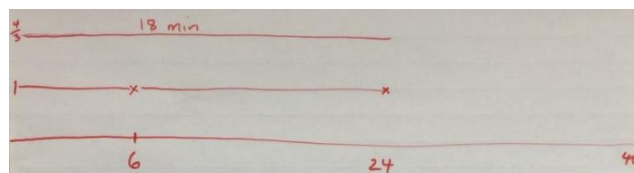
Figure 8.
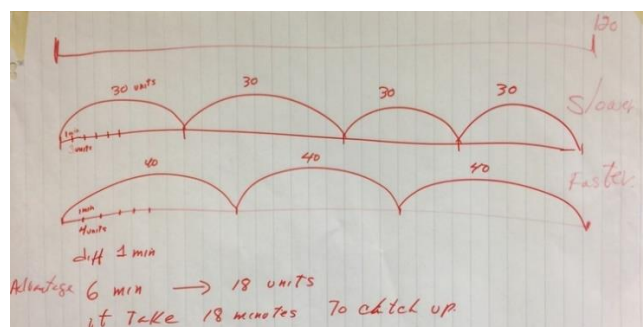
Figure 9.

Finally, some groups used algebraic conventions to establish a relationship and to solve the problem.

Figure 10.

After completing the problem and sharing solutions, we discussed if, how, and when quantitative reasoning was employed by attending to how the context of the problem was analyzed into a quantitative structure or set of relationships, and when the focus was on the relationship between quantities rather than the numbers themselves. In general, our group reached general agreement that when problem solving, we tend to move in and out of quantitative reasoning. That is, the initial analysis or mathematizing of the problem requires attention to the relationship between quantities. In this specific problem, the relationships between time, speed and distance traveled were needed for each individual and, depending on the solution method, between the individuals walking as well. However, once the initial relationships were established, applying quantitative reasoning was often minimal. People often employed either arithmetic or algebraic operations to compute a solution without the need to keep the quantitative relationships in mind. Upon arriving at a solution, quantitative reasoning was often used to analyze the situation again to ensure that the result was reasonable in the context of the actual problem.

To further our understanding of potential distinctions of quantitative reasoning and other forms of thinking, we examined sets of problems used by Thompson (1993) in a teaching experiment on complex additive structures. Three sub-groups were formed to examine a set of problems and discuss how the modifications to the problems—from numeric to generalized—prompted learners to attend to the relationship between the quantities. Each problem led to various ways to represent the relationships.

Problem 1	Problem 2	Problem 3
Jim, Sue, and Tom played two games of marbles together. Sue won 6 marbles from Jim and 5 marbles from Tom. Jim won 3 marbles from Tom and 4 marbles from Sue. Tom won 12 marbles from Jim and 2 marbles from Sue. Compare the number of Tom's marbles before and after these two games.	Jim, Sue, and Tom played two games of marbles together. Sue won 6 marbles from Jim and 4 marbles from Tom. Jim won 5 marbles from Tom and 3 marbles from Sue. Tom won 2 marbles from Sue, and altogether he came out ahead 4 marbles. How many marbles did Tom win from Jim?	Jim, Sue, and Tom played two games of marbles together. Sue won ___ marbles from Jim and ___ marbles from Tom. Jim won ___ marbles from Tom and ___ marbles from Sue. Put numbers in the blanks so that altogether, Sue came out behind by 3 marbles.

Table 1. Marbles Problem (Thompson, 1993, p. 169).
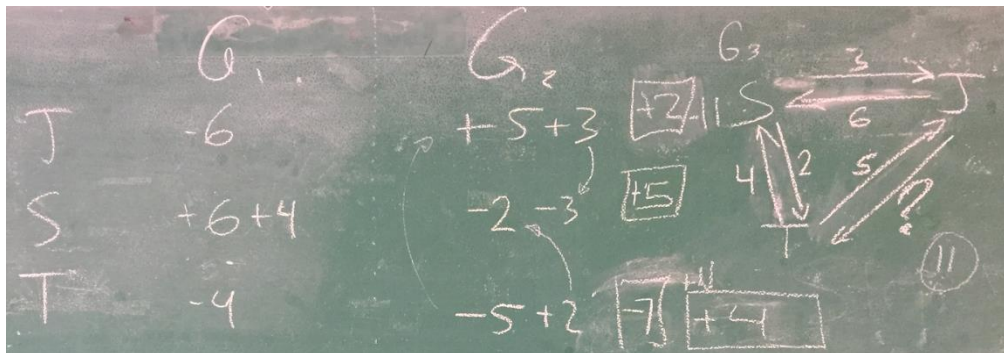
Figure 11. Solutions for the Marbles Problem.

Problem 1	Problem 2	Problem 3
Metcalf has two third grade rooms (C and D) and two fourth grade rooms (E and F). Together, rooms E and F have 46 children. Room C has 6 more children than room F. Room D has 2 fewer children than room E. Room F has 22 children. How many children are there altogether in rooms C and D?	Metcalf has two third grade rooms (C and D) and two fourth grade rooms (E and F). Together, rooms E and F have 46 children. Rooms C and D have 50 children together. Room C has 6 more children than room F. Room D has 2 fewer children than room E. There are ___ children in room E. What number or numbers can go in the blank so that everything works out?	Metcalf has two third grade rooms (C and D) and two fourth grade rooms (E and F). Together, rooms E and F have 46 children. Rooms C and D have 48 children together. Room C has 6 more children than room F. Room D has 2 fewer children than room E. There are ___ children in room E. What number or numbers can go in the blank so that everything works out?

Table 2. Classroom Problem (Thompson, 1993, p. 171).
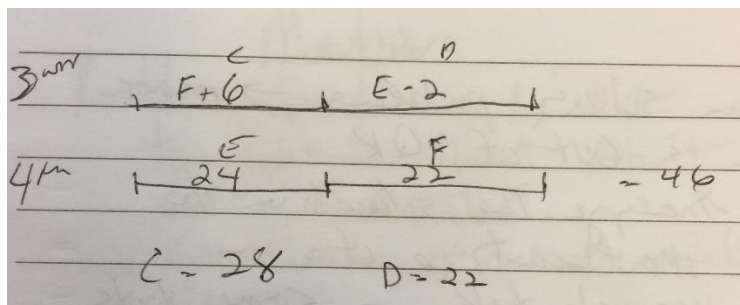
Figure 12. Solutions for the Classroom Problem.

Problem 1	Problem 2	Problem 3
Two fellows, Brother A and Brother B, each had sisters, Sister A and Sister B. The two fellows argued about which one stood taller over his sister. It turned out that Brother A won by 17 centimeters. Brother A was 186 cm tall. Sister A was 87 cm tall. Brother B was 193 cm tall. How tall was Sister B?	Two fellows, Brother A and Brother B, each had sisters, Sister A and Sister B. The two fellows argued about which one stood taller over his sister. It turned out that Brother A won by 17 centimeters. Brother A was 186 cm tall. Sister A was 87 cm tall. Brother B was ___ cm tall. Sister B was ___ cm tall. Put numbers in the blanks so that everything works out.	Two fellows, Brother A and Brother B, each had sisters, Sister A and Sister B. The two fellows argued about which one stood taller over his sister. It turned out that Brother A won by 17 centimeters. Brother A was ___ cm tall. Sister A was ___ cm tall. Brother B was ___ cm tall. a) Put numbers in the blanks so that everything works out. b) What has to be true about the numbers anyone puts in the blanks if everything is going to work out?

Table 3. Brother and Sister Problem (Thompson, 1993, p. 175).
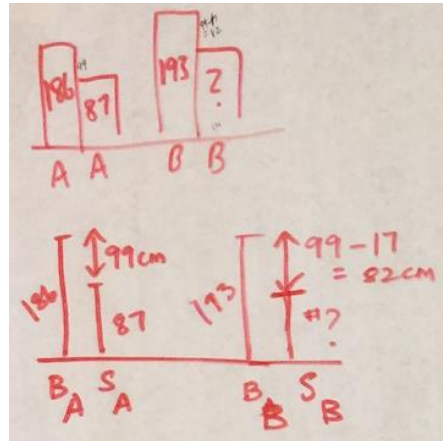
Figure 13. Solution for the Brother and Sister Problem.

DAY THREE

For day three, working group members were asked to share a problem or task that might elicit quantitative reasoning in children. Several examples were shared during the final morning. A sample of tasks is provided below along with some points of interest for each one.

ALTERING A TRADITIONAL ARITHMETICAL PROBLEM (SIMON LAVALLÉE)

Simon suggested that altering a traditional arithmetical problem so that it became more open-ended, allowed more opportunities for quantitative reasoning.

> *Arthur a 2 pommes et Boris en a 3. Ensemble, combien en ont-ils ?*
> *VS*
> *Arthur et Boris ont chacun un sac qui contient des pommes. Ensemble ils ont 10 pommes. Combien de pomme y a-t-il dans le sac d'Arthur et combien y en a-t-il dans le sac de Boris ?*

En ouvrant le problème, il est possible de discuter avec les élèves de la relation entre le nombre de pommes d'Arthur de Boris et le total.

MATHVILLE PROBLEM WITH REVISIONS (AUDREY GARNEAU)

Initially, the Mathville task was posed as follows:

> *À Mathville, le dernier hiver a été très rigoureux. Au début de l'hiver, Augustin avait 2005 buches. Les jours chauds, il a brulé 25 buches pour chauffer sa maison. Les jours froids, il en a brulé le double. À la fin de l'hiver, il lui reste 205 buches. Sachant qu'il y a eu autant de jours chauds que de jours froids, combien de jours l'hiver a-t-il duré ?*

Cette tâche correspond à l'énigme *Long hiver* sur le site de la semaine des maths http://www.semainedesmaths.ulaval.ca.

En discutant en équipe de travail, il a été réalisé que cette tâche permet plus ou moins de travailler le raisonnement quantitatif, puisque la question est fermée, une seule réponse est possible et cette réponse ne peut être obtenue que par calcul. En ouvrant le problème, il est alors possible de poser d'autres questions aux élèves afin de les amener à se centrer sur les relations.

> *À Mathville, le dernier hiver a été très rigoureux. Augustin avait 1800 buches au début de l'hiver et il les a toutes brulées. Les jours chauds, il a brulé 25 buches pour chauffer sa maison. Les jours froids, il en a brulé le double. Combien de jours l'hiver a-t-il duré ?*

Les élèves peuvent obtenir des résultats différents puisque l'hiver peut durer un nombre pair de jours entre 36 et 72. L'enseignant pourrait alors demander à ses élèves

- Pourquoi ta solution fonctionne-t-elle ?
- Pourquoi y a-t-il d'autres solutions possibles ?
- Quelles situations sont impossibles ?
- Pourquoi les nombres impairs ne fonctionnent-ils pas ?

Cette nouvelle tâche semble davantage pertinente pour développer le raisonnement quantitatif puisqu'il s'agit d'un problème ouvert dans lequel les relations entre les nombres sont importantes et où plusieurs réponses sont possibles.

SIMON'S NEIGHBORHOOD (VALÉRIANE PASSARO)

In this problem Valériane offered a problem involving the relationships of time, distance and speed that would be accessible to elementary school students and introduce them to the idea of covariation.
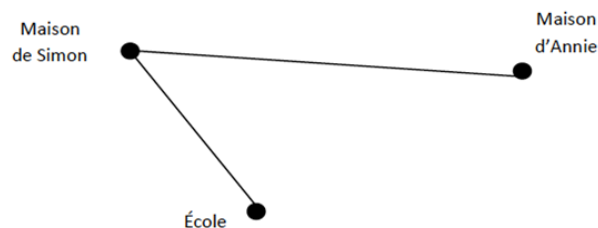

Figure 14. Le plan du quartier de Simon.

1. La distance entre la maison de Simon et celle de son amie Annie est-elle plus courte ou plus longue que la distance entre la maison de Simon et l'école ?
2. Si Simon marche pour aller chez son amie Annie, est-ce que ça lui prendra plus ou moins de temps que s'il marchait pour aller à l'école ?
3. Si Simon se rend chez Annie en courant est-ce que ça lui prendra plus ou moins de temps que s'il y allait en marchant ?
4. Si Simon prend 5 minutes pour marcher de chez lui jusque chez Annie, combien de temps cela devrait-il lui prendre approximativement pour aller à l'école en marchant ?
5. Si Simon prend 5 minutes pour aller chez Annie en courant, combien de temps cela pourrait lui prendre pour y aller en courant ?
6. Supposons que Simon se déplace toujours en marchant, complète les phrases suivantes :
   Plus la distance parcourue par Simon est grande, _____ ça prend de temps.
   Moins la distance parcourue par Simon est grande, _____ ça prend de temps.
7. Simon se rend à l'école, complète les phrases suivantes :
   Plus Simon se déplace rapidement, _____ ça prend de temps.
   Moins Simon se déplace rapidement, _____ ça prend de temps.

JELLY BEANS PROBLEM

The Jelly Bean problem is known to many people and was also used as a task to emphasize relationship between quantities:

> *A jar of jelly beans sat on the teacher's desk. Suhong was very hungry so he helped himself to half of the jelly beans. Angela forgot to bring a snack for recess so she ate a third of what was left in the jar. Fatima saw Angela eating jelly beans at recess so when she came back into the room she took a fourth of the remaining jelly beans. The teacher noticed that when Tim returned from recess he quickly sat down and got organized for the afternoon math class, so she said that Tim could be rewarded with 6 jelly beans, and he ate them. Barbara was hoping to get some too, but she was having trouble settling down, and the teacher said that she could only have some when she was quiet. Chris told Barbara that it's no big deal because there are only 3 left in the jar anyway. So, how many jelly beans were there to begin with?*

CONCLUDING REMARKS

As a synthesis, we asked participants to construct a concept map that could help us understand what quantitative reasoning is in primary school. First, we all agreed that quantitative reasoning is linked to Number and Operational sense development, a crucial element of every Canadian curriculum. By focusing our attention on relationship, we help children explore numbers and operations: this is more difficult when the focus is on the calculation and answer. Second, we were able to link quantitative reasoning to a variety of processes: using referents from experience, justifying, analyzing relationships, and comparing quantities, to name a few. Third, we concluded that relations between quantities and patterning activities were at the junction of quantitative reasoning and algebraic reasoning.

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SOCIAL, CULTURAL, HISTORICAL AND PHILOSOPHICAL PERSPECTIVES ON TOOLS FOR MATHEMATICS[1]

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INTRODUCTION

In this working group we examined tools for doing mathematics within the social, cultural, historical, theoretical and philosophical contexts of both their use and their origins. By 'tools', we mean any of the things people imagine using as part of their mathematical activity—and we spent a good deal of time in the working group exploring differing conceptualizations of 'tools' and their effects. During the working group sessions, we used a variety of tools to solve mathematical problems at all educational levels (elementary, secondary and post-secondary) and discussed what takes place in light of different theoretical perspectives. We explored the 'enabling constraints' of approaching a particular mathematical relationship using historical mathematical tools from a broad range of cultures and in contemporary practices. We tried to contextualize each tool as much as possible in the philosophical and sociocultural milieu in which it developed. From a theoretical perspective, we looked at Leontiev's ideas on actions, operations and tools; we got inspired by Heidegger's work on tools as essentially non-functioning devices; we talked about Vygotsky's perspectives on how tools carry with them the perceptions and thoughts of people who made/used/modify them over time; and discussed McLuhan's analysis of the cultural effects of tools and technologies (where making tools

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[1] This text was written based on notes from the group leaders, but also on reflections shared by the participants, for example on the blog we created to accompany our working group: https://cmesgmathtools.blogspot.ca.

immediately leads to tools remaking us). Participants also brought their own favourite 'mathematical tools'.

DAY 1 THEORETICAL BACKGROUND: McLUHAN'S APPROACH TO 'TOOLS' (LED BY SUSAN GEROFSKY)

In the first session we started by posing the question of "What do we mean by tools?" and focused on the theoretical approaches of Canadian media theorist Marshall McLuhan to explore this question. McLuhan was an English professor at the University of Toronto from the 1950s to the 1970s who became an influential theorist of culture and technology, and his work is both highly-regarded and provocative to this day.

McLuhan saw tools as extensions of our bodies that both enhance and simultaneously numb the part of the body they extend. For example, automobiles extend the powers of our legs and feet, but simultaneously immobilize us and numb our legs and feet when we travel by car. McLuhan saw that "we shape our tools and our tools immediately shape us" (McMahon & Sobel, 2002), as human individuals and infrastructures are wholly reshaped by each new pervasive technology, with technology defined broadly to include spoken and written language, the printing press, cinema, radio and television, computers and the Internet, and much more.

In a time when other theorists were focused on a metaphor for technologies as 'pipelines' delivering content, McLuhan controversially pointed out that "the medium is the message" (McLuhan, 1964/1994, Ch. 1)—that is to say, the effects of a new tool or communication medium in restructuring society and people are indescribably greater than the effects of any particular content of that medium. McLuhan theorized that a radical shift in a society's media and tools also remakes the balance of our senses (call the sensorium). For example, McLuhan argued that widespread alphabetic literacy from the 16th to the 20th centuries, through the effects of the movable type printing press, skewed the human sensorium toward the visual; and the advent of electric and electronic media, from the introduction of the telegram in the 19th century up to and beyond our current networked computer environment, is taking us rapidly 'back through the Renaissance' to a global village dominated by new forms of oral culture, where everyone is involved in everyone else's business (McLuhan, 1964/1994, 1962/2011).

For McLuhan, there is no distinction between tools, communications media, technologies, and cultural extensions of the self. He explains that, since new technologies simultaneously extend and numb us, we are as unconscious of our media environment as fish are of water (McLuhan, 1968/2001). McLuhan's insights include the fact that every new medium surrounds the old and immediately makes it visible, obsolete and an art form; that new media take old media as their content; and that all media work us over completely. Marshall McLuhan with his son Eric developed a four-part methodological frame, the McLuhan tetrad, for analyzing the effects of media, tools, technologies (Figure 1; McLuhan & McLuhan, 1988).
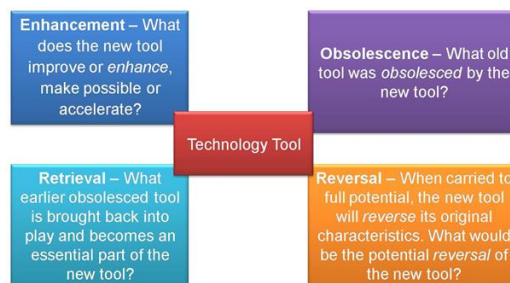

Figure 1. McLuhan's tetrad, from https://mcluhangalaxy.wordpress.com.

ACTIVITIES DAY 1: QUIPU, STICK MAPS, GREENLAND COASTAL MAPS: HOW CAN WE CONNECT WITH MATHEMATICAL TOOLS FROM OTHER TIMES AND CULTURES?

With McLuhan's theoretical approach in mind, we worked with three hands-on activities from three different cultural contexts, to explore how tools are formed by us and at the same time form us.

Our first question was a concern around cultural appropriation. We have material evidence of many mathematical technologies that have been used in other places, times and cultures. In many of those cultures, there may not have been a term for 'mathematics'—or for 'art', or for 'technology'—in the way we frame these entities. So these objects and the techniques that created them come from cultural contexts that are almost certainly quite different from those of contemporary urban Canada, for example. Can we connect with these technologies as mathematical tools in meaningful ways that respect the cultures that made and used them?

We acknowledged that we would certainly be reframing things according to *our own* understandings of mathematics and tools. We can begin to conjecture and intuit how they may have been significant in other cultures, and years of study and/or cultural immersion may help us do this better, but we can never fully live other cultures, especially those that have changed or are gone.

The organizers of this working group discussed this issue. Our collective take on it is this:

We can approach these objects with a sense of wonder. Jean-François writes:

> *Things we are curious about, things we try to understand, make sense of, things that surprise us, things in which we also recognize ourselves, but mostly things to be curious about! So we do not pretend to know what they are, what they do, what they are for, what they mean, we don't appropriate them. They fascinate us, and we share that fascination!*

Yasmine writes:

> *I always see it as a form of getting to know one another and getting closer to each other. ...We are not saying 'what the tools are' we are only sharing 'how the tools make sense to us' and invite others to do the same thing.*

Marcia Ascher spent over ten years studying Inca quipu. She writes, in the seminal work, *Mathematics of the Incas: Code of the quipu*:

> *Using material things as a source of knowledge does not, however, do away with distortion.* […] *There are some things in one culture for which there are no counterparts elsewhere. When this happens, understanding becomes even more difficult for someone outside the culture.* […] *And the problem increases when an attempt is made to know about a culture that is remote in time as well as in space.* (Ascher & Ascher, 1981/1997, pp. 13-14)

With that proviso, we began to explore these objects from cultures somewhat distant from our own, with a sense of curiosity and wonder, and from a starting point in our own sense of what mathematics and technologies are. We acknowledged that, although we would no doubt find some surprises and fascinating things, we cannot claim to have a full sense of these lifeways and cultures—but would take a small step towards approaching them.

One group formed to work around Marshall Islanders' Mattang, Rebbelith and Meddo stick charts for navigation by ocean swells (Figure 2). Stick charts are a significant element of the Marshallese South Pacific navigation tradition. Skilled navigators have used the maps for many generations as training devices, marking the interplay of oceanographic phenomena and land

masses. A detailed and intimate knowledge of prevailing ocean swells, and the ways that they were reflected and refracted on either side of even the smallest island, along with a thorough knowledge of navigating by the stars at night, allow traditional Marshall Islander navigators to cross the huge expanses of the south Pacific Ocean in small sailing canoes through a series of mid-course corrections, without getting lost or off-course (Davis, 2009; Hutchins, 1983).

With an activity, we tried to focus on the mathematical ideas of modeling and mapping embodied in these charts, as well as on the ideas about wave dynamics that they incorporate, not in the ocean, but in a sloped and bumpy back yard of the Education building at McGill University. A group of 10 participants went outside to feel the nuances of the contours of the ground (as a land-based analogy to ocean swells), noticing its slopes and curves by walking the terrain—sometimes uphill and downhill, feeling the slope, and sometimes across the hill to sense lines of equivalent altitude. Participants walked both forward and backward, sometimes with eyes shut to give attention to their sense of balance, the touch of feet on ground, and the way that sounds directed them on the hill. They noticed 'desire lines'—the places that would be easiest and most welcoming to walk—and saw that a gravel path sometimes veered away from the actual desire lines. They observed students traversing the hill on their way to class and saw that their feet moved off the gravel path to follow the same desire lines the group had identified. They looked for the reasons behind the bumpy terrain of this small hill and realized that it was a result of tree roots lifting the soil—and so the group was able to map the unseen tree roots beneath the surface of the ground. They speculated on the path that water would flow along and down the hill in a heavy rain, and wondered whether it would follow the same desire lines/paths of least resistance as people's feet.

Throughout this process, the group gathered sticks to map their growing knowledge of the terrain. Conventions were decided and agreed up for ways to indicate slope, pathways and flows, and these conventions were materially enacted in the stick map that was developing on the ground beside the hill. As the map did not interlock as thoroughly as the Marshall Island ocean maps, it was necessary to take reference photos and sketches of the original map of the hillside to translate and re-make it accurately indoors. Finally this group re-constituted their terrain-mapping stick chart on the classroom floor, and introduced their work and its results to the rest of the working group.

Figure 2. An example of a Marshall Island stick chart; collecting wood and mapping the terrain.

As one of the participants reflected:

> *The activity challenged us to visually record the characteristics of an outdoor space using the stick maps of the Marshall Islanders to record wave movements. This tool was striking in three significant ways: 1) We attend to what we value; 2) As much as the user adapts the function of the tool, the tools also changes the user; 3) As a tool constructed for one purpose by a culture is appropriated into a different culture it reveals new insight on the new culture through the lens of the old.*

A second group of participants simultaneously used twine to represent a mathematical problem with knots of Quipus (Figure 3).

Quipus are constructions made of knotted strings in different colours that recorded numbers and possibly words (in abbreviated, coded form) as well. They were used in the Incan kingdom that extended over most of the western portion of South America from the 12th C-16th C CE. String and rope technologies were highly developed in Andean cultures and continue to be used to construct (and re-construct annually) sturdy pedestrian suspension bridges across mountain valleys. (See, for example, the fascinating film of contemporary collective rope bridge construction in the Andes at http://nmai.si.edu/inkaroad/engineering/video/bridge-qeswachaka.html.)

In the Incan kingdom, communications networks were maintained by human runners who could traverse these rope bridges much easier than llamas or other pack animals. Records of financial and military interactions, supply movements and political decisions had to be light, weatherproof and easily carried by runners in adverse conditions. Where the Roman empire used papyrus scrolls, Incan rulers and bureaucrats used quipus—string constructions that used a notation system of knots and colours to indicate numbers for accounting, and no doubt much more that has not yet been decoded. Incan culture used a base-10 system in their numerical notation and calculation, and base-10 was also very important in the administration of the empire. For example, each leader on a particular level of the imperial administration was responsible for 10 sub-units of administration beneath them. The knots formed patterns of base-10 numbers, with special knot formations indicating the units column, and these can be read by knowledgeable interpreters to this day. It is very unfortunate that the 16th C Spanish conquest, and the resultant burning of most of the quipus existant at the time, has meant that much of the culture and knowledge of the Incan notation system has been destroyed, but there are still about 600 surviving quipus that are available for study. (See, for example, the Harvard Khipu Database Project at http://khipukamayuq.fas.harvard.edu/WhatIsAKhipu.html, and the US National Endowment for the Arts Quipu project at https://50.neh.gov/projects/quipus-inca-language-knots). Marsha and Rosbert Ascher's book (Ascher & Ascher, 1981/1997) continues to be an important source of information about the ways numbers were coded on the quipu.

Figure 3. Quipus made out of yarn treads telling various stories.

In our working group, participants working on quipu-making started by trying to learn about the tool and its functionality from Ascher & Ascher (1981/1997) and began by considering some suggested exercises for recording financial transactions or timetables on a quipu. The group decided to diverge from these exercises, though, and took on the task of recording their own family relationships via the quipu they were constructing. Each person's immediate family was coded with different coloured twine, and 'offshoot' strings were knotted into the pattern to represent different families and generations. Knots were tied into the quipu to represent numbers of siblings, parents, children, aunts, uncles and cousins in different generations of their families. The group used their own interests, their perceptions of the affordances and pleasures of using the quipu, and information that they found on different websites, to create representations of something they cared about—their own families.

A third group work on creating edge-carved wooden maps inspired by Greenland Inuit Ammassalik shoreline and island charts (Figure 4). These traditional maps, carved from driftwood in lands above the treeline, have the advantage of buoyancy (so that they would float if accidentally dropped in the water) and being both visual and tactile (so that they can be read even in the dark). To conserve wood and make these traditional maps more efficiently usable, Inuit mapmakers would carve around all the edges of an oblong piece of driftwood to represent a linear, crenulated shoreline. The map would be read 'all the way around', but interpreted as the more or less linear shore. (See figure 4, with an example of an Ammassalik carved wooden chart and its correspondences to a map in the style we are more familiar with).
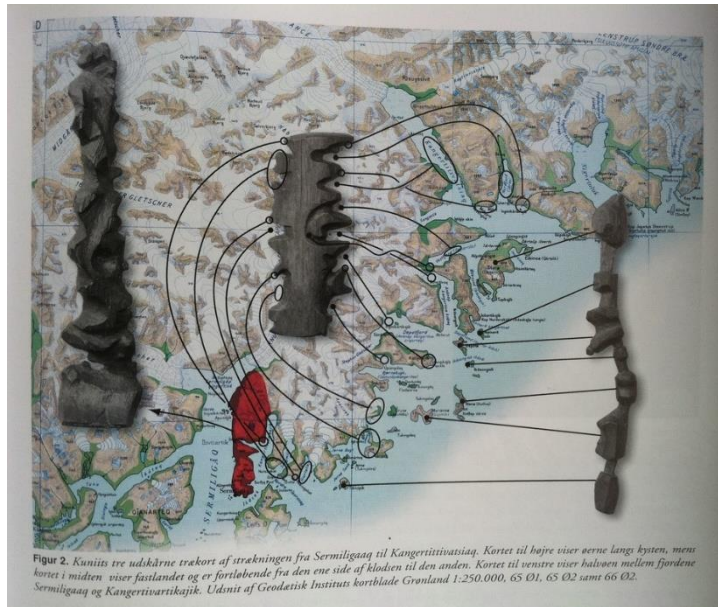
Figure 4. An Greenland Inuit Ammassalik map (from http://nuukmarluk.weebly.com/uploads/7/4/2/3/7423064/2044082_orig.jpg?667).

A trek outdoors inspired a carved wooden vertical contour map of a rocky waterfall in Mount Royal Park, located on the edge of the McGill University campus. The waterfall group made the decision to carve the crenulated vertical elevation of the rocks along the edge of the waterfall and stream, rather than a bird's eye view of the path of the stream, and so diverged from the sample Ammassalik maps by repurposing the orientation of the edge-carved map to suit their new purposes. Another group described the pathway, street crossings and stoplights between two buildings we were using on campus for the conference. When someone from outside the working group happened to come by and ask directions to the place we would have lunch, this group handed their edge-carved basswood map to the inquirer and told them that everything was included there!

Both the map-carving groups were struck by the importance of actually walking the terrain or the elevation they were representing, rather than simply looking at is. As with the stick mapping group, participants found that the multisensory information and the counting of steps and other uses of body measurement were necessary to the task of carving an accurate map—even though they also used mobile phone cameras to record the scene they were mapping as an additional check once the maps were complete. As one of the participants noted, doing this difficult work of mapping and carving "*made us appreciate the detail and delicate nature of this Inuit tradition*". The mathematical skills required to produce (and even use) such maps also very quickly became apparent.
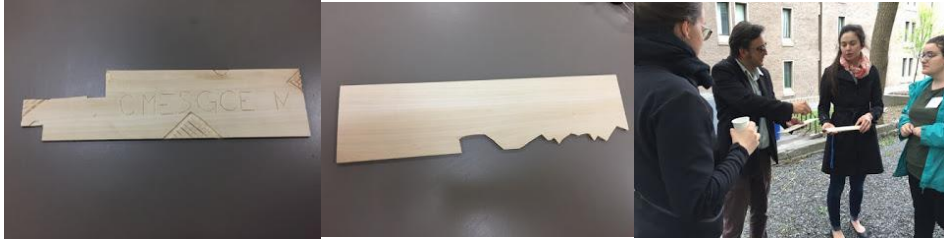
Figure 5. Carved maps inspired by Greenland Inuit Ammassalik shoreline and island charts.

DAY 1 REFLECTIONS: ATTENDING, KNOWING, EXPERIENCING AND MEANING

We attend to what we know, and know what we attend to. The ways we used tools on this first day revealed that relationship with the words and preoccupations of our cultures. We saw that we mapped and recorded what was important to us as an exploration and a recording of what is useful. Whether it was the map of a terrain, a pathway to follow, or a series of quantities or relationships, there was the tendency to select things to represent what was important to us and perhaps served a useful function. But we also found that we learned in-depth about whatever we mapped as we mapped it. The mathematics involved was an intense core of the process of experiencing and representing the ways we made sense of the world, as we changed and were changed by the tools, and by the things we did as we worked with these tools. Walking the path and scaling at the same time, we expanded our senses and our mathematical tools.

As we struggled to make meaning of the world through a map or quipu, a tension also became apparent between other cultures' tools and how we tend to viewed space, quantities, relationships, and through this tension, questions emerged. As a tool constructed in one culture is appropriated into a different culture, it necessarily reveals new insights. The tool becomes a lens through which our attention is filtered. Stickmaps reflect wave movement, and the group creating them had to shift their attention from visible modern pathways to gradients and tree roots. The group edged-carving a wooden map realized how we needed to travel at a different speed to use and create such map and realized the importance of touch in the emerging textured symbolism they started to create. Such maps really communicate an alternate perspective on the world—an alternate world.

DAY 2 THEORETICAL BACKGROUND: VYGOTSKY AND THE MAKESHIFT OF TOOLS (LED BY YASMINE ABTAHI)

In the second day of this working group we examined how we interact with tools to create artefacts that make sense to us, in terms of thinking about mathematics. A general overview of the Vygotskian idea of tools and artefacts was discussed.

- Tools, such as a hammer are "are defined as means of external activity with which humans can influence the objects" (Vygotsky, 1978, p. 55).
- Signs, such as language and the various systems used for counting,

CONTEXTE THÉORIQUE DU JOUR 2: VYGOTSKY ET LA FORTUNE DES OUTILS (DIRIGÉE PAR YASMINE ABTAHI)

Dans la deuxième journée de ce groupe de travail, nous avons examiné comment nous interagissons avec des outils pour créer des artefacts qui ont un sens pour nous en termes de réflexion sur les mathématiques. Un aperçu des idées vygotskienne d'outils et d'artefacts a été discuté.

- Les outils, tels que le marteau, sont « définis comme des moyens d'action extérieure avec lesquels les humains peuvent influencer les objets » (Vygotsky, 1978, p. 55).

are "means of internal activity that affect human behaviour" (p. 55).

To Vygotsky, the most essential difference between signs and tools is the ways in which they orient human behaviour. A tool's function is *externally* oriented: "aimed at mastering nature" (p. 55). A sign, on the other hand, is *internally* oriented: "aimed at mastering oneself" (p. 55). This looks a bit dualistic—external/internal, back and forth—but it is not meant to indicate dualism. Vygotsky further expanded his thinking on signs and tools to note, "distinctions between tools as a means of labor […] of mastering nature, and language [signs] as a means of social intercourse become dissolved in the general concept of *artefacts*" (p. 57). To Vygotsky artefacts are created by attaching signs to the tools. So, for example, children tie signs to mathematical tools by, for instance, talking about what they do, about a mathematical concept at play, or by writing mathematical symbols as they interact with a tool. Yasmine explained that for her, an artefact is not a 'thing', an object independent of the perception of the child who is using it and independent of what he/she is using it for. Rather, artefacts are created as children tie signs to their interactions with tools to work on or solve particular tasks.

In this session, we watched a three-minute video of a 5-year old child showing her mother how she used a TV remote control to count by threes. As we watched the video, we tried to focus on the properties of the remote control as a tool. How, for example, did the layout of the numbers that are organized in rows of three guide the child's counting? In the short discussion, we brought up and reflected on a few issues such as the embodied action of the child, the format of the tool, the non-mathematics design of the tool, the relation between the tool and ways of thinking about it, and more. As one participant put it,

> *I see interaction of multiple tools as a kind of system. For example, the child interacts with the remote, including the buttons, the layout of the buttons, the symbols on the buttons, her fingers,*

- Les signes, tels que le langage et les différents systèmes de comptage, sont des « moyens d'action interne qui affectent le comportement humain » (p. 55).

Pour Vygotsky, la différence la plus essentielle entre les signes et les outils est la manière dont ils orientent le comportement humain. La fonction d'un outil est orientée vers l'extérieur : « visant à maîtriser la nature » (p. 55). D'un autre côté, un signe est orienté vers l'intérieur : « destiné à se maîtriser » (p. 55). Cela semble un peu dualiste—externe/interne, va-et-vient—mais ce n'est pas ce que Vygotsky met de l'avant. Ainsi, Vygotsky élargit sa réflexion sur les signes et les outils en notant que « les distinctions entre les outils comme moyen de travail […] de maîtriser la nature, et le langage [signe] comme moyen de relations sociales se dissolvent dans le concept général d'artéfact » (p. 57). Les artefacts sont créés en attachant des signes aux outils. Ainsi, par exemple, les enfants lient des signes à des outils mathématiques en parlant par exemple de ce qu'ils font, d'un concept mathématique en jeu, ou en écrivant des symboles mathématiques lorsqu'ils interagissent avec un outil. Yasmine a expliqué que pour elle, un artefact n'est donc pas une « chose », un objet indépendant de la perception de l'enfant qui l'utilise et indépendant de ce pour quoi il l'utilise. Les artefacts sont plutôt créés lorsque les enfants lient des signes à leurs interactions avec des outils pour travailler ou résoudre des tâches particulières.

Dans cette session, nous avons regardé une vidéo de trois-minutes d'un enfant de 5 ans montrant à sa mère comment elle utilisait une télécommande de télévision pour compter par trois. En regardant la vidéo, nous avons essayé de nous concentrer sur les propriétés de la télécommande en tant qu'outil. Comment, par exemple, la disposition des nombres organisés en rangées de trois a-t-elle guidé le comptage de l'enfant ? Nous avons ainsi abordé quelques questions telles que l'action incarnée de l'enfant, le format de l'outil, la conception non mathématique de l'outil, ainsi que la

language of counting, other language, her mother's questions, the camera. Making meaning with a tool—significance—transforms the tool into an artefact. In most mathematical situations, multiple tools are used together (such as symbols, algorithms, methods, diagrams, language). But rather than multiple artefacts emerging, maybe we should talk about them as single artefact, or an artefact system. Something that assembles at a particular moment, and continually changes, reassembles, disassembles, according to what is going on. Meaning arises from the intertextual relations between the tools in the artefact system. 'Artefactness' arises from this intertextuality. I can fold a piece of paper to help add fractions, but the task already involves the written problem, using fraction symbols, so intertextuality arises across an artefact system comprised of the symbols, the folded paper, various gestures, counting, etc. Conjecture: developing broader, more complex artefact systems is a valuable aspect of mathematical thinking.

We see how the web of relations among the child with her previous knowing of mathematics and of the tool, her mother, and the particulars of the designs of the tool come together here.

For the second activity on this day we first watched another short video, within which two children in grade 7 worked with sets of tools—ones that have some sort of mathematical thoughts of the designer incorporated into them (fraction strips, Cuisenaire rods), and tools with no specifically mathematical design (ribbons, tapes and papers) to add two fractions. After a short reflection, the participants were asked to work with a variety of different tools to think about adding two fractions. The (mathematical) tools used were pencils, markers, grid papers, transparent papers, green and blue pieces of Poster board, blank A4 print paper, scotch tape, red masking tape, glue, ribbons, three different sizes of rulers (1 m, 50 cm, 15 cm), and pairs of scissors. The challenge the participants further took up was to imagine that they did not know the concept of common denominator. During and after interactions

relation entre l'outil et les façons de le penser. Comme l'a dit un participant,

L'interaction de plusieurs outils peut être vue comme une sorte de système. Par exemple, quand l'enfant interagit avec la télécommande, plusieurs aspects de l'objet interviennent : les boutons, la disposition des boutons, les symboles sur les boutons, les doigts de l'enfant, le langage de comptage, le langage ordinaire, les questions de sa mère, et même la caméra qui filme l'événement. Dans la plupart des situations mathématiques, plusieurs outils sont utilisés ensemble (symboles, algorithmes, méthodes, diagrammes, langage). Mais plutôt que de multiples artefacts émergents, nous devrions peut-être parler d'eux comme d'un artefact unique, ou d'un système d'artefacts. Quelque chose qui s'assemble à un moment donné, et change continuellement, se rassemble, se démonte, selon ce qui se passe. Le sens provient des relations intertextuelles entre les outils du système d'artefact : faire du sens avec un outil—la signification—transforme l'outil en artefact. Vu ainsi, l'artéfact est le produit de cette intertextualité. Je peux plier un morceau de papier pour aider à additionner des fractions, mais la tâche implique déjà le problème écrit, utilise des symboles, etc.; l'intertextualité survient à travers un système d'artefacts composé des symboles, du papier plié, de divers comptages, etc. Développer des systèmes d'artefacts plus larges et plus complexes est un aspect précieux de la pensée mathématique.

Nous voyons par-là comment le réseau de relations entre l'enfant avec sa connaissance précédente des mathématiques et de l'outil, sa mère, et les particularités des conceptions de l'outil se rejoignent dans cet épisode.

Pour la deuxième activité de ce jour, nous avons d'abord regardé une autre courte vidéo dans laquelle deux enfants de 7e année travaillaient avec des outils pour additionner des fractions. Ces outils étaient de différents types : certains 'contenaient' une sorte de pensée mathématique de la part du concepteur (bandes de fractions, bâtonnets de Cuisenaire), d'autres étaient des outils sans conception spécifiquement

with the tools, we discussed how the interactions with the tools were mediated by the feedback from the different artefacts that they created throughout their interactions with the tools.

We also noted that in interactions with the tools and in all problem solving stages, both attaching mathematical meaning(s) to the tools and the consequential process of creating an artefact were a gradual and complex process, which were closely related to: (a) The ways in which the participants perceived the mathematical affordances of the tools—through the feedback provided by the tools—to create and to use the artefacts; (b) their previous mathematical knowing of fractions in general and of the addition of fractions in particular; and (c) the task of adding two fractions.

There were many thoughtful discussions that resulted on the use of tools to develop mathematics from tools that have been designed for the purpose of teaching such as Cuisenaire rods, geometric shapes and graph paper, to less obvious but still valuable tools such as a television remote control, our fingers, and conversation.

mathématique (rubans, bandes et papiers). Après une courte réflexion, les participants ont été invités à travailler avec ces outils pour réfléchir à l'addition de deux fractions. Les outils (mathématiques) offerts étaient des crayons, des marqueurs, des papiers quadrillés, des papiers transparents, des affiches vertes et bleues, du papier blanc A4, du ruban adhésif transparent, du ruban adhésif rouge, de la colle, des rubans, trois tailles de règles (1m, 50 cm, 15 cm), et des paires de ciseaux. Le défi particulier proposé aux participants était de faire l'addition de deux fractions avec ce matériel en faisant comme s'ils ne connaissaient pas le concept de dénominateur commun. Pendant et après le travail, nous avons discuté de la manière dont les interactions avec les outils ont été influencées par les réactions des différents artefacts qu'ils ont créés tout au long du processus.

Nous avons également noté que dans les interactions avec les outils et dans toutes les étapes de résolution de problèmes, attacher une ou deux significations mathématiques aux outils et au processus consécutif de création d'un artefact est complexe et se fait graduellement. Il est intéressant de souligné comment ceci est étroitement lié à : (a) les façons dont les participants ont perçu les affordances mathématiques des outils—à travers le retour d'information fournie par les outils—pour créer et utiliser les artefacts; (b) leur connaissance mathématique préalable des fractions en général et de l'addition de fractions en particulier; et (c) la tâche d'additionner deux fractions. De vives discussions bien réfléchies ont suivi l'activité, concernant l'utilisation d'outils pour développer les mathématiques, que ce soit des outils conçus pour l'enseignement (tels que les bâtonnets Cuisenaire, les formes géométriques et le papier millimétré), ou des outils moins évidents, mais néanmoins précieux comme une télécommande, nos doigts et la conversation.

DAY 3 THEORETICAL BACKGROUND: RE-THINKING (MATHEMATICAL) TOOLS (LED BY JEAN-FRANÇOIS MAHEUX)

In this third session, we looked into mathematical tools from a historical perspective, keeping in mind perhaps a reflection on what this tells us about the nature of mathematical activity. We are very familiar with those Babylonian tablets collating important results, as well as heuristics in the form of problems and their solutions. These clay tablets can probably count as one of the oldest known mathematical 'tools' that might also have been designed to teach or learn mathematics.

Dans les sections précédentes, nous avons pu voir et expérimenter comment la matérialité des outils, leur matière même, peut être inséparable de leur fonction, et du type d'expérience (mathématiques) qu'ils offrent. Dans le cas des tablettes, on peut se demander comment la matière (argile, stylet de bois) a pu influencer le travail mathématique de l'époque et le choix, par exemple, d'un système de notation positionnelle utilisant seulement deux symboles.

Le terme « outil » est en fait lié à une foule de mots intéressants pour penser, et repenser ce que nous entendons par « outils mathématiques » … et ce que nous attendons d'eux, peut-être ! Dans la même veine, nous nous sommes arrêtés un moment pour penser à nos outils en relation avec ceux d'autres métiers, ou pratiques. Qu'ont donc de particulier ces outils que l'on dit « mathématiques » ?
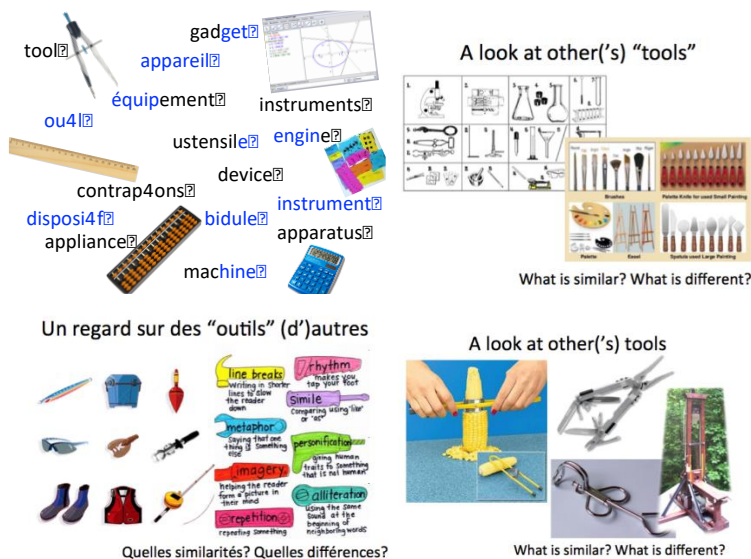
Figure 5. Re-thinking (mathematical) tools.

Certainly, an important part of what makes a tool a mathematical one is its inclusion, its participation into some mathematical practices or activity (which, of course, opens the question of what makes them mathematical, but that is for another time!). Many tools present mathematical features, and many tools without 'specific' mathematical features can certainly enter mathematical activity, and then become mathematical tools for that reason (a pen, a rope, words, gestures, etc.). This becoming part can also simply be this moment when an object becomes the object of mathematical attention. Is this not an interesting way to see what we did in the previous sections?

Suite à nos discussions à propos de ce qui fait d'un outil un outil mathématique, nous avons proposé aux participants d'examiner en sous-groupes quelques propos encore une fois un peu dérangeants à propos d'outils. Lors du partage qui s'en est suivi, nous nous sommes rendu compte que l'exercice avait encore une fois permis de spécifier, d'illustrer, de nuancer nos rapports aux outils, et à leur rôle dans l'activité mathématiques. Voici quelques-unes de ces idées dans la figure 6.

L'homme se découvre quand il se mesure avec l'obstacle. Mais, pour l'atteindre, il lui faut un outil. Il lui faut un rabot, ou une charrue. A. de St-Exupéry	*Humans find themselves when they measure against obstacles. But to engage the earth, we need tools. We need a spade or a plough.* A. de St-Exupéry
La fabrication et l'utilisation d'outils, d'instruments et de machines font partie de ce qu'est la technique, mais la technique n'est pas que des moyens (à maitriser). Elle un mode de dévoilement : une provocation par laquelle la nature est mise en demeure de livrer. M. Heidegger	*Making and using tools, instruments and machines is part of technology, but technology is no simply means (to master). It a mode of revealing: a challenging which order nature to deliver.* M. Heidegger
Human are not something natural that then develop technology: both co-emerge. And in fact technology comes from behind. It is within the (back of the) human, and it arrives by chance. D.Wills	*L'humain n'est pas quelque chose de naturel qui développe ensuite la technologie : ils co-émergent. Et en fait, la technologie provient de l'arrière. Elle est dans (le dos) de l'humain, et elle arrive par hasard.* D. Wills
We carry out actions to achieve a goal, assisted by tools (instruments, symbols, signs, language) that mediate them. The fate of operations (actions that have become unconscious) is to become functions of a machine. A. Leontiev	*On réalise des actions pour atteindre un objectif, aidé d'outils (instruments, symboles, signes, langage) qui servent de médiation. Le sort des opérations (actions devenues inconscientes) est de devenir des fonctions d'une machine.* A. Leontiev

Figure 6. Prompts to new ways to specify, illustrate, nuance our relationship with tools, and their role in mathematical activity.

WHAT ABOUT 'CONCEPTUAL' TOOLS?

The second part of this last session was dedicated to somehow different kind of tools: conceptual mathematical tools. More precisely, we explored a few methods to figure the square root of 200, as it was done in the past.

Quand on pense à des outils mathématiques, on a souvent en tête quelque chose de plus abstrait que les règles, compas, et autres bidules du genre : on pense aussi aux méthodes par exemple ! Dans la seconde partie de cette session, nous nous sommes intéressés à quelques manières (tirées de l'histoire) d'obtenir la racine carrée de 200.

Find the side of a square whose surface is 200 hops. Solve this problem without using any algorithm (or any instrument) can be quite challenging… but not impossible. Although you do have to immediately take a stand in terms of what you will count as an instrument or not. Can you draw using a pencil and some paper? Can you use counters of any sort?

A Babylonian method

Les Sumériens avaient développé, semble-t-il, différentes méthodes leur permettant de « résoudre » des équations du second degré. Résoudre, ou en tout cas trouver des résultats

satisfaisants pour eux. Leur technique la plus simple consiste à faire une moyenne. C'est la même méthode que présente Héron d'Alexandrie quelque 1700 ans plus tard. Il s'agit d'appliquer la formule suivante, dans laquelle « e » est une approximation de la valeur recherchée :

$$\sqrt{n} \;=\; \frac{\frac{n}{e} + e}{2}$$

We asked, "allowing yourself to use our symbols as well as your favorite ways to do additions and divisions solve the problem of 200 hops with this formula, where *e* is an approximation of the root you are looking for." Amazingly enough, this simple formula gives very good results even with a pretty rough approximation of the root. With a good one (e.g., reusing results given with the formula), we can be more precise than ever needed!

La formule en tant qu'outil est un objet intéressant, par exemple du point de vue de sa réutilisation. Une des propriétés communes des outils est justement cette réutilisabilité (*reusability*). Le marteau peut servir à la construction de plusieurs projets, mais il sert aussi plusieurs fois dans le travail d'enfoncer un clou. Les plus perspicaces auront déjà reconnu dans la formule sumérienne l'essentiel de la méthode de Newton : nos outils mathématiques se raffinent, mais gagnent aussi en généralité (avec Newton, résoudre $x^2 = y$ est simplement un cas particulier). Ce gain s'accompagne cependant d'une difficulté à les manipuler. Le cheval tirant une charrette s'arrête de lui-même devant un obstacle, une voiture… pas encore ! Quelque chose d'intéressant se dégage de l'utilisation et de l'observation d'outils « simples », dont on peut bien suivre bien la mécanique (la racine de *n* est forcément entre *e* et *n/e* !). Comparons avec la méthode suivante, par exemple.

In India

An interesting elaboration of the Babylonian formula was found in India, in a Bakhshali manuscript dating from circa -400[2]. Taking into consideration the difference between what our first approximation gives and the number we want to square, the method (in modern notation) looks like this:

Let $n = e^2 + b$, where *e* is the number we want to square (e.g., 28), *e* is our approximation (e.g., 5) and *b* is, of course, the difference between *n* and e^2 (28-5=3).
Then *e′* is given by

$$e + \frac{b}{2e} - \frac{\left(\frac{b}{2e}\right)^2}{2e + b/e}$$

The tool here is a more complex and opaque; more prone to errors, more mysterious. And more efficient too, provided that one would be using it with the same as what we saw with the Sumerian approach. Unpacking it, figuring out how the tool works, is still possible. But clearly a much more difficult exercise than observing how the root of a number is necessarily between some number and its multiplicative inverse. And as much as we can find beauty in simple tools, more intricate ones (baroque?) also have their charm, if or when we are willing to see it. Tools are beautiful objects, and beauty has many faces (e.g., Sartwell, 2013). There is also pleasure,

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[2] I recently discovered that this manuscript might actually be even a few centuries older that previously thought.

different kinds of pleasures, in using simple or complex tools. There are different kinds of struggles with each tool. And probably also, especially if we think about these two examples, there is the possibility of an enriching sense of doing things in the 'old fashioned way', the possibility for wonder in a sense similar to what we described when discussing various cultures' tools. These more or less ancient mathematical ways to go about the (mathematical) world invite us to reconsider, through tools, who we are, what we do, where we come from, and where we go.

<u>Pas à pas</u>

Les outils n'ont pas tous la même précision, donc ! On peut imaginer que cette précision dépend souvent de la fin qu'ils servent. L'intérêt mathématique d'un calcul précis, exact, de la racine d'un nombre a aussi donné naissance à son outil : un algorithme dont la trace la plus ancienne se trouve également dans un vieux manuscrit indien (daté d'il y a environ 500 ans, il est signé de la main d'un certain Aryabhata). Cet outil, complexe, pénible à utiliser, et opaque à souhait a néanmoins traversé l'histoire. On le voit dans les travaux des mathématiciens arabes puis européens, et il fut même assez longtemps enseigné dans les écoles du 20e siècle. Il ressemble un peu à l'algorithme de division que nous connaissons (encore) bien (mais dont plusieurs prévoient la 'disparition' prochaine !). La page web créée pour le groupe de travail en donne une illustration assez facile à suivre (https://cmesgmathtools.blogspot.ca/2017/05/session-3-fr.html).

I can *illustrate* it for you in the fairly simple case of the calculation of $\sqrt{150}$. So, let's first observe that 150 is also …000150.0000… What we do then is to split the number in two-digit packets, starting from the decimal point: 00 01 50 . 00 00 00 … We are now interested in the number packet different from 00 at the highest position: here it is 01.

- We are looking for the root r of the closest perfect square c such as $c \leq 01$. Here, it is simply $r = 1$ since $1^2 = 1 \leq 01$. This r is the first digit of $\sqrt{150}$.
- For the next step, we now compute the difference d between r^2 and our packet: $01 - 1^2 = 0$. We can then move on to the next packet.
- Note that with another number (e.g., 2842) we could obviously have had a remainder different from 0 (e.g., 3 because we would have 5 as the first digit: $5 \times 5 = 25 < 28$ and $28 - 25 = 3$).
- We put the next packet onto the right of our difference d: 050. We multiply by 20 the number formed by the digits identified so far (regardless of the decimal point if there is one, see below): $1 \times 20 = 20$. And then we look (again) for the digit r giving us the closed number c such that $(20 + r) \times r \leq 050$. Here, we take $r = 2$ since $22 \times 2 = 44 < 50$ (and $23 \times 3 = 69$ is too large). This r is, as previously noted, the next digit of $\sqrt{150}$.
- For the next step, we compute the difference d: $50 - 44 = 6$, etc. Up to here we have $\sqrt{150} = 12$.

There is (at least) one 'shortcut' we can use to find r relatively easily even with large numbers. For example, if we need to solve: $(24480 + r) \times r \leq 190400)$, we can notice that r is, *of course*, near the floor value of the division of our 'augmented' d by the product of our digits by 20. Here we have $r \approx [190400 \div 24480] = [7.77...] = 7$. So if doing division is not a problem, we can actually save ourselves some time/work. We can even get a pretty good guess with simple approximation, e.g., $19 \div 2.4 = 7.9166…$ Here, 7 is in fact the next digit.

Learning to use these kinds of tools really emphasizes what we often call the procedural aspect of doing mathematics. But here again we can see that being procedural is not a given. Remembering and managing apparently meaningless, intricate rules (the illustrated case does

not show some of the 'difficulties' we can come against) can be hard mathematical work. There can be a real sort of craftsmanship there, and we do know of individuals who, in the past, were celebrated for such skills. Historically speaking, this all changed when de Prony created the first 'mathematics factory' (a few years after the French revolution), training unemployed hairdressers to fill sheets of additions and subtractions used to compute logarithmic and trigonometric tables. The creation was at the origin of Babbage's invention of his difference machine, one of the first computers.

<u>By the rules</u>

Le côté lent et pénible de cet algorithme explique bien qu'on lui préfère souvent des méthodes approximatives. En termes d'efficacité, on a en fait tout avantage à retourner vers des outils au sens où on l'entend généralement. La règle à calculer est un de ces ingénieux instruments ! Selon les règles offertes, il est possible de connaitre des valeurs trigonométriques par exemple, mais aussi des racines carrées, cubiques, etc. Mais contrairement aux calculatrices modernes, ceci demande un peu plus de travail mathématique. Et puis évidemment, il existe plusieurs variations de règles. Comment ça marche ?

Disons simplement que l'idée générale est d'avoir une règle graduée de manière à ce qu'on puisse avoir en vis-à-vis les nombres 1,2,3,4... et leur racine 1, 1.414, 1,732, 2. On utilise ensuite la règle en se basant sur le fait que $\sqrt{an} = \sqrt{a}\sqrt{n}$ en prenant $a = 10$ et $a = 100$ (ou une puissance de 10 ou 100). De cette manière, on peut toujours « ramener » un nombre à une valeur entre 1 et 10. Avec 200, on aura $a = 100$ et $n = 2$. Avec 14.3 on aurait $a = 10$ et $n = 1.43$. Pour des nombres plus grands (ou plus petits, car on peut aussi diviser par 10 ou 100), on est encore toujours dans un cas soit on a un facteur de 100, soit on a un facteur de 10. Et comme $\sqrt{10}$ et $\sqrt{100}$ ont des valeurs différentes, la règle présente deux graduations.

La règle nous donne en général que la valeur de $\sqrt{n}$ est $0 > n > 10$. Si le nombre qui nous intéresse est plus grand ou plus petit, on compte les chiffres pour savoir si on va lire sur la règle de « impair » (pour 20, 2000, etc.) ou « paire » (pour 2, 200, etc.). C'est l'utilisateur qui doit ensuite « placer la décimale » en fonction de l'ordre de grandeur attendue. Ainsi, pour $\sqrt{200}$ on va chercher sur la règle « paire » la valeur de $\sqrt{2}$, et on multiplie par 10. On peut voir sur la figure suivante, la racine à extraire est choisie sur la graduation A, et la racine obtenue est lue sur la graduation C. Ainsi, à gauche on reconnait $\sqrt{2} \approx 1{,}41(4)$ et à droite $\sqrt{20} \approx 4{,}47$.
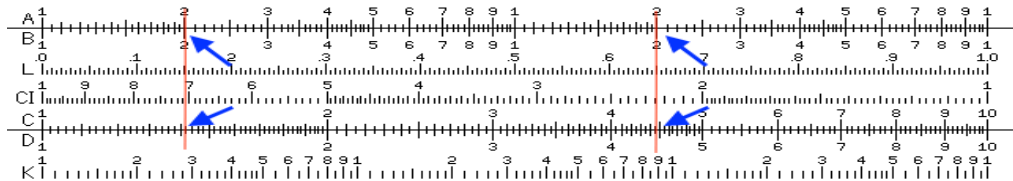
Figure 7. On a slide rule, we read 2 values for $\sqrt{2}$.

Similarly we see on the left $\sqrt{4.5} \approx 2.12$ and on the right $\sqrt{45} \approx 6.70$. It is possible to increase the level of precision, by 'amplifying' the rule by using the central part of the ruler in the same way Vernier calipers work.

Participants explored with slide rules made of paper. The ruler itself is a fascinating object for many, and manipulating them showed how dexterity can also be an issue in 'higher' mathematics! But besides that, the tool offers an interesting mixture of needing to understand what is going on (to choose the appropriated scale), and the mindlessness of instrumented activity in the sense that the tool is taking care of the work and provides the answers. This is a

quite refined version of the mind-enlarging observation that (mathematical tools) need us a much as we need them. It is through our actions that they become part of mathematical activity. Once started, they will support us just as much as we support them in this dance we call doing mathematics.

Si on accepte que toutes ces instances soient malgré tout bel et bien…activité mathématique, comment se comparent-t-elles? Que signifie, très concrètement, faire des maths dans chaque cas?

CONCLUSION

Mathematical tools are things we are curious about, things we try to understand and make sense of. They are things that surprise us, things in which we also recognize ourselves! We do not pretend to know what they are, what they do, what they are for, what they mean. We do not appropriate them. They fascinate us, and this working group was, first of all, about sharing that fascination. This is one of the ways in which mathematical tools are also forms of getting to know one another and getting closer to each other. Sharing how the tools make sense to us, and inviting others to do the same, these tools gave us occasion to do mathematics together. Each offered us mathematical experiences of a kind, and each carried traces of the mathematical work of others. They all presented themselves as standing proofs of the often-forgotten existence of the mathematicians who created them, as Pimm (2007) puts it.

Looking back, we realize how our discussions around tools tended to personify them, as if they even had agency, and perhaps intentions different from our own. Tools shape us? Tools need us? Tools learn? Can they go wrong? Make mistakes? Refuse to work? Mostly metaphorical perhaps, these images come with a transformation in our languaging, our thinking. Do they layer the ground for the upcoming time where the biomechanical fusion of humans and instruments will, perhaps, change everything?

Another idea that emerged from our three days together examining mathematical tools concerns the materiality and aesthetics of physical tools and the ways they are constructed. Most of the typical manipulatives used in elementary mathematics classrooms are made industrially (by others) from coloured plastics and cost money. This reveals something of our society's ways of integrating and processing experience through our interactions with the material world.

It is a very different learning experience to work with purchased industrial plastic objects in a classroom or to go outdoors and work with natural materials that we forage or grow in our own gardens. We took pleasure in working with sticks, wood, hand-made twine and other materials we could collect and make for ourselves. But we also felt pleasure and surprise as we experimented with Cuisenaire rods and fraction circles in unusual ways, and when we worked with the Babylonian square root algorithm and the slide rule. Working in new and surprising ways with particular tools may be also a way to re-discover our own (mathematical) roots.

CONCLUDING REFLECTIONS

To recap some insights from our first day's questions about tools, cultures and appropriation,

JF writes:

> *Things we are curious about, things we try to understand, make sense of, things that surprise us, things in which we also recognize ourselves, but mostly things to be curious about! So we do not pretend to know what they are, what they do, what they are for, what they mean, we don't appropriate them. They fascinate us, and we share that fascination!*

Yasmine writes:

> *I always see it as a from of getting to know one another and getting closer to each other....We are not saying "what the tools are" we are only sharing " how the tools make sense to us" and invite others to do the same thing.*

Susan reflects at the end of Day 3:

> *I am struck by the differences between industrial/ Modernist and post-modern ideas about tools that we saw in the quotations about tools that Jean-François brought today, and throughout the working group sessions.*

Modernist writers use the language of master and servant: tools *serve* us as we *master*, control and use the Earth and natural things (for example, by industrial and mathematical means). Goals are linear and clearly-defined. There are no unplanned side effects, or at least, they are considered to be minimal and not worth noting. The image is muscular, efficient, deliberate and a one-way action of humans taking and making what they want.

The post-modern theorists, including McLuhan and David Wills, use the language of mutual shaping and co-emergence: humans *are shaped* by tools as much as we shape them, and things arise *unnoticed* or *by chance* as much as by design. Whatever goals we might think we have are a kind of distraction from the really big effects of interactions between people and technologies. The unintended side effects are always much greater than the intended effects. The image is of a complex co-emergent ecology where no one is in control, intentions and human 'will' are constantly reshaped by interactions, and where *mastery* is impossible and, in fact, undesirable.

Even the post modern is very human-centric, as Richard Barwell pointed out in one of our discussions. Perhaps a next move might be to recognize the being-ness of all beings in the more-than-human, greater-than-human world. These beings could be as diverse as trees, rocks, fellow animals, cell phones, hammers and language…

So I am struck by some of the ideas Yasmine, JF and others introduced where tools are personified and have agency (and perhaps intentions different from our own!):

"Tools need us as much as we need them"—JF

"Do tools learn?"—Richard Barwell

"Tools accumulate knowing."—Yasmine

"I am interested in what happens when tools go wrong or don't work. Does this open up new possibilities?"—JF

"What if we had a calculator that sometimes (randomly) gave wrong answers?"—Nat Banting

"What is interesting is when a tool is used for unintended purposes—as in the video where Laila used the TV remote control for counting by 3s."—Yasmine

"We shape our tools and then our tools immediately shape us"—Marshall McLuhan

I am also interested in the materiality and aesthetics of physical tools, and the ways they are constructed. So many of the typical manipulatives used in elementary math classrooms are made industrially (by others) from coloured plastics and cost money. I am interested in how this differs from tools we make ourselves, outdoors, from more natural materials that we find, forage, grow in our gardens, et cetera.

What effect does this have on us as physical, earthly creatures? What effect does this have on our senses and sensory perceptions and awareness/noticing? How might it affect in-depth understanding? Certainly people who walked a terrain, grew to love a certain place, became friends with the rocks of a waterfall, closed their eyes and walked backwards to feel the balance and texture of the land, who became intimate with ropes and knots, who saw their family appear symbolized in their quipus…felt emotions, pleasure, depth of knowledge, surprise.

We also felt pleasure and surprise as we experimented with Cuisenaire rods, with Babylonian square root algorithms, with slide rules, with fraction circles and with drawings. Perhaps it is that freshness of knowing (and suddenly not-knowing!) that we hope to reach by taking up the constraints as well as affordances of particular tools?

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COMPRÉHENSION APPROFONDIE DES MATHÉMATIQUES SCOLAIRES

DEEP UNDERSTANDING OF SCHOOL MATHEMATICS

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INTRODUCTION

The focus of this working group was to explore what the group's notions of 'deep understanding' was and to see if we could articulate what that looked like and felt like through

engaging in mathematical tasks. Ensuring that students develop a 'deep understanding' is a phrase often used to describe the goals of both K-12 and post-secondary mathematics education. We framed our session around the following questions:

- What does having a deep understanding of school mathematics mean?

- How is it different from deep understanding of mathematics?

- How can we teach for deep understanding of school mathematics?

- How do we work with pre-service and in-service teachers to teach for deep understanding?

- Qu'est-ce que veut dire « avoir une compréhension approfondie des mathématiques scolaires » ?

- En quoi est-ce différent d'avoir une compréhension approfondie des mathématiques ?

- Comment peut-on enseigner pour provoquer une compréhension approfondie ?

- Comment travaille-t-on avec les futurs enseignants et les enseignants pour enseigner cette « compréhension approfondie » ?

These are open questions within the field of mathematics education that we did not expect to arrive at a complete answer to but to explore our understandings of. Throughout the three days, we posed mathematical problems to participants that we hoped would stimulate our thinking as researchers, teachers, and teacher educators exploring pedagogical, didactic, curricular, and evaluative implications for practice. We were not disappointed!

Il semble que les questions proposées dans la description du groupe ont intéressé plusieurs participants qui avaient des expériences/expertises variées. En effet, le groupe était formé d'enseignants du secondaire, du collégial et de l'université, d'enseignants en formation des maîtres, de chercheurs et d'étudiants gradués. Ils ont également mentionné diverses raisons qui les ont amenés à participer au groupe. Parmi les raisons mentionnées, la vision de ce qu'est la compréhension, en général, et ce que sont les mathématiques scolaires ont été particulièrement soulevés.

We drew on Liljedahl's (2016) work on Building Thinking Classrooms to promote engagement with the tasks we were posing. We had participants working on vertical surfaces in groups of three with the problems we posed. We were unable to provide non-permanent surfaces for all participants but had two groups for each problem working on the larger white-boards. Our intent was to provide common mathematical experiences for participants in the working group that we could draw on to frame 'deep understanding'.

JOUR 1 : QU'EST-CE QUE LA COMPRÉHENSION APPROFONDIE?

Pour la première rencontre du groupe, il nous est apparu important d'aborder en premier lieu le thème de la compréhension approfondie. Nous pensions qu'avant d'aller de l'avant sur les thèmes de l'enseignement pour une compréhension approfondie, il était nécessaire de connaître les différentes visions de ce qui est entendu par « compréhension approfondie ».

The problem that we posed for the participants to engage with was the following:

> An escaped prisoner finds himself in the middle of a square swimming pool. The guard that is chasing him is at one of the corners of the pool. The guard can run faster than the prisoner can swim. The prisoner can run faster than the guard can run. The guard

does not swim. Which direction should the prisoner swim in order to maximize the likelihood that he will get away?

Un prisonnier évadé se retrouve au centre d'une piscine carré. Le gardien qui est à sa poursuite se trouve à un coin de la piscine. Le garde peut courir plus vite que le prisonnier peut nager. Le prisonnier peut courir plus vite que le garde peut courir. Le garde ne sait pas nager. Dans quelle direction le prisonnier devrait-il nager pour maximiser ses chances de pouvoir s'évader?

Participants worked in groups to create solutions to the problem. Four groups' solutions are shown in Figure 1 below.
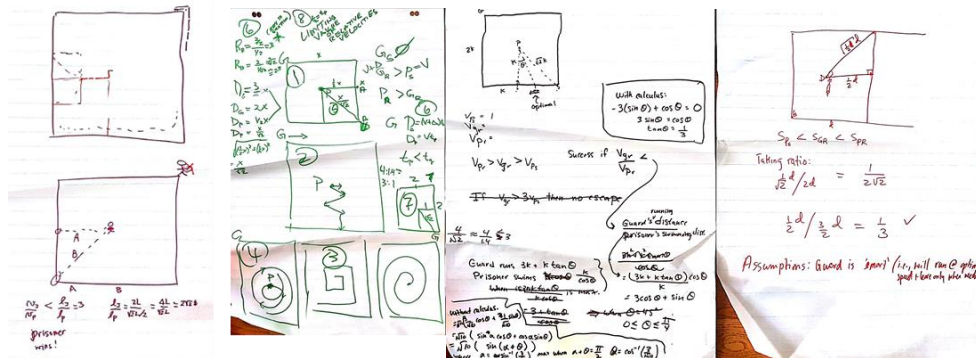
Figure 1. Solutions to the prisoner problem.

Participants then engaged in a Gallery Walk to view their other groups' solutions. We then came back as a whole group and asked the question

- Qu'est-ce qu'une compréhension approfondie des mathématiques scolaires (de l'école) ?
- What is deep understanding of school mathematics?

Plusieurs idées intéressantes sont ressorties de cette discussion. D'abord, quelques concepts mathématiques qui pourraient être utiles pour résoudre ce problème ont été identifiés : par exemple, les concepts de taux de variation, de vitesse, de dérivée. Cela nous a amenés à parler plus généralement « d'outils mathématiques » à posséder pour pouvoir résoudre des problèmes. Également, l'importance de l'habileté à choisir les « bons » outils pour résoudre des problèmes a pu être identifiée comme un aspect d'une compréhension approfondie. Ceci a également permis de soulever l'aspect dynamique de la compréhension. Il s'agit de l'idée que la compréhension serait en constant changement : de nouveaux concepts mathématiques, de nouvelles façons de représenter des situations et des concepts, de nouveaux problèmes à résoudre qui nécessitent une nouvelle stratégie de résolution, etc.

De plus, le groupe a également souligné l'importance de la schématisation ou de pouvoir se faire une image de la situation. Particulièrement dans ce problème, le schéma était un point de départ presque inévitable pour aller plus loin dans la situation. La discussion autour de la schématisation nous a amenés à identifier la créativité en mathématique comme un autre aspect possible de la compréhension approfondie. Par exemple, la créativité dans la schématisation de la deuxième équipe dans la figure 1 les a amenés à voir le problème différemment. Aussi, l'ouverture du problème et le questionnement par rapport au caractère complet de la réponse (par exemple : est-ce que ma démarche est suffisante ? Puis-je aller plus loin ?) ont également été identifiés comme une certaine créativité qui pourrait être associée à une compréhension approfondie.

Enfin, nous avons pris la liberté de formuler d'autres questions qui nous ont semblé ressortir de la discussion :

- Is there a turning point where we can say that we will need deep understanding in a problem? (The 'is that enough' part of the problem was a suggestion for being a manifestation of a deep understanding.)
- Are there characteristics of a task that might provoke or develop a deep understanding?
- Is that deep understanding: selecting which tools we need, produce a schema or a picture that represent the situation?
- Are the mathematical concepts in themselves a part of a deep understanding? In which way?

En deuxième partie de journée, nous avons proposé le problème suivant : $6 \div 2\,(4-1)$. Nous avons demandé de résoudre ce problème en ayant en tête la question suivante : est-ce qu'une compréhension approfondie est nécessaire pour résoudre ce problème ? Si oui, comment se manifeste-t-elle ? What is deep understanding in this problem, if present? Les participants ont encore une fois été invités à travailler en équipe de trois sur de grandes affiches collées aux murs.

La discussion qui a suivi a été très animée. En effet, tous n'étaient pas d'accord sur la place de la compréhension approfondie dans ce problème. Certains semblaient assez catégoriques comme quoi ce problème n'avait « rien à voir » avec une compréhension approfondie. L'utilisation de règles ou de conventions a été un enjeu. Ces conventions font-elles partie d'une certaine compréhension approfondie ? Sont-elles nécessaires ? Le fait que ces conventions soient parfois liées à un contexte culturel a mis en lumière l'aspect « humain » des mathématiques et a soulevé l'importance de la communication en mathématiques. Ainsi, on peut se questionner sur la place de la communication en/avec les mathématiques dans ou pour une compréhension approfondie.

Plusieurs ont aimé le fait qu'il n'y ait pas de consignes liées au problème. Ils y ont vu une occasion de se demander d'où venait cette opération, de rechercher des exemples de problèmes écrits qui pouvaient être représentés par cette opération. Certains participants ont d'ailleurs remarqué que cette tâche n'était pas si simple. Le travail de deux équipes est présenté ci-après dans la Figure 2.
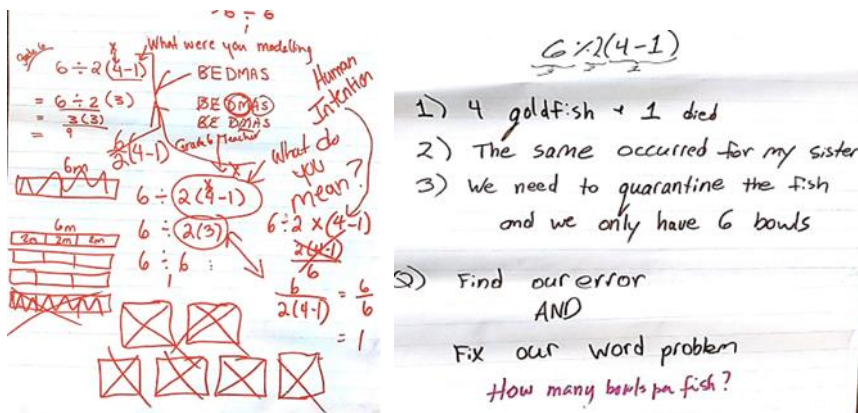

Figure 2. Sample participant responses to the order of operations question.

Again, after groups worked on this problem, they participated in a Gallery walk where they viewed other groups' solutions. This activity prompted much discussion around mathematical conventions and communication. Questions arose regarding the 'deep mathematics' that students might need to know in order to have a discussion regarding the mathematical conventions that are at play in this question. Additionally, there was agreement that knowing the conventions is not enough to qualify for 'deep understanding'.

At the end of Day 1, for homework, we asked participants to respond to the following prompts in a Google Doc in order to get insight into what they were thinking regarding the following topics:

- State 5 words that you think of when you hear 'deep understanding'
- State 5 words that you think of when you hear 'school mathematics'

DAY 2: WHAT IS SCHOOL MATH? TEACHING FOR DEEP UNDERSTANDING OF SCHOOL MATH?

At the beginning of Day 2, we took all of the contributions that participants made to the Google Doc and illustrated the responses to each question visually using a word cloud generator. Though we gave instructions as to how participants were to respond, some wrote fewer than 5 words and some chose to write paragraphs. The word clouds that were presented are shown in Figure 3.

School mathematics   Deep understanding
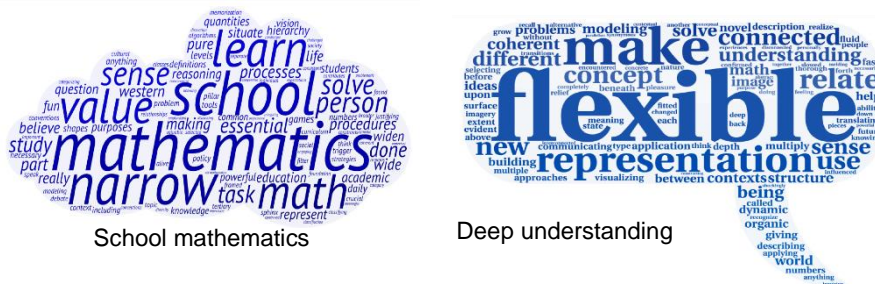
Figure 3: Word clouds of group contributions.

These visual representations caused much discussion among the participants. Many noted that the words that were being used to describe 'school mathematics' were harsh and cold, while the words used to describe deep understanding were more collaborative in nature. This led to a question posed by one of the participants regarding school mathematics and defining what it is specifically. This prompted a series of generalities being produced about school mathematics but nothing that anyone could agree on specifically.

The next phase of Day 2 was to address teaching for deep understanding. We used resources from the National Council of Teachers of Mathematics (NCTM) professional development materials in order to highlight specific teaching practices that either support or inhibit the development of deep understanding. We posed the Candy Jars Problem shown in Figure 4 (Retrieved from http://www.nctm.org/Conferences-and-Professional-Development/Principles-to-Actions-Toolkit/The-Case-of-Mr_-Donnelly-and-the-Candy-Jar-Task/) with the intent of then using the teaching scenarios based on the same problem.

Figure 4. The candy jars problem.

Groups of 3 worked together to come up with solutions to the problem. Group solutions to the problem are in Figure 5 below.

Figure 5. Group solutions to the candy jar task.

After engaging in the task, participants were placed in six random groups of five, with three groups given one teaching scenario, and three groups given a second teaching scenario. The teaching scenarios illustrated the ways in which two different teachers enacted this task in their classes. (Videos found at http://www.nctm.org/Conferences-and-Professional-Development/Principles-to-Actions-Toolkit/The-Case-of-Mr_-Donnelly-and-the-Candy-Jar-Task/). Groups were asked to analyze the scenarios and were asked to discuss the following:

- What did you notice about elements of the teaching that led to student deep understanding? Or that got in the way of developing deep understanding in the students?

Once groups had a chance to explore their scenario, participants were then placed in five random groups of six with three participants that had been experts on each of the scenarios. They were to present their scenarios to each other and to discuss what they noticed about

teaching for deep understanding. We then did a large group debrief on what is needed to teach for deep understanding and the ideas were captured on the whiteboard in Figure 6 below.
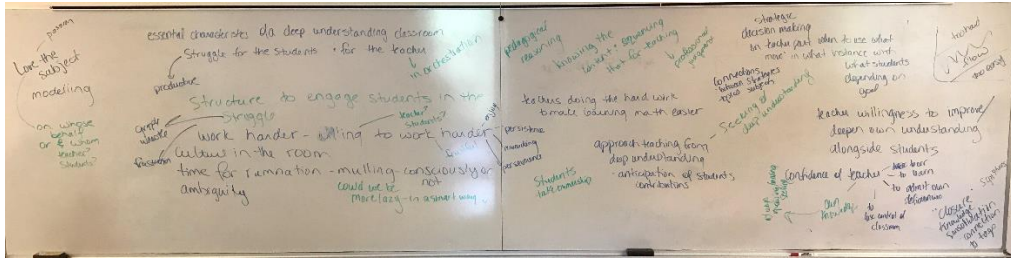
Figure 6. Debrief of teaching for deep understanding.

Plusieurs aspects d'une compréhension approfondie, qui semblaient être parlant pour les participants, ont été identifiés. Par exemple, la confiance en eux et en leurs élèves, leur humilité, leur habileté à anticiper ce que les élèves feront, leur amour des mathématiques et de l'enseignement ont été reconnus comme des qualités aidantes pour enseigner pour une compréhension approfondie. Aussi, la propension des enseignants à pouvoir délaisser ou modifier en temps et lieu les plans qu'ils avaient prévus a également été discutée. Certains ont parlé d'être à l'aise avec une certaine « perte de contrôle ». Or, d'autres ont précisé que de rester confiant et à l'aise dans une situation qui dévie d'un plan initial, ou d'accepter de suivre les élèves vers des avenues imprévues, était tout sauf une perte de contrôle. Au contraire, cela demanderait un très grand contrôle pour rester ouvert et confortable dans ce type de situations.

As a culminating activity for the day, we had participants individually reflect on the understandings that they now had about what it means to teach for deep understanding using a Four-Square Frayer Model. Sample participant responses are below in Figure 7.
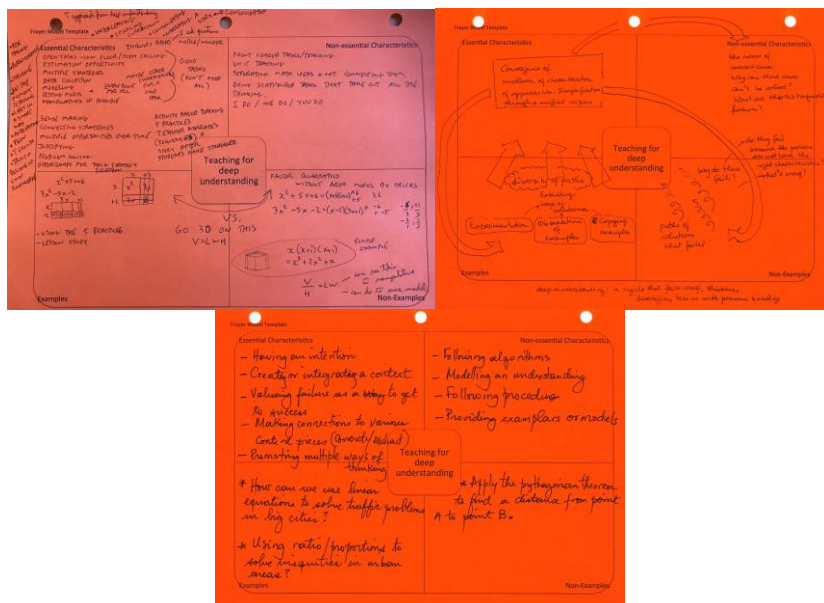
Figure 7. Participant Four-Square Frayer Model responses.

Overall, participants identified the following elements of teaching for deep understanding in each of the sections of the Four-Square Frayer Models:

Essential Characteristics
- Multiple representations and connections between the representations
- Deep understanding by the teacher of the concepts or willingness to work with students to further their own understanding
- Anticipating student responses and approaches—both accurate and inaccurate ones
- Well-designed sequencing/scaffolding
- Deep understanding comes from discovering the concepts through engaging with them with very little 'covering' of the curriculum
- Understanding that deep understanding is a marathon not a sprint and that you will only be with the student for a very short part of the process
- Valuing failure as a way to get to success
- Having clear objectives related to possible learning trajectories
- Class culture that is conducive to cooperative learning
- Provide thinking time—mulling over time
- Having students reflect on their misconceptions and why they changed their thinking
- Promoting diversity
- Synthesis of concepts through discussion
- Teachers create activities that allow students to develop/inquire/discover/create/ explore the mathematics themselves
- Students are the ones 'doing' the mathematics
- Modeling honesty and the integrity of someone seeking the truth
- Humility to listen, understand, and be surprised and/or taught by student thinking

Non-Essential Characteristics
- Tasks 'after the teaching' that match the math skills learned in the lesson
- Requiring specific algorithms
- Suggesting specific solutions to 'move on' in the lesson
- Teacher does not need to have all of the answers or need to anticipate all possibilities
- 'Real life' applications
- Copying class examples or 'notes' prescribed by teacher
- Giving directions of what to do
- Rote learning/memorization/surface learning
- Providing exemplars or models
- Prior knowledge or prior successful positive experience with the content can add an extra layer to deep understanding; however, it is not essential, since deep understanding is not linear where one needs to master previous level to develop the next level
- Using conventions and mathematical terminology

Examples
- Open tasks that allow for students to struggle
- Teacher moves that allow for students to work and communicate together
- Having students connect their solutions to each other's and then the teacher connects student thinking to the specific focus of the lesson
- Giving questions that enable the student to model with the use of the concept needed to be built upon
- Allowing multiple representation questions and discussing them in the whole class discussion
- Connections between different strategies: it is this connection that makes/takes basic understanding of a concept to a deeper level

- Reflecting on teaching—if lesson did not go as expected or learning was not as expected asking oneself what did not work and why

Non-Examples
- 'Chalk and talk'
- Checklist approach to teaching concepts/skills
- Teaching procedure only
- Have students apply an algorithm in a new situation
- Questions like "What is π?" "What is multiplication?"
- Aimless tasks—tasks without a learning intention

DAY 3: TEACHER EDUCATION

Day three would not be complete unless we started with a problem! The problem we posed was

Start with two numbers. The third number is the sum of the first two. The fourth is the sum of the second and third, and the fifth number is the sum of the third and the fourth (yes it is Fibonacci).	Commence avec deux nombres. Le troisième est la somme des deux premiers. Le quatrième est la somme du deuxième et du troisième, et le cinquième est la somme du troisième et du quatrième (oui, c'est Fibonacci).
We want the fifth number to be 100. Find all the whole number pairs for the first two numbers that will make this so.	Nous voulons que le cinquième nombre soit 100. Trouve toutes les paires (premier et deuxième nombres) de nombres entiers qui remplissent cette condition.

We had participants think about how this problem could be used with pre-service and/or in-service teacher educators to promote deep understanding in their students.

Figure 8.

The conversation kept coming back to the samples of teacher practice from the day before. The question that arose was: How do we work with pre-service and in-service teachers to help move their practice from the 'blue sheet (Sandra Pascal) to the green sheet (Mr. Donnelly)'? This was something that we puzzled over and did not come to a conclusion on, but suggestions were made that included teachers need to see the value in engaging students in tasks that evoke deep understanding and that they also need to experience engagement in tasks that elicit deep understanding from them so that they could have a sense of what being a student in that environment would be like. The points that were raised in our conversation are as follows:

- Paradigm shift is scary to think about—shifting gradually is maybe 'easier' to consider
- Having an occasion to 'see', 'feel' deep understanding and to reflect on that practice
- Teachers have deep understanding learning—learning is an organic process—learn, practice, assess—teachers need time to reify

- Collaborative conversations between university, in-service teachers, pre-service teachers, field instructors
- Nudged to think about experience to make sense of it
- Safe place for teacher reflection—publicly or privately
  - Safe place does not necessarily mean free from criticism but a sense of "working together"
- Teachers noticing kids learning—see the deep understanding—then identifying what do we (teachers) or they (students) do next
- Teaching in isolation inhibits the potential for reflection and the opportunity to engage with others
- Structured observation and sharing and opportunities for growth
- Rethinking professional growth opportunities; i.e., math residency—working together with others over an extended period of time in a teacher's space, sharing ideas, lessons, and modelling
- For pre-service teachers, get them to solve math problems, engage in case studies of teacher practice, watching videos of teacher practice
- Instructors of pre-service teachers also could have deep understanding and teaching for deep understanding
- A sense of freedom for in-service teachers to do what they need to do—working with department heads or principals to help them know how to work with teachers
- Unpack own experiences of doing math and teaching math
- Maybe moving from 'Add on learning', which is comfortable, to uncomfortable learning which involves rearranging or letting go of currently held conceptions—put at risk complacencies about roles in education—push for dissonance or disequilibrium—embrace that nagging voice in one's head to not continue in the 'stuck' for both pedagogy and content—letting go of the notions of 'I've got to cover…'

After this discussion, we offered up a sample set of student solutions to the previous problem and asked the participants to 'mark' those solutions. An exciting and intense conversation followed before groups were comfortable starting—a conversation that we did not expect but really enjoyed! Participants wondered what the criteria were that we were assessing and what outcomes the task was designed to assess. Questions were raised regarding why we needed to assess this task. An agreement was made that groups would 'mark' the student work based on what they thought in their groups and then we would discuss the different grading options that were used.

Un autre aspect qui a été soulevé est que la langue anglaise semble permettre de distinguer un « assessment » d'une « evaluation ». Une participante a appuyé cette distinction par la racine latine des deux termes: « assessment » est « be aside » comme dans le sens d'accompagner, alors que « evaluation » est « to give a value ». Il a été souligné qu'en français, il ne semble pas avoir de mot qui traduirait directement l'idée du terme « assessment ». En effet, le mot « évaluation » est la traduction « officielle » des deux mots. Or, une distinction peut se faire si l'on qualifie l'évaluation de « formative » par exemple. Dans ce cas, on retrouverait l'idée portée par le terme anglophone « assessment ».

Different groups approached the evaluation of student work in different ways based on their understanding of assessment and their own contexts. This discussion was shorter than the rest and mainly raised issues of ensuring that students know what they are being assessed on, that the assignments are aligned to the curriculum, and that students know the criteria for evaluation before or during the completion of the assessment.

The last section of the third day we posed our original questions back to the group, had them reflect on the questions and provide 'answers' or commentary on the questions. As we had hoped, some of our questions were revised to be better questions and participants were very thoughtful in their responses.

A summary of their responses and adjustments to the questions follow.

- How is having a deep understanding of school mathematics different than having a deep understanding of mathematics?

  o Deep understanding for whom?
  o Is there a difference? Should there be a difference?
  o School math is developing—it is a present tense verb, a personal one
  o Deep understanding of school math includes knowing what the problem aims to evaluate or help you practice. The understanding of the school context allows the student to concentrate his/her solution on the 'available maths' and in the 'available time'. Deep understanding of maths requires one to learn to cope with liberty and discover the solution's characteristics by himself/herself
  o Deep understanding of school mathematics involves the understanding of the mathematics in school curriculum of specific level while deep understanding of mathematics involves understanding of mathematics in all levels
  o Deep understanding of school mathematics includes a wise familiarity with the sorts of ways in which students come to understand school mathematics
  o Math should be math everywhere. The notion of school math is a way to make a difference between what is taught and what could be acquired
  o Beside institutional constraints 'time' or 'perceptions of time' there should not be a difference
  o What is deep understanding? Definition can be subjective/relative to its environment!!
  o Deep knowledge of mathematics taught in schools and deep knowledge of students and how they interact with content *plus* recognition of and awareness of some of the constraints that doing math within a school context may impose (e.g., curriculum predefined, time concerns, need to evaluate)
  o 'School math' is equivalent to 'school curriculum' + 'students' cognition' + 'pedagogical strategies'; 'math' does not need to consider students and pedagogy

- How can we teach ~~for~~ *continue developing* deep understanding of ~~school~~ mathematics?

  o Uncover curriculum instead of cover curriculum
  o Do not give out answers—make/help/guide/inspire students to do the work
  o Encourage struggle
  o Go beyond the original task with questions such as "what if…" or "why…"
  o Make connections with what students already know; pour qu'ils aient quelque chose sur lequel s'appuyer
  o Give the opportunity to all students to walk the 'understanding road' at their own pace
  o Restructure our course design from unit based to activity based interleaving big ideas—teaching through tasks
  o Focus on processes in our classes—communicating, reasoning and proving, connecting, representing, selecting tools and strategies, reflecting, problem solving

- o We can teach as if we were on a personal quest for deep understanding of school mathematics
- o Make connections that are meaningful to childrens' ways of knowing, being, and doing
- o Actively reinvest concepts from previous courses so that objects are not isolated

- How do we work with pre-service and in-service teachers to teach for deep understanding?
- Comment travaille-t-on avec les futurs enseignants et les enseignants pour enseigner cette « compréhension approfondie » ?

  - o Nous pourrions les amener à promouvoir certaines valeurs intellectuelles telles que : l'honnêteté, la curiosité, la recherche de la vérité, l'humilité, le courage.
  - o Permettre aux enseignants de rendre transparente leur propre expérience à faire des mathématiques et à les enseigner.
  - o Avec l'humilité de reconnaitre que peu importe quelle « compréhension approfondie » pour enseigner avec ou pour une compréhension approfondie, nous sommes tous à une distance infinie de notre but.
  - o Fournir des tâches qui permettent une compréhension conceptuelle avec plusieurs représentations, solutions et approches.
  - o Les engager dans des expériences riches et variées dans lesquelles, ils peuvent réfléchir aux mathématiques, approfondir leur propre expérience et penser à la pédagogie. Apprendre à le faire pour eux-mêmes.
  - o Amener les futurs enseignants et les enseignants à développer leur propre compréhension approfondie.
  - o Développer des expériences en communauté qui démontrent à ces communautés qu'elles apprennent avec les savoirs développés dans ces communautés.
  - o Les mathématiques ont besoin d'être conceptualisées de façon plus large que le modèle « Eurocentralisé ». Le raisonnement mathématique est présent dans une variété de cultures autour du monde et dans différents systèmes de connaissances.
  - o Différentes façons de comprendre doivent être prises en compte en plus de la façon plus traditionnelle d'apprendre.
  - o Varier les approches et les outils, les laisser tenir les mathématiques dans leurs mains—apprendre avec la personne dans son entièreté.
  - o Comprendre les concepts et leurs applications dans la vie de tous les jours.
  - o Placer les enseignants et les étudiants ensemble dans un contexte d'immersion où ils vivront des expériences riches et excitantes (ex., Camps mathématiques)

CONCLUSION

En conclusion, nous croyons que le groupe de travail a été très enrichissant. Les discussions étaient très animées et nous ont permis de réfléchir plus en profondeur au concept de compréhension approfondie et des contextes dans lesquels développer un tel type de compréhension. Aussi, la question des mathématiques de l'école a semblé intéresser tout le monde. La difficulté à trouver un consensus sur les éléments pouvant décrire ce concept est en fait très parlante sur sa complexité. Enfin, toutes ces discussions ont mené à la formulation de plusieurs questions très intéressantes qui pourront continuer à être explorées.

In conclusion, we believe that the working group has been very rewarding. The discussions were very lively and allowed us to think more deeply about the concept of deep understanding and the contexts in which to develop such a type of understanding. Also, the issue of school mathematics seemed to interest everyone. The difficulty of finding a consensus on the elements

that can describe the concept of school mathematics is very complex. Finally, all these discussions have led to the formulation of several very interesting questions that can continue to be explored.

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Topic Sessions

Séances thématiques

MODELLING MATHEMATICAL MODELLING

MODÉLISER LA MODÉLISATION MATHÉMATIQUE

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Mathematical modelling is a powerful way for anticipating or getting insight into real-world situations and phenomena. It thus seems appropriate to approach the integration of modelling in mathematics education from a modelling perspective. In this session, I started with a representation of the modelling process, more detailed than what we typically see, and I showed how this representation has helped me engage in discussion with students and professors, design learning activities, analyse student projects and assist students in gaining autonomy with respect to modelling. I then shared recent refinements to this model that have been introduced to reflect in more detail the complexity of simulation-based engineering and help plan a new training program for this specialization.

As the topic session consisted of a collage of distinct experiences that revolved around a common model, I give myself permission here to alternate between French and English.

MODÈLES ET MODÉLISATION

Puisque les mots « modèle » et « modélisation » ont acquis des sens différents dans certains discours éducatifs, il peut être utile de préciser le sens qu'on leur donne autant en mathématiques appliquées qu'en épistémologie des sciences.

Un modèle est d'abord une représentation simplifiée d'une situation qui met en évidence les aspects essentiels de cette situation (objets, variables et relations) au regard d'un certain but (Blum et al., 2002). Une telle représentation peut se faire dans les registres graphique (ex., un diagramme), symbolique (ex., une équation), numérique ou simplement verbal, ou même avec du matériel concret, dans la mesure où elle résulte d'un processus d'abstraction.[1] Un programme informatique peut aussi être vu comme modèle, dont la description permet la résolution du problème, la simulation d'un système ou l'exploration de la situation ou de scénarios possibles.

Dans cette perspective, modéliser consiste à élaborer, utiliser, valider et ajuster un modèle mathématique pour comprendre une situation réelle ou anticiper un phénomène qui lui est associé. Il s'agit donc d'un type particulier de résolution de problèmes, dont la raison d'être

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[1] Lorsqu'on utilise du matériel concret pour expliquer un modèle mathématique plus abstrait, il m'apparaît plus juste de parler de contextualisation. Le modèle est toujours plus abstrait que ce qu'il représente.

procède d'une situation réelle et dont le succès se mesure à son applicabilité à cette situation pour atteindre le but poursuivi. Ce processus complexe a été lui-même modélisé par différentes représentations cycliques (ex., Blum et Leiß, 2007; Giordano, Weir et Fox, 1997) où selon ce qu'on cherche à mettre en valeur, on inclut typiquement de deux à sept phases envisageables à l'intérieur d'un même cycle et où l'on ne suppose pas que ces phases s'organisent selon une séquence linéaire. Dans ces différents modèles, on fait ressortir l'importance de la *simplification* de la situation. Blum et al. (2002) en font une étape où l'on extrait et l'on organise en *structure* les éléments essentiels de la situation en fonction du but poursuivi et des connaissances disponibles. Il en résulte un *modèle réel* (ou modèle conceptuel), pas encore tout à fait mathématisé, à partir duquel on peut formuler plus précisément le problème à résoudre.

En vertu du but particulier poursuivi à l'intérieur d'un projet de collaboration consacré à la modélisation chez des étudiants universitaires (Caron et Bélair, 2007), nous avons cru utile d'y aller de notre propre représentation de ce processus.

EXPLORING MODELLING COMPETENCIES

Another representation of the modelling process came about in a study (Caron & Bélair, 2007) that aimed at exploring the competencies displayed by undergraduate students of mathematics in modelling a given situation. We also looked for reasons for the different degrees to which they engaged in the situation and achieved the intended goal. The study was done in a third-year mathematical modelling course at Université de Montréal with a rather heterogeneous student group. It revolved around a modelling project that was part of the course assignments. The project was to be chosen from a list of ten topics, most of which came with an initial reference. The work could be done in pairs, and students were encouraged to use technology, if deemed relevant.
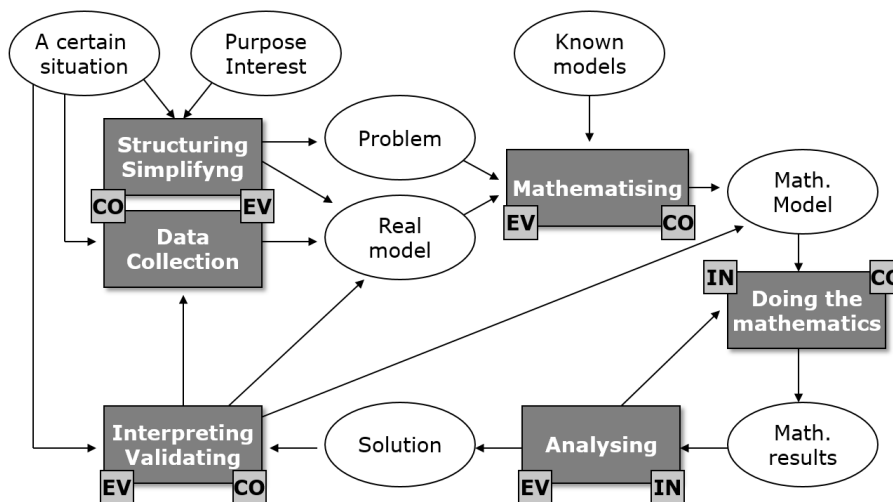
Figure 1. Our model of the modelling process.

While incorporating many elements of the description of the modelling process by Blum et al. (2002), our model shown in Figure 1 made explicit the fact that one does not reinvent the wheel every time a new problem emerges and that known models are part of the resources that one uses, adapts, or extends when modelling. This is particularly true when modellers already have a substantial mathematical tool box as well as an experience of applied mathematics; building

an adequate model can sometimes be more a matter of adapting and combining known models than of building a completely original one only from abstraction.

Another property of our model is that each of the phases of the modelling process had been linked to the modelling competencies that play a key role in that phase. The three-tier classification of work sociologist De Terssac (1996) was used to that end, as was done previously with the more general process of problem solving (Caron, 2004):

- CO: *communication skills* ('compétences d'explicitation'): to translate, represent, interpret what the context is, what is to be done, and what has been done;
- IN*: intervention skills* ('compétences d'intervention'): to act upon a situation by using available knowledge and by transforming encountered situations into reusable knowledge;
- EV: *evaluation skills* ('compétences d'évaluation'): to identify, choose, and justify whatever is being engaged into action.

For instance, *mathematising* a problem requires both *evaluation skills* to identify appropriate mathematical objects to be included in the model and *communication skills* to translate the structure of the situation and the associated problem into mathematical language with which the mathematical work will be done. With that model, we were considering computer programming languages as extensions of mathematical language.

The model was used first to analyze the organization or the modelling project reports produced by nine students; the consecutive elements of the report (sentences, equations graphs, etc.) were mapped to the different stages of modelling, as shown in Figure 2.
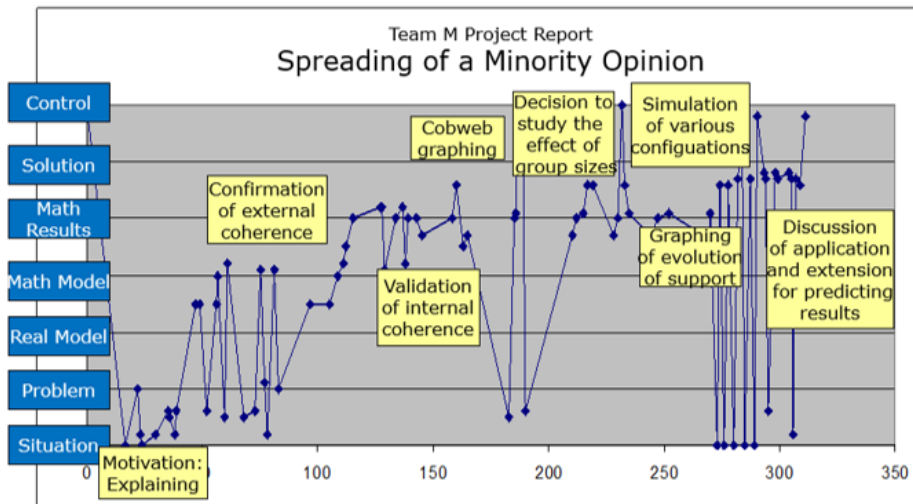
Figure 2. Sequential structure of a project report.

For each of the phases of the modelling process, we then looked for connections between the competencies that the students displayed and their educational background, interests in mathematics (that we captured with a questionnaire), the motivation that they showed for the project, and the modelling paradigm they seemed to have adopted—*theoretical*, for understanding, or *empirical*, for predicting (Maull & Berry, 2001).

The analysis made us see possible connections, which could become hypotheses to be tested with a larger group of students. Interest in reading appeared connected with *communication skills* mobilized in modelling the project. Experience of complexity, understanding of the role of the model, and familiarity with the context or underlying principles emerged as possibly favoring stronger *evaluation skills*. And the level at which technology had been integrated in one's mathematical practice appeared to play a significant role in the *intervention skills* displayed.

CONTRIBUER AU DÉVELOPPEMENT DE LA MODÉLISATION

Le même modèle a été utilisé dans des cours de didactique pour permettre aux étudiants de s'approprier le processus de modélisation mathématique à travers l'étude d'une situation réelle de leur choix (évolution de la population étudiante, prix des billets d'avion, etc.).

Il est frappant de voir les étudiants utiliser un tel diagramme un peu comme une carte routière, pour mieux se situer dans le processus et comprendre les actions nécessaires pour passer d'une phase à l'autre. Le modèle permet notamment de voir que le simple passage d'un ensemble de données représentant la relation entre deux variables au modèle fonctionnel qui ajuste le mieux ces données a souvent pour effet d'escamoter la phase de *structuration* de la situation, limitant, au mieux, à un rôle *prédictif* le modèle ainsi construit. Pour envisager une *explication* de la relation entre les deux variables, il faut souvent creuser davantage, en remontant et en combinant des relations déjà connues ou qui s'appuient sur des principes acceptés. En d'autres mots, il faut passer du *paradigme empirique* au *paradigme théorique* (Maull et Berry, 2001).

Une situation proposée par deux étudiants a permis de l'illustrer clairement (Caron, sous presse). S'intéressant à la durée de vie d'un site d'enfouissement, ils disposaient pour chacune des années considérées du volume de déchets qui avaient été acheminés à ce site et de la capacité restante du site ($C(n)$), telle qu'évaluée à partir de photographies aériennes. Connaissant la capacité totale du site, ils pouvaient aisément déduire le volume de déchets accumulés ($V(n) = C_{\max} - C(n)$) pour chacune des années. Mais ces valeurs ne pouvaient être liées par de simples relations additives avec la quantité de déchets acheminés par année : le volume accumulé était toujours en-deçà de la valeur anticipée ($V(n) < V(n-1) + Q(n)$). Une fonction logarithmique paraissait le mieux ajuster les données du volume en fonction du temps, mais elle n'offrait ni garantie de validité pour la suite des choses ni pouvoir d'explication.

C'est en cherchant une explication à la perte de volume (la compression des déchets) qu'on a pu rebâtir un modèle relativement simple, basé sur une nouvelle relation de récurrence : $V(n) = a\,V(n-1) + b\,Q(n)$. Dans cette relation, interviennent deux facteurs de compression distincts : un premier (a) pour le volume de déchets déjà présents à l'année précédente, et un second (b) pour le volume de déchets frais acheminés dans l'année. Il devenait ensuite possible, par un ajustement itératif de ces deux facteurs sur le tableur, de reconstruire une excellente approximation de la courbe du volume occupé par les déchets dans le site d'enfouissement. La recherche des valeurs optimales pour ces deux paramètres a d'ailleurs constitué un moment intéressant où un authentique questionnement interdisciplinaire a émergé : quels déchets sont appelés à être comprimés davantage ? Un plus grand facteur de compression signifie-t-il une pression plus forte ?

SUPPORTING PROGRESSIVE INTEGRATION OF MODELLING

For many decades, three streams have been offered to Québec students for completing their secondary studies in mathematics. Starting in 2008, one of these streams was to promote learning mathematics via applications, instruments, case studies and experimental data. From a

lack of resources for embracing such a direction, a collaboration was put in place between Montréal school boards and École de technologie supérieure (ÉTS). Voluntary teachers and instructors regrouped in teams to codesign learning activities in mathematics that would feed from scientific investigation and use of technology. They were progressively joined by CEGEP teachers from the Montréal area. The learning activities that were developed are now in use in secondary schools, cégeps and in the transitional mathematics course at ÉTS.

To varying degrees, the learning activities that have been produced involve some aspects of modelling (Caron & Savard, 2012). Yet, because they are centered on specific mathematical content to be learned, they do not have the openness typically associated with a modelling task. The diagram of the modelling process became instrumental in locating what aspects of modelling were being addressed. This was done in particular for one of the learning activities in its implementation at ÉTS (Caron & Pineau, 2017).

The activity is based on L'Hospital's Weight Problem, a classical optimization problem that was presented in 1691 by Johann Bernoulli to the Marquis de L'Hospital as an introduction to differential calculus. It aims at predicting the equilibrium position of a weight in an apparatus that includes two strings, three bars and a pulley. Prior classroom experiments (Drijvers, 1999; Van Maanen, 1991) tended to suggest that modelling was at the core of some of the difficulties experienced by the students.

The problem thus was transformed into a hands-on activity where all students could interact with their own apparatus and use measuring tools to assist them in mathematising the system (Figure 3). The activity was introduced in the ÉTS transitional course to consolidate knowledge and use of trigonometry, functions, and the recently encountered derivative, while developing skills for constructing and validating a model.
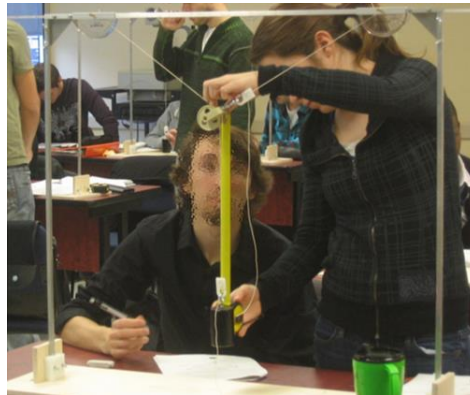
Figure 3. Measuring on a hands-on version of L'Hospital's problem.

The interaction with the physical apparatus was shown to enable students to explore and anticipate the behaviour of the system, to formulate a real model and to interpret and validate the formulas that they were led to build. It helped them distinguish between parameters and variables in *structuring* the situation.

In addition to the use of the apparatus, scaffolding has revealed very useful for students to progress in *mathematising* the situation, along the axes of interest for the course and within the allotted time. A partially defined model with a schematic representation of the situation is given for students to complete. The idea is for them to experience how an elaborate model can be built from combining simple relations, through variable substitution (or function composition)

and simple arithmetic operations. For some students, this exercise was a moment of revelation, something different from what they had been used to doing for defining a model, an entry into the *theoretical paradigm*, which can help explain as much as it can help predict.

Making explicit the contribution to modelling of a learning activity that might otherwise be looked as an application has been an interesting use of the modelling diagram. It has shown the possibility of introducing elements of modelling in a content-driven course and of progressively reducing scaffolding as students are exposed to more activities of that nature.

ORGANISER LA FORMATION AU GÉNIE PAR SIMULATION

Le développement de la capacité à résoudre des problèmes d'ingénierie complexes en tirant parti des outils d'ingénierie modernes constitue l'une des « qualités » à développer dans la formation au génie (Ingénieurs Canada, 2014). Cela renvoie notamment au rôle croissant que la simulation joue dans le travail d'ingénieur (Caron et Garon, 2013).

À Polytechnique Montréal, une spécialisation en génie s'appuyant sur la simulation est en voie de développement dans les départements de génie mécanique et de génie chimique. Dans le but de soutenir l'organisation cohérente des cours d'une telle spécialisation et de son prolongement aux cycles supérieurs, une nouvelle version du cycle de modélisation a été élaborée pour servir de cadre de référence.

Cette autre version était rendue nécessaire en raison de l'importance du travail informatique pour résoudre les problèmes dont la complexité commande un passage par la simulation. La seule petite « boîte » associée au traitement mathématique, à la droite du diagramme de la Figure 1, ne rendait pas justice à l'ensemble des étapes impliquées pour passer du modèle mathématique à sa solution. On a donc choisi de distinguer le modèle informatique du modèle mathématique, et d'intégrer dans la partie de droite du diagramme les principales étapes qui sont impliquées dans la résolution numérique par simulation. De plus, à l'instar de Roache (1998), nous avons convenu de distinguer la *vérification* (s'assurer que les équations sont bien résolues, qu'il y a *cohérence interne*, mathématique) de la *validation* (s'assurer qu'on a résolu les bonnes équations, qu'il y a *cohérence externe* entre le modèle utilisé et la situation de départ).
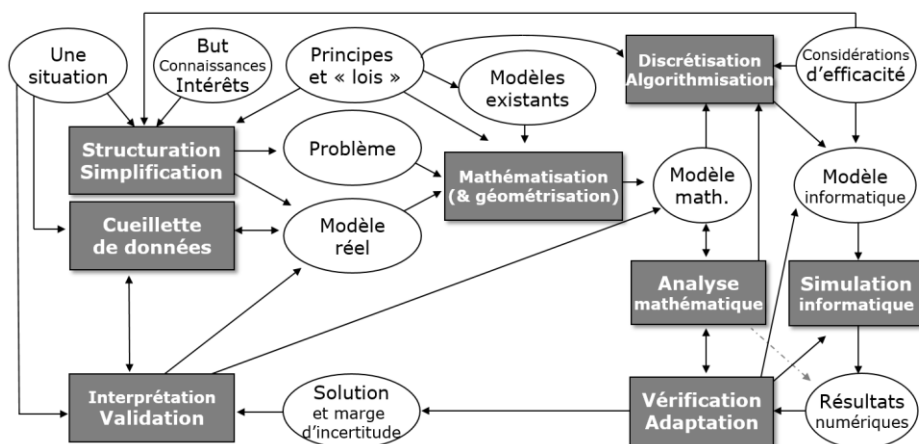
Figure 4. Le processus de modélisation dans la pratique du génie par simulation.

L'élaboration de ce nouveau modèle s'est faite à partir d'échanges avec des professeurs impliqués dans la conception de cette nouvelle spécialisation liée au génie par simulation. Le modèle a ensuite été testé, d'abord à l'aide d'un étudiant à la maîtrise qui l'a utilisé pour résumer un article, à la fin de l'exercice, il a spontanément exprimé qu'il aurait bien aimé connaître ce modèle avant. La validation et l'exploitation du modèle se sont poursuivies dans une séance de travail avec d'autres professeurs qui devaient identifier pour chacune des phases les savoirs importants qu'il convient de mobiliser, les difficultés auxquelles se heurtent typiquement les étudiants, et les activités qui peuvent ou pourraient développer leurs compétences au regard de cette phase. Les échanges ont conduit notamment à réaliser qu'on ne revenait peut-être pas assez aux principes et aux lois quand on utilise les équations qui en rendent compte, et qu'il y aurait avantage à le faire de façon plus systématique.

Nous avons parcouru ce cycle avec des étudiants au premier cycle et aux cycles supérieurs, en leur demandant, pour chacune des phases du cycle, d'identifier des cours ou des expériences où ils avaient le sentiment d'avoir travaillé cette phase et d'évaluer ensuite leur sentiment de compétence au regard de cette phase. Cela a permis d'identifier (ou de confirmer) que la *mathématisation*, la *discrétisation* et l'*algorithmisation* ainsi que les *considérations d'efficacité* informatique étaient relativement peu couvertes dans les cours actuels et mériteraient une attention particulière dans cette nouvelle voie de spécialisation consacrée au génie par simulation. Des séances d'enseignement et des questions d'évaluation sont aussi examinées à l'aune de ce diagramme.

Le modèle sert maintenant de cadre à la rédaction d'un guide de modélisation appelé à servir dans l'ensemble des cours de cette spécialisation.

CONCLUSION

As was provocatively asserted by Box (1979), "all models are wrong, but some are useful" (p. 202). Our model of the modelling process is no exception to the rule. It has been designed, used, adapted or extended according to the goal pursued with each project. Yet, despite (or because of) its inherent simplifications, such schematic representation has helped engage dialog, analyse tasks and courses, identify or explain phenomena, assess progress, define a course of action or promote a change of perspective. More generally, it has shown that the use of a model to talk about modelling indeed appears a sound approach to get to the core of the matter.

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COLLECTIVE LEARNING: RE-THINKING THE ENVIRONMENT, ARTIFACTS AND CLASSROOM INTERACTIONS

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INTRODUCTION

Over the past two decades, a group of CMESG members have been working on understanding the ways in which mathematics learning occurs in classrooms viewed as collective systems. That is, how mathematics classrooms can be seen as complex systems in which agents spontaneously interact and adapt to each other organizing and sustaining learning processes in a collaborative way. In this paper I draw from a groups' work on SSHRC funded collaborative research titled *Advancing Research Methodology for Mathematics Education*. Specifically in the paper I offer a discussion about how artifacts such as a classroom boards play a vital role in the coordination of behaviours or actions in a mathematics classroom viewed as a complex system. The source of inspiration for this discussion comes from studies in cognitive stigmergy which have led to a better understanding of how agents in a complex system communicate indirectly through their environment which is articulated and is typically composed of artifacts, which build up the social workspace, or field of work. I begin the paper with introducing the notion of stigmergy in complex systems followed by a discussion about cognitive stigmergy, role of artefacts, and the environment. Then I discuss how we might conceptualize cognitive stigmergy, artefacts and environment in mathematics classrooms. Finally I conclude with some implication of this discussion for mathematics education practice and research.

COMPLEX SYSTEM BEHAVIOUR: SELF ORGANIZATION AND THE COORDINATION PARADOX

Complex systems can self-organize themselves to produce organized patterns resulting from localized-neighbouring interactions within the components of the system, without any central control. This complex system behavior known as decentralized control raises the so called 'coordination paradox'. As Varela (1997) notes

> *What I wish to insist upon here is the relatively recent (and stunning!) conclusion that lots of simple agents having simple properties may be brought together, even in a haphazard way, to give rise to what appears to an observer a purposeful and integrated whole, without the need for a central supervision.* (Varela, 1997, p. 83)

While in a complex system each agent appears to pursue its own agenda, somehow the collective as a whole exhibits high levels of organization or coordinated actions or behaviour (Theraulaz & Bonabeau, 1999). Another key aspect of complex systems is the dialectical entanglement of the system and its environment. That is, the system both shapes and is shaped by its environment. In 1959 a French zoologist, Grassé, sought to understand the mechanisms underlying decentralized control in social insects. Specifically, Grassé inquired into the so-called 'coordination paradox'. Grassé found out that in the coordination and regulation of

termite colonies there is the phenomenon of indirect communication mediated by modifications of the environment, i.e., insects interact indirectly: each insect (ants, bees, termites) affects the behaviour of other insects by indirect communication through the use of the environment, which is made of objects and artifacts, such as material for the nest or chemical traces. In this sense, the environment of the system is not a simply passive 'container', but in contrast, it embeds dynamical mechanisms and processes that promote the emergence of local and global coordinated behaviours. Grassé coined this phenomenon or mechanism stigmergy. The term stigmergy is formed from the Greek words *stigma "sign"* and *ergon "action"* and captures the notion that an agent's actions leave signs in the environment, signs that it and other agents sense and that determine their subsequent actions.

COGNITIVE STIGMERGY, ROLE OF ARTIFACTS AND THE ENVIRONMENT

Despite the differences between social insects and other animal systems (flocks of birds, schools of fish, etc.), these animal systems appear to exhibit similar collective behaviours suggesting the possibility of stigmergy as a mechanism underlying the collective coordination of actions within human systems. However, ant-like agents are not humans, and therefore, they do not have same cognitive ability as humans. Moreover, the ant environment is quite different and elementary, including pheromone-like signs/signals compared to that of humans (Ricci, Omicini, Viroli, Gardelli, & Oliva, 2007). As Maturana (2002) notes unlike other animal systems, human beings exist in language as consensual coordinations of coordinations of behaviours.

> As we [humans] *language, objects arise as aspects of our languaging with others, they do not exist by themselves. That is, objects arise in language as operations of coordinations of coordinations of doings that stand as coordinations of doings about which we recursively coordinate our doings as languaging beings.* (Maturana, 2002, p. 29)

Thus for human systems, the environment includes signs or signals that are subject to an interpretation in the context of a shared, conventional system of signs. Also the environment is articulated and is typically composed of artifacts, which build up the social workspace or field of work (Ricci et al., 2007). There are many examples of human-human stigmergy, in fact "it would be more difficult to show a functioning human institution that is not stigmergic, than it is to find examples of human stigmergy" (Parunak, 2005, p. 163). Parunak (2005) proposes a framework that can be used to analyze a stigmergic system. A stigmergic system comprises of a population of agents and an environment in which they are immersed. Each agent has an internal state, which generally is not directly visible to other agents; sensors that give it access to some of the environment's state variables; actuators that enable it to change some of the environment's state variables; a program (its 'dynamics') that maps from its current internal state and its sensor readings to changes in its state and commands given to its sensors and actuators. The environment has a state, certain aspects of which generally are visible to the agents; a program (its 'dynamics') that governs the evolution of its state over time. One example of human-human stigmergy is that of trail formation (Parunak, 2005). In trail formation the environment is comprised of vegetated terrain. Its state can be seen as degree of ground cover, while its dynamics might be articulated as in dying of trodden vegetation and/or regrowing on the untrodden areas. The agents are the people—pedestrians who sense the smoothness to path and act by walking the next step. The emergent system behavior is walking trails (Figure 1). Another example of human-human stigmergic system is that of document editing (Parunak, 2005)

Figure 1. Image of walking trail ("Cornish Hill walking trail," n.d.).

The environment—artifact is the document (see Figure 2). Its state is the current content and its dynamics is internal semantic propagation. Agents are writers/editors who sense the current state of the document and act by adding new content and/or strikeouts and highlights. The emergent system behaviour is the joint expression of the content.

Figure 2. Image of document editing (Cactus, n.d.).

COGNITIVE STIGMERGY, CLASSROOM INTERACTIONS, ENVIRONMENT AND ARTIFACTS

In the field of mathematics education, there has been an interest in the importance of classroom interaction in the teaching and learning of mathematics. The impetus for this focus comes from a general realization that classroom interactions—through students' discussion and other forms of interactive participation—has potential to enhance mathematical understanding. Within mathematics education there is extensive literature dealing with interaction stemming from social interactionism (e.g., Bauersfeld, 1980) and socio-constructivism (e.g., Cobb & Yackel, 1996). Indeed the field of mathematics education has witnessed a proliferation of research issues, topics and visions such as cooperative learning, collaborative inquiry and communities of practice (Balacheff, 1991; Bauersfeld, 1995; Cobb, Wood, & Yackel, 1993; Lampert & Cobb, 2003; Yackel & Cobb, 1996). A common thread through this research is a need to better

understand, describe and define collective learning systems both in theory and methodology (Roth & Lee, 2002; Saxe, 2002; Sfard & Kieran, 2001). In our research, we use the word collective rather than collaborative (Martin, Towers, & Pirie, 2006), as we realize that not all collaborative classroom interactions and actions lead to the emergence of a collective learning system. In this paper, I am interested in thinking about a classroom as stigmergic (complex) system comprising of agents and the environment in which they are immersed as articulated by classroom artifacts that play a vital role in the coordination of behaviours/actions. In the following I try to articulate some of the components of classroom as a stigmergic system using the Parunak (2005) framework as previously discussed in the paper.

WHO ARE THE AGENTS IN MATHEMATICS CLASSROOMS?

Agents interacting in mathematics classrooms are not bodies—they are emergent (global patterns) cognitive selves. As Rudrauf, Lutz, Cosmelli, Lachaux, and Van Quyen (2003) notes

> *Thus, from both the biophysical and the concrete experiential points of view, there is no central "I," other than the one sporadically actualized in a linguistic, self-referential mode in communication. The "I" can only be localized as an emergence but it acts as the center of gravity of the subject himself/herself, of his/her real-life experience (Varela, 1993).* (p. 42)

WHAT ARE ARTIFACTS IN MATHEMATICS CLASSROOM ENVIRONMENT THAT MEDIATE COORDINATION OF ACTIONS FOR COLLECTIVE LEARNING

Flip chart paper, board (white, green, black, interactive), post it notes, social networks, computational environments, et cetera are examples of artifacts. The state and the dynamics of these artifacts will depend on the type of artifacts.

WHAT ARE THE POTENTIAL SIGNS THAT AGENTS IN MATH CLASSROOMS CAN SENSE AND ACT ON?

Signs or markers might include students' mathematics productions, mathematical ideas, et cetera. I prefer to call these signs as cognitive trails:

> *Cognitive Trails are in the environment, certainly, but they are also cognitive objects. A trail isn't just an indentation in a physical surface, but a marking of the environment; a signposting for coordinating sensation and movement, an experiential line of force. Hence the marking is both experiential and environmental.* (Cussins, 1992, p. 674)

Figures 3 and 4 are some of mathematics cognitive trails that might be found in mathematics classrooms.
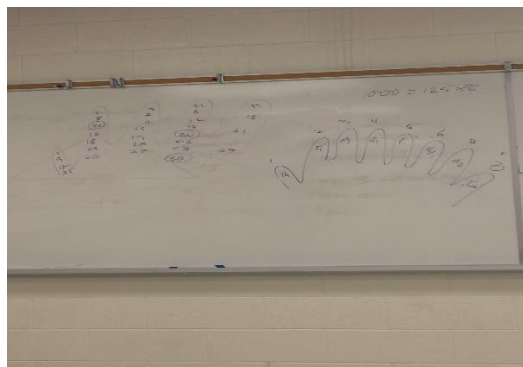
Figure 3. Mathematics cognitive trails example.
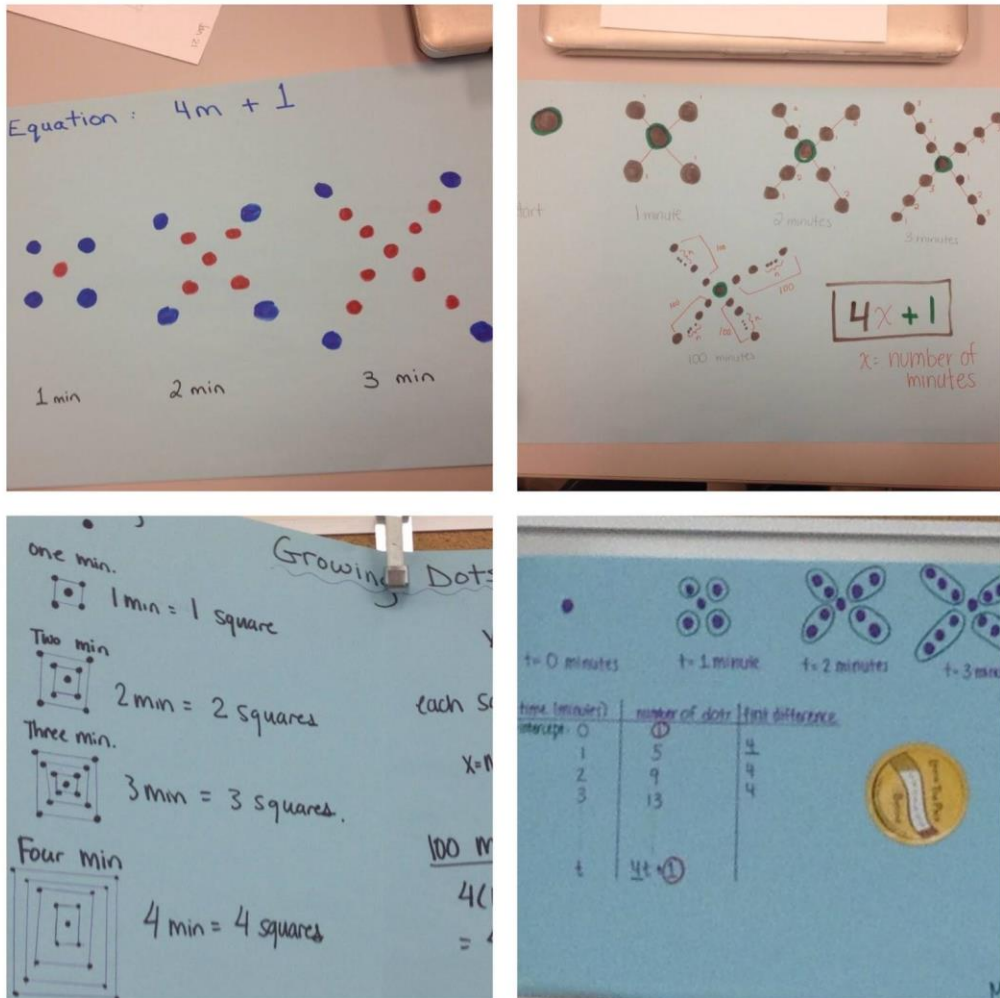
Figure 4. Mathematics cognitive trails examples.

WHAT EMERGENT SYSTEM/COMPLEX/COLLECTIVE BEHAVIOR MIGHT WE OBSERVE IN MATHEMATICS CLASSROOMS?

Collective mathematics understanding; solution for problem; a coherent mathematics lesson; coherent idea, et cetera might be observed.

CONCLUDING REMARKS

The role of artifacts in human cognition continues to be a central focus in activity and distributed cognition theories (Suzi & Ziemke, 2001). Since the early 1990s there has been a growing interest in the role of artifacts as they relate to collective behaviour in complex systems. In this paper I have introduced the notion of stigmergy, its relation with the environment, and interaction through artifacts. I have tried to show how we might begin to think about the role of artifacts in coordination collective learning behaviour in mathematics classrooms. This has implications for both practice and research. As practitioners (whether designing or selecting), we might think about what and in what ways classroom artifacts have more potential for mediating collective mathematics behaviour. In his work on building thinking mathematics

classrooms, Liljedahl (2016) explored the potential for alternative work surfaces, such as poster board, flipchart paper attached to the walls, and smaller whiteboards laying on desks and vertical white boards in supporting students' thinking (thinking classroom). He observed that

> *Groups are more eager to start, there is more discussion, participation, persistence, and no-linearity when they work on the whiteboards. However, there are nuances that deserve further attention. First, although there is no significant difference in the time it takes for the groups to start discussing the problem, there are a big difference between whiteboards and flipchart paper in the time it takes before groups make their first mathematical notation. This is equally true whether groups are standing or sitting. This can be attributed to the non-permanent nature of the whiteboards. With the ease of erasing available to them students risk more and risk sooner.* (Liljedahl, 2016, p. 371)

Clearly the board in the classroom can be used in various ways ranging from a static space for conveying information as in chalk and board traditional way of teaching to a dynamic intellectual commons for coordinating actions. The following excerpts from *Blackboard: A personal history of the classroom* by Lewis Buzbee (2014) says it all:

> *Once I start on the board, I often can't stop and continue to add phrases, strange pictures, the titles of books, sometimes just marks, a kind of visual punctuation. The ham of my left hand will be covered with red or blue or green dry-erase marker by the end of the evening, and when I stand back to look over what I've written, nothing makes any sense. My board work looks more like a foreign language than literary criticism. But it's still effective board work. I've been able to draw connections; I've been able to drive home key points. I've made the students look beyond me, themselves, and our little room.* (para. 29)

In our SSHRC collaborative research we are in are interested in

- Exploring new ways of observing and visualizing dynamic phenomena in collective learning systems, particularly in classroom contexts
- Identifiying data collection techniques suitable for collecting data from and about complex systems
- Crafting methodological tools that reveal collective learning rather than individual learning
- Developing means for representing and analyzing data collected from collective learning systems

Given this, we choose to focus on and develop a tool for observing (non)actions/activity on the board (i.e., its state and dynamics) as one of vital signs of classroom life.

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A PSYCHOLOGICAL VIEW
OF TEACHING PROOF CONSTRUCTION

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INTRODUCTION

For over 10 years, Annie Selden and I have jointly taught a small (4 to 10 students, 3 hours per week) graduate course for incoming mathematics graduate students who think they need a little help in constructing proofs. The course is taught from notes containing statements of theorems, definitions, requests for examples, questions, and some advice.

The notes include some theorems about sets, functions, real analysis, abstract algebra (in the form of semigroups), and if time permits, some topology. The abstract algebra is about semigroups so that the students are not likely to have seen proofs of the theorems. The notes and advice are always available, but nothing else outside the course is. The students autonomously (i.e., independently) construct proofs for the theorems in the notes and present them at the board, and we provide comments and advice where needed. Our goals include that students learn to write proofs acceptable to the other faculty and begin to build an ability to construct proofs of some difficulty. In adjusting the course to student needs, we call on both of the mathematics education research and psychological literatures.

Logic is handled in context in brief lectures/discussions as the need arises. Also, early in the notes, theorems and definitions are written in the most unpackable/formal way, as opposed to the most informal/memorable way. For example, in real analysis, "For all $a \in \mathbb{R}$ and all functions f, if f is differentiable at a, then f is continuous at a", rather than "Differentiable functions are continuous". We earlier found that students often had difficulty unpacking the meaning of informally written statements (Selden & Selden, 1995). Such unpacking would needlessly interfere with maximizing students' early successes. We want students to have early successes to help maximize their sense of *self-efficacy* (Bandura, 1994, 1995), which can be helpful in constructing the harder parts of proofs towards the end of the course.

I have included the above paragraph because it illustrates how purely psychological information (e.g., about self-efficacy) can inform the early notes in a way that may well aid a student in later (autonomously) constructing a difficult proof near the end of the notes. This kind of teaching would probably be very difficult to arrange when lecturing with a book.

We have almost always taught this course to very small numbers of students. However, occasionally we have taught larger classes. I suggest that at or above 10 students, it would be a good idea to have the students responsible for constructing proofs in small autonomous groups with some mechanism to be sure no student is left out.

BEHAVIORAL SCHEMAS

ACTIONS

In discussing proof construction, I use the term *action* broadly, as a response to a situation in a partly constructed proof. An action might be physical, such as making a sketch in scratch work, or writing an additional part of the proof. It could also be mental, such as trying to recall an earlier result or saying something in inner speech. It could even be a 'meta-action' meant to alter one's own ability to act, such as taking a break in an effort to seek a fresh approach.

If, in several proof constructions, similar situations in partly completed proofs, have led to similar reasoning, which in turn, have led to similar actions, then a *link* may be learned between the kind of situation and the kind of action. Later, other similar situations will evoke the action without the need for the intermediate reasoning. This kind of associative learning can be implicit and has been studied by psychologists interested in the psychology of everyday life (Bargh, 1994; Bargh & Chartrand, 2000). Using such *situation-action* links strengthens them, so they eventually become automated. We call such automated situation-action links *behavioral schemas*.

I first noticed behavioral schemas in my own thinking when I was writing out a key without stating explicit reasons in preparation for grading an examination. I now suggest that behavioral schemas can occur naturally, that is, without explicit instruction. They can be detrimental or beneficial. Beneficial ones can express appropriate inferences, reduce needless burdens on working memory, and reduce confusion. (For more details, see Selden, McKee, & Selden, 2010, pp. 204-207.)

The associative learning of situation-action links and behavioral schemas is reminiscent of the conditioning of behaviorism in the early 20th century. At the time, behaviorism was a major perspective in psychology, at least in the US. It was later surpassed by the *cognitive revolution* that began around 1950. The major difficulty with behaviorism seems not to have been with conditioning per se, but with an accompanying philosophical perspective. That perspective maintained that to be 'scientific' everything involved in an explanation should be observable and measurable. Such specificity is still not currently possible in studying the functioning of the brain, but this is not regarded as a fundamental research flaw. Indeed, in teaching, one can often design associative learning experiences and be reasonably sure that such learning will occur. Despite this, the idea of associative learning does not seem to be used with constructivist perspectives.

EXAMPLES OF BEHAVIORAL SCHEMAS

A person driving a car typically stops at a red light in an automated way. That is, without examining any warrant for doing so and with little awareness of the constituent actions. Here is a more mathematical example. A person familiar with simple algebra might solve

$$x + 1 = 7 - 2x$$

using a sequence of five behavioral schemas with outputs:

$$x = 7 - 1 - 2x$$

$$x = 6 - 2x$$

$$x + 2x = 6$$

$$3x = 6$$

$$x = 2$$

Note that, in the above, consciousness of the result of each behavioral schema seems to be required as input for the next. One cannot just see the original equation, hold one's mind blank for some time, and then provide the solution, $x = 2$.

THE NATURE OF BEHAVIORAL SCHEMAS

One might see behavioral schemas as largely conceptual in nature. This is because one must recognize a situation as something one is familiar with, perhaps in a new setting. Also, one might see a behavioral schema as *knowledge-in-action*, perhaps only a small extension of the idea of *theorems-in-action* (Vergnaud, 2009, p. 88).

I will now turn to two ways of using situation-action links and behavioral schemas to alleviate two early proving difficulties we have noticed—getting started and using definitions.

PROOF FRAMEWORKS

A feature that can help students get started constructing a proof is what we call a *proof framework*, of which there are several kinds and often a *first-level* and a *second-level* framework. Given a theorem of the form, "For all real numbers x, if $P(x)$, then $Q(x)$", a first-level framework would be "Let x be a real number. Suppose $P(x)$. … Therefore, $Q(x)$.", with the remainder of the proof replacing the ellipses.

A second-level framework can often be obtained by 'unpacking' the meaning of $Q(x)$ and putting the framework for its proof between the lines already written for the first-level framework. This will cause the proof to 'grow' from both ends towards the middle, instead of from the top down.

EXAMPLE: USING A PROOF FRAMEWORK

Below is a first-level proof framework for a very elementary number theory theorem.

> **Theorem:** For all integers n and m, if n is even and m is odd, then $n + m$ is odd.
>
> **Proof:** Let n, m be integers. Suppose n is even and m is odd.
>
> . . .
>
> Therefore, $n + m$ is odd.   QED.

After adding the second-level framework based on the meanings of odd and even, the emerging proof looks like:

> **Theorem:** For all integers n and m, if n is even and m is odd, then $n + m$ is odd.
>
> **Proof:** Let n, m be integers. Suppose n is even and m is odd.
>
> Then there exist integers i and j such that $n = 2i$ and $m = 2j + 1$.
>
> . . .
>
> Then $n + m = 2k + 1$, where k is an integer.
>
> Therefore, $n + m$ is odd.   QED.

To complete the proof, one only needs to replace the above ellipses with something like, "Now let $k = i + j$ so that $n + m = 2i + 2j + 1 = 2(i + j) + 1 = 2k + 1$. (For more information and another example, see Selden, Selden, & Benkhalti, 2018.)

PROOFS INVOLVING INFINITE SETS

There is a bit more to say about the above apparently 'straightforward' proof. The theorem starts "For all integers n", treating n as an integer variable. But the proof starts "Let n be an integer", which means that n represents an unspecified integer constant, such as 6 or -17. This technique is generally accepted in mathematics as proving something about all integers because any integer can replace n to yield a correct proof. That is, n is often said to be "fixed, but arbitrary".

Some students feel there is something peculiar about writing "Let n be an integer", even though they may carry out the action. Doing an action without understanding the reason violates the idea/norm that students should understand and agree with an action before executing it. I suggest that this idea should be moderated somewhat. What is needed is that students eventually agree that the action is legitimate, not that they agree before using it in any particular proof.

Below is an interview excerpt from a graduate student, Mary, in which she reported her experiences taking a real analysis course taught by Dr. K two years prior to the interview.

> **Mary:** *At that point* [early in Dr. K's real analysis course] *my biggest idea was, well he said to 'do it', so I'm going to do it because I want to get full credit. And so I didn't have a real sense of why it worked.*
>
> **Interviewer:** *Did you have any feeling ... if it was positive or negative, or extra ...*
>
> **Mary:** *Well, I guess I had a feeling of discomfort ...*
>
> **Interviewer:** *Did this particular feature* [having to fix x] *keep coming up in proofs*?
>
> **Mary:** *... it comes up a lot and what happened, and I don't remember* [exactly] *when, is that instead of being rote and kind of uncomfortable, it started to just make sense ... By the end of the semester this was very comfortable for me.*

> *Mary appears to have developed both the behavioural schema and the associated feeling of appropriateness only after executing the <situation, action> pair numerous times. In early executions of this <situation, action> pair, Mary carried it out partly based on Dr. K's authority. In addition, after completing each such proof, Mary reported that she had attempted to convince herself that considering a fixed, but arbitrary element resulted in a correct proof. Only after repeatedly executing this <situation, action> pair, and convincing herself that the individual proofs were correct, did she develop a feeling of appropriateness.* (Selden et al., 2010, p. 209)

This agrees with the idea that students can perform certain actions procedurally and only later understand the reasons for them.

While proof frameworks are very useful when students are beginning to learn to construct proofs, they can also be useful later. It turns out that considerably later, Mary, who had also taken our 'proofs' course, was taking her real analysis PhD comprehensive examination consisting of eight (original to her) theorems to prove in three hours. Upon reading one of the theorems she panicked, but then controlled herself and decided to write a proof framework. Just this simple act oriented her to the 'real problem' at hand, relieved her anxiety, and allowed her to begin to construct a proof.

OPERABLE INTERPRETATIONS

In addition to helping students start proofs, some students appear to benefit from early suggestions on how to use definitions and previously proved results. What I mean by this, and call an *operable interpretation*, is perhaps best conveyed by an example. Given a function $f: X \rightarrow Y$ and $A \subseteq Y$, one defines $f^{-1}(A) = \{x \in X \mid f(x) \in A\}$. An operable interpretation would say, "If you have $b \in f^{-1}(A)$, then you can write $f(b) \in A$ and vice versa".

One might think translation into an operable form would be unnecessary or easy, especially because the symbols in $\{x \in X \mid f(x) \in A\}$ can be translated into words in a one-to-one way. But for some students it requires practice. Perhaps the most pedagogically useful question is not: Can a student see this without help? But rather: Will a student use such an operable interpretation in the midst of a proof construction without it having been previously mentioned? Based on our experience helping students in our 'proofs' course, the answer to the latter question often seems to be: *No*.

DIFFICULTIES IN PROOF CONSTRUCTION

At the beginning, I mentioned our goal that students in our 'proofs' course build the ability to construct proofs of some difficulty. The meaning of *some difficulty* may change from teacher to teacher, and from course to course. However, articulating *some* meaning can allow a teacher to include advice or other material early in a course which can prime a student's thinking later in the course to autonomously overcome some difficulties.

Currently in our 'proofs' course, we think of proof construction difficulties as of four types:

> *Type 0* involves difficulties with using definitions.
>
> *Type 1* calls for using a result in the notes.
>
> *Type 2* needs a lemma, not in the notes, but relatively easy to discern, formulate, and prove.
>
> *Type 3* needs a lemma for which at least one of discern, formulate, and prove should be difficult.

Types 0, 1, and 2 can occur in any course requiring many introductory definitions. Type 3, however, consists of theorems whose proof construction calls for ideas that are fundamentally different in kind, somewhat like dimensions in a vector space. I will give just one example of this kind of theorem, but first I need some background.

UNGUIDED EXPLORATION

In constructing some proofs, one may reach a point where there is no 'natural' way forward. In *unguided exploration*, one may need to find, or define, an object and prove something about it, with no idea of its usefulness. That is, one may need to 'explore' the situation (in an unguided way). Below I give an example of a Type 3 proof near the end of our 'proofs' course. The proof entails unguided exploration, but first I need some definitions for semigroups and groups.

EXAMPLE OF A THEOREM WITH A TYPE 3 PROOF

A *semigroup* is a nonempty set, S, together with an associative binary operation (often written as if it were multiplication). The operation on S is *commutative* provided for all $x, y \in S$, $xy = yx$. An *ideal* of S is a nonempty set $I \subseteq S$ so that $SI \cup IS \subseteq I$. Also, I is *proper* if $I \neq S$.

A *group* is a nonempty set, *G*, together with an associative binary operation (often written as if it were multiplication). There is an *identity* element $e \in G$, that is, for each $g \in G$, $eg = ge = g$. Finally, for each $g \in G$, there is a $g' \in G$ so that $g'g = gg' = e$.

STARTING THE PROOF

Given the theorem, "A commutative semigroup with no proper ideals is a group", this can be rewritten as an if-then statement with a first-level proof framework:

> **Theorem:** If *S* is a commutative semigroup with no proper ideals, then *S* is a group.
>
> **Proof:** Let *S* be a commutative semigroup with no proper ideals.
>
> . . .
>
> Then *S* is a group. QED.

Adding the second-level proof framework yields:

> **Theorem:** If *S* is a commutative semigroup with no proper ideals, then *S* is a group.
>
> **Proof:** Let *S* be a commutative semigroup with no proper ideals.
>
> **Part 1:** (Find an identity element *e*.) Let $s \in$ S.
>
> . . .
>
> Then $es = s$ so *e* is an identity of *S*.
>
> **Part 2:** (Find an inverse t' for each $t \in S$.) Let $t \in S$.
>
> . . .
>
> Then $t't = e$.
>
> Then *S* is a group. QED.

Unfortunately, the proof framework is not much help in finding an identity element. But eventually, one may see there are two objects to work with—*S* itself and an element, $a \in S$. Now about the only action available is multiplication, so one looks at *aS* and wonders whether this might be an ideal. It turns out that $(aS)S \subseteq SS \subseteq S$ (and *S* is commutative) so *aS* is an ideal. But there are no proper ideals. So $aS = S$ and if $b \in S$ then there is an $x \in S$ so $ax = b$. This means any equation of the form $ax = b$ can be solved for *x*. This should be good for something. To find out, one needs to 'explore' the situation.

If one 'plays with' the equation $ax = b$ and replaces *b* by *a*, one obtains an element *e* so that $ae = a$ and $ea = a$. Now *e* 'looks like' an identity element, but unfortunately only 'works' for *a*. If one next notes that, for any $s \in S$, there is a solution $c \in S$ of the equation $s = ax$, one can write $s = ac$. Then $es = e(ac) = (ea)c = ac = s$. Thus, *e* is an identity for *S* and this finishes Part 1. Part 2 can also be proved by clever instantiations of the equation $ax = b$ and is left to the reader.

We are considering adding some heuristic advice about unguided exploration early in the course. However, even if students recall such advice, we suspect some might be hesitant to act on it because it can seem a lot like guessing. We conjecture some students have implicitly learned to avoid guessing because, on normal (timed) tests, doing so can use up considerable time and reduce one's grade. To overcome this, students should have some self-efficacy. It might be helpful to see what a mathematician did when proving this theorem (alone and for the first time).

DR. G PROVES THE THEOREM

Dr. G, a very accomplished mathematician, took various 'twists and turns' in his proof attempt. He was working from a very brief set of notes on semigroups cut down from our course notes in which this semigroup theorem was labeled *Theorem 20*. Dr. G's work was documented using a *Livescribe* pen that records sound and writing with time stamps, which can be played back on a computer. Below is a very abbreviated transcript of Dr. G's proving.[1]

7:20 am: Dr. G starts by considering Theorem 20. He has breakfast and goes for a walk.

8:07 am: He returns from the walk, realizes gS is an ideal so $gS = S$. He then thinks about inverses. Then he strikes through everything that he has written.

8:09 am: Dr. G notes that he needs an identity element.

9:44 am: He is suspicious that Theorem 20 is not true, but has few examples.

9: 48 am: He "tosses around" the idea that a commutative semigroup with no proper ideals must have an identity. But why? He thinks about translating by a fixed element. This seems not to be helpful.

9:54 am: Dr. G can neither prove Theorem 20 nor find a counterexample. So he looks ahead to Question 22 (final task). He sees how to answer this. He looks at Theorem 21 *A minimal ideal of a commutative semigroup is a group*. He thinks he can prove that, but does not believe there are minimal ideals.

10:08 am: Dr. G again attempts Theorem 20. He sees for $a \in S$, there is $e \in S$ so $ae = a$, which acts like an identity for a. Does it work for any b?

10:12 am: He finds e' so $be' = b$, but that's no help. He needs $e = e'$.

10:13 am: He sees there is f so $b = af$. So $be = afe = aef = af = b$. So e is an identity.

10:18 am: By a similar method, he shows S has inverses and is a group.

OBSERVATIONS ON DR. G'S WORK

Perhaps the most important thing about Dr. G's work is what is *not* there. There is no evidence Dr. G thought there was anything wrong with having gone in all those unhelpful directions. What mattered to him was the generation of ideas. He exhibited persistence and a willingness to try argument directions that he didn't know ahead of time would be helpful. Dr. G's persistence was probably supported by a well-developed sense of self-efficacy. Also, he did not seem to need any prior knowledge of semigroups to succeed in proving the theorem. Sometimes it is suggested that what students need is additional mathematical knowledge, but here that is not the case. What is needed is the kind of persistence and self-efficacy that Dr. G exhibited. We think persistence and self-efficacy, and perhaps other positive psychological habits, can be 'taught' to students.

IN CONCLUSION

I have mentioned a few capabilities/habits that can be useful for constructing proofs, but that receive little explicit attention in transition-to-proof or other proof-based courses: (1) how to get started—our suggestion is proof frameworks; (2) not always writing top-down; (3) looking up definitions and learning how to use them in an operable form; (4) looking for possibly useful

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[1] These data come from the PhD dissertation of our student Milos Savic (2012).

previous results; (5) how and when to 'explore'; (6) how and when to persist; and (7) self-efficacy. Teaching/facilitating/engendering these capabilities/habits seems to call as much on psychology as on mathematics.

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New PhD Reports

Présentations de thèses de doctorat

THINGS KIDS THINK WITH: THE ROLE OF THE PHYSICAL PROPERTIES OF MATHEMATICAL TOOLS IN CHILDREN'S LEARNING IN THE CONTEXT OF ADDITION OF FRACTIONS

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Within the field of mathematics education, there is a strong impetus to incorporate different mathematical tools into the teaching and learning of mathematics in general and into the teaching and learning of fractions in particular (Abrahamson, 2009; Abrahamson, Gutiérrez, Lee, Reinholz, & Trninic, 2011; Bartolini Bussi, 1996, 2011; Bartolini Bussi & Mariotti, 2008; Friedman, 1978; Prince & Felder, 2006; Sowell, 1989). Nevertheless, the concerns that have been articulated with regard to the role of the physical properties of mathematical tools and their constraints and affordances for the learning/knowing of fractions remain largely unanswered.

I define *mathematical tool* as any *tool-like object* for which its mathematical affordances are perceived by a child who is using it to solve a mathematical task. Examples include a piece of paper, an apple, fraction circles, or Java-Bars. I use Engeström's (2009) conceptualisation of *object* as any focus of attention. I emphasise here that an object becomes a mathematical object if the person(s) perceive(s) its mathematical *affordances*. For example, a paper clip can become a mathematical object if we can perceive the 'measuring' affordances that are provided by the paper clips for non-standard measurement. To examine the role that is played by the physical properties of mathematical tools in the learning of mathematics, I selected the mathematical concept of fractions.

WHAT IS THE ISSUE?

The incorporation of mathematical tools into the teaching and learning of fractions is recommended by both research and different ministry documents, which conceptualise tools as being useful. Yet, these tools also have limitations. The usefulness (or not) of a mathematical tool depends not only on its physical properties and affordances of the tools, but also on the child's perceptions while interacting with them. A tool that is useful for one child to solve a particular task may not be useful for another child for the same task or for the same child for a different task (Pimm, 2002). Similarly, a tool that is useful for working on one task may not be useful for working on another (Clements & McMillen, 1996; Cramer & Wyberg, 2009). For example, it is one thing for a child to show $\frac{1}{2}$ or $\frac{1}{3}$ using Cuisenaire rods (see Figure 1) but it is quite another to use the rods to add $\frac{1}{2}$ and $\frac{1}{3}$.
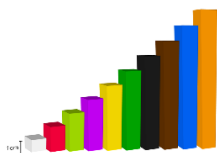
Figure 1. Cuisenaire rods ("Cuisenaire rods," 2018).

The literature in mathematics education includes such unanswered questions as "What is it about pattern blocks that did not support students' thinking on fraction order tasks?" (Cramer & Wyberg, 2009, p. 14) and "Is it the sensory characters [of the manipulative] that make mathematical tools helpful?"(Clements & McMillen, 1996, p. 270). These questions are posed to identify the strengths and limitations of the tools used in learning fractions. However, they are not addressed in terms of the role that is played by physical properties or in terms of the feedback from the mathematical tools in solving a mathematical task. These questions, the questions which arose from my interim report, along with my own experience as a teacher and a learner of mathematics, led me to look more closely at the interrelationships between the physical properties of the mathematical tools used to teach the addition of fractions and children's knowing of and learning about the addition of fractions.

THE SPECIFICS OF MY STUDY

This study is a continuation of my interim report within which I looked at the role of feedback from mathematical tools used in children's problem solving. I describe feedback as the physical properties of a mathematical tool and/or the interrelationships among its various components that make the mathematical affordances of the tool more apparent. For example, in fraction circles, I consider the relationships among the sizes of the pieces to be a form of feedback (see Figure 2).

Figure 2. The relative sizes of pieces in fraction circles are feedback (Abtahi, 2016, p. 6).

My purpose in this study was to investigate how children use the physical properties of the mathematical tools to think about and/or solve addition of fractions problems and the role of feedback from the tools in this process. I investigated these interrelationships with respect to the Zone of Proximal Development (ZPD). The ZPD is described by Vygotsky as "the distance between the actual developmental level (independent problem solving) and the level of potential development (problem solving under adult guidance or in collaboration with more capable peers)" (Vygotsky, 1978, p. 69). Following Vygotsky's view, in the field of mathematics education, the more knowledgeable others are usually conceptualised as agents such as teachers, adults and peers. Furthermore, the communication and interactions that take place within the ZPD have usually been referred to as sign-mediated and inter-subjective (Lerman & Meira, 2001; Roth & Radford, 2010), meaning that within these studies the notion of ZPD is looked at between children and adults or between children themselves.

To look at the interrelationships between the feedback from the mathematical tools, what children did and said, and the mathematical task I examined

- the children's perceptions of the mathematical tools;
- the children's interactions with the tools, which lead them to work on or to solve the mathematical task at hand; and
- the children's talk about the mathematical tools and their actions, which might indicate the emergence of the ZPD.

The two research questions I addressed were

1. How does the feedback from the mathematical tools play a mediating role between the physical actions of the child with respect to the mathematical affordances of the tools and the child's thinking about and learning and knowing of solving addition of fractions problems?
2. What role is played by mathematical tools in the emergence of a Zone of Proximal Development during the child's solving of addition of fractions problems?

To address these research questions, I used Vygotsky and Gibson as my theoretical background.

THEORETICAL FRAMEWORK

In order to systematically analyse the role of feedback in the dialectic process of the children's interaction with tools, I looked at: tools as physical 'things'; the dialectic process of children's modifications to the mathematical tool and the modifications to their mathematical thinking and ways of problem solving; and children's learning during their work on the addition of fractions problems. In order to explore tools as physical 'things', I used Gibson's view of affordances. To investigate the dialectic modifications to the tools and to the children's thinking as they interact with the tools, I used a Vygotskian perspective on the tie between tools and signs as well as Vygotsky's perspective on perception. To investigate children's learning I focused on the emergence of the Zone of Proximal Development.

Gibson: Concerned with how the environment supports thinking and action, Gibson (1977) contended that "in any interaction involving an agent with some other system, conditions that enable that interaction include some properties of the agent along with some properties of the other system" (p. 68). For my study, the "system" was the mathematical tool and the agents were the children. Therefore, using Gibson's (1977) analogy, my reference to the term *affordance* would be to whatever it was about the mathematical tool that contributed to the kind of interaction that happened. In consequence, I referred to *perception* as whatever it was about the child's thinking/knowing (of the tools and the addition of fractions) that contributed to the kind of interaction that happened.

Vygotsky: Two important assumptions by Vygotsky provide the theoretical foundation of my study: 1) a child's development and learning depend on the presence of the mediating agents of tools and signs; and 2) the tie between tools and signs creates artefacts. The notion of tool I refer to is based on Marx's view of working tools whereby that man uses the physical and mechanical properties of objects to reach his goals (Marx & Engels, 1865). The Marxian view of tool was then extended by Vygotsky as a means of external activity (i.e., labour) with which humans influence the environment. Hammers, nails and chairs are examples of tools. Signs, on the other hand, are means of internal activity that affect humans internally. Languages, various systems for counting, and algebraic symbol systems are examples of signs. For the purpose of this study, I consider both x^2 and an abacus as mathematical objects, though an abacus is a mathematical tool and x^2 is a mathematical sign. Vygotsky (1978) believed that an essential difference between signs and tools are the ways in which they orient human behaviour. A tool's function is *externally* oriented: "It is a means by which human external activity is aimed at mastering nature" (Vygotsky, 1978, p. 55). A sign, on the other hand, is *internally* oriented: "It

is a means of internal activity aimed at mastering oneself" (Vygotsky, 1978, p. 55). It is in conjunction with the child's solving of a mathematical problem that the mathematical tool acquires a new meaning and form. The mathematical tools acquire a new form because it is in solving a mathematical task that the child needs to perceive mathematical affordances provided by the tools. These affordances become apparent to the child by the help of more knowledgeable others. The mathematical tools acquire new meaning because it is in solving a mathematical task that mathematical tools are tied to the signs—especially in talking about the task, drawing and using symbols (Arzarello, 2010; Bartolini Bussi, 2011; Bartolini Bussi & Mariotti, 2008).

To go about thinking about my research questions, I collected data by interviewing 13 children from two grade 7 classes, in Ottawa, Canada. The interviews were conducted in teams of two (and one team of three) and in three parts. In each part of the interviews, conducted over three different days, children were given different types of tools to work with in order to solve two addition of fractions problems. Interviews were designed so that the children could interact with specific types of tools to think about and solve addition of fraction problems. In the second round of interviews, the children used tools that have some fraction related properties already designed into them. These tools include Cuisenaire rods, fraction strips and fraction board. In the third round of interviews, the children used tools that did not have any fraction related properties incorporated in their design, such as, papers, adhesive and masking tapes and ribbons. Data collected from the interviews was then analysed to answer both research questions.

THE EMERGENCE OF THE ZONE OF PROXIMAL DEVELOPMENT

Lerman and Meira (2001) stated that "the ZPD is not something that pre-exists; it is not carried around, like a box, by the child" (p. 203). The ZPD emerges as a field for "interaction and communication where learning leads development" (p. 204). How then does one recognize the emergence of the ZPD? Radford (2013) referred to a social and sign-mediated process of becoming acquainted with historical and cultural forms of expression, action and reflection. Hence, one possible way of looking at children's learning is to look at the ways in which children express, reflect and act change as they participate in an interaction (among themselves, with the teachers, or with the tools). How does learning happen? For Lerman (2014), the zone of proximal development is "the mechanism through which learning happens" (p. 22). Vygotsky (1978) proposes that a fundamental feature of learning is that it creates the zone of proximal development because "learning awakens a variety of internal developmental processes that are able to operate only when the child is interacting with people in his environment" (p. 67). In my study, I focused on what children did with the tools and say about them to see if and how their interaction with the tools provided for them newer possibilities of expression, reflection and action.

FINDINGS

In general, the analysis of data showed the following two findings in relation to the role of feedback from the tools in children's interaction with the tools to solve addition of fraction tasks:

1. Children's interactions with the tools were mediated by the feedback from the different artefacts that they created throughout their interactions with the tools. One of the theoretical foundations of this study was that instead of acting directly, in unmediated ways in a social and physical world, all our actions are indirect and mediated (Wertsch, 1993) and that from a Vygotskian perspective, these actions are mediated by tools and signs (Vygotsky, 1978). With this finding, I propose to extend the notion of mediation to show that it was neither the tools nor the signs that independently mediated the children's interactions with the tools. Rather, it was the children's created artefacts

that played a mediating role between the children's modifications to the tools as well as the modifications to their mathematical thinking.

2. In the children's interactions with the tools and in all problem-solving stages, both attaching mathematical meaning(s) to the tools and the consequential process of creating an artefact were a gradual and complex process, which were closely related to (a) the ways in which the children perceived the mathematical affordances of the tools—through the feedback provided by the tools—to create and to use the artefacts; (b) the children's mathematical knowing of fractions in general and of the addition of fractions in particular; and (c) the task of adding two fractions.

Vygotsky, in his description of ZPD, explicitly referred to "problem solving under adult guidance or in collaboration with more capable peers" (Vygotsky, 1978, p. 69), referring to the more knowledgeable other(s) as an "adult" or "more capable peers". Following Vygotsky, other researchers also looked at the 'more knowledgeable others' as being people (Goos, 2014; Graven & Lerman, 2014; Lerman & Meira, 2001; Roth, 2014; Roth & Radford, 2010; Steele, 2001; Tudge, 1992; Valsiner, 1984).

In this study, I am expanding the inter-subjective notion of ZPD to include the guidance provided by the tools. I build on Roth and Radford's (2010) conceptualisation of "participation"—in which the zone of proximal development is viewed as an interactional achievement that allows all participants to become teachers and learners (p. 305) to carefully describe the more knowledgeable others. By the more knowledgeable others, I mean the participant(s) in an interaction whose knowing or feedback are used in different stages of the process of the problem-solving activity to think about the problem and/or to solve it. In children's interaction with the tools, I consider the 'participants' to be the children and the tools, and I see the role of the more knowledgeable other as alternating between the children and the tools.

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EXAMINING MATHEMATICS ANXIETY AMONG CLASSROOM TEACHERS

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INTRODUCTION

This paper focuses on the part of my dissertation that describes my research with in-service elementary school teachers on the nature and causes of mathematics anxiety that they experienced and how the anxiety differs in terms of various demographic factors. I also examined the relationship between mathematics anxiety and mathematics teaching anxiety, as well as the types of mathematics teaching anxiety among teachers. Convenience sampling was used to select two school boards in Southern Ontario. The data were collected through an online survey completed by 111 elementary in-service teachers and face-to-face interviews. A brief discussion on the findings from this study is provided, together with recommendations on strategies that could be used by teachers and school boards to reduce mathematics anxiety and break its re-occurring cycle.

MATHEMATICS ANXIETY AND TEACHERS

Mathematics anxiety continues to receive increasing attention in various countries, including Canada and the United States, in recent years as it appears to be an impediment to the learning of mathematics. Mathematics anxiety is described as a person's negative affective reactions to situations that involve mathematics, numbers, and calculations (Ashcraft & Moore, 2009). It may weaken an individual's state of mind and eventually progress to mathematics avoidance and mathematics phobia (Tobias, 1978), resulting in low achievement in mathematics (Ashcraft & Kirk, 2001). Individuals with mathematics anxiety steer away from mathematics courses and careers that involve mathematical competence, thus limiting their career potions in lives.

Studies have shown that some teachers possess levels of mathematics anxiety (Hadley & Dorward, 2011; McAnallen, 2010) and that their anxieties could be transferred to their students (Vinson, 2001). Teachers who have mathematics anxiety have been shown to be less successful at conveying important mathematical concepts that are requisites for further academic growth to their students. In Canada, mathematics requirements for students majoring in elementary education are minimal (TEAS, 2012), indicating that one can pursue a career as an elementary school teacher despite having mathematics anxiety and a history of avoiding the subject. Having teachers who are anxious about mathematics in the classroom could have detrimental effects on the students' achievement and attitude towards mathematics (Beilock, Gunderson, Ramirez, & Levine, 2010).

Gender differences in mathematics anxiety have been reported by some researchers (Khatoon & Mahmood, 2010; Ma & Cartwright, 2003) who argue that, although males and females

experience some level of anxiety towards mathematics, the level of anxiety in females appears to be greater than in males. Despite the fact that researchers have investigated the possible causes of mathematics anxiety, "there is still no clear answer as to the nature and causes of this anxiety" (Balogˇlu & Kocak, 2006, p. 1331). Some studies offered that mathematics anxiety does not originate from mathematics itself but rather from the way the subject is taught in school and the way it might have been presented to teachers when they were children (Fiore, 1999; McAnallen, 2010; Stuart, 2000). According to Hadfield and McNeil (1994), the causes of mathematics anxiety can be categorised into three factors—environmental, intellectual, and personality. Environmental factors include negative school experiences, parental pressure, insensitive teachers, non-democratic and non-supportive class environment. Intellectual factors include negative attitude, low persistence, self-doubt, learning style, and lack of confidence in mathematical ability. Personality factors include reluctance to ask questions due to shyness, low self-esteem and gender bias. Teachers are categorized under environmental factors and they have been reported to have an influence on students' mathematics anxiety levels through their teaching methods (Arem, 2003; Tobias, 1998; Vinson, 2001). Amongst these teaching methods are those that place emphasis on getting the right answers rather than concept development; repetition and rushing through materials, or 'timed tests', rather than understanding; and continuous lecturing rather than providing hands-on mathematics experiences, together with accommodating different learning styles (Arem, 2003; Geist, 2010; Harper & Daane, 1998; Popham, 2008; Scarpello, 2007). Teachers with high mathematics anxiety often use traditional teaching methods, and as a repercussion, mathematics anxiety limits the teachers' skill in doing or teaching mathematics.

MATHEMATICS ANXIETY AND MATHEMATICS TEACHING ANXIETY

There are indications that teachers who are anxious about mathematics are likely to be anxious about teaching mathematics (Hadley & Dorward, 2011). The construct, mathematics teaching anxiety, is defined as the teachers' feelings of tension and anxiety that happen in the course of teaching mathematical concepts, theorems, and formulas or problem solving (Peker, 2006). Mathematics teaching anxiety could be due to subject knowledge, self-confidence, attitude towards teaching mathematics, and subject teaching knowledge. Although some studies (Brown, Westenskow, & Moyer-Packenham, 2011; Peker, 2006) have looked into mathematics anxiety alongside mathematics teaching anxiety, they focussed on pre-service teachers and reported inconsistent results. Similarly, while most studies on mathematics anxiety have been conducted with pre-service teachers (e.g., Brady & Bowd, 2005; Gresham, 2009; Liu, 2008; Vinson, 2001), college students (e.g., Ashcraft & Kirk, 2001; Hembre, 1990; Wilder, 2012; Shields, 2006), adolescents (e.g., Khatoon & Mahmood, 2010) and school children (e.g., Ma & Cartwright, 2003), few studies, conducted in the United States, have involved elementary school teachers (McAnallen, 2010; Hadley & Dorward, 2011). Therefore, my dissertation was inspired by the observed limited research in the literature that addressed mathematics anxiety and mathematics teaching anxiety among in-service elementary school teachers, particularly in Canada. Thus, the purpose of the study was to examine the nature (levels, causes, and effects) of mathematics anxiety that exists among in-service elementary school teachers in Southern Ontario and how this anxiety differs by gender and other demographic factors, including socio-cultural factors. The study also investigated the relationship between mathematics anxiety and mathematics teaching anxiety, as well as the types of mathematics teaching anxiety that may exist among in-service elementary school teachers.

THEORETICAL FRAMEWORK

Vygotsky's (1981) sociocultural theory was one of the theories that was used as a framework in the study. Sociocultural theory positions an individual in specific historical, cultural, and institutional contexts. It highlights the influences of adults, siblings, and peers (environmental factors) on an individual's development and formation of concepts. It was used in the study to

understand the social and cultural influences that may contribute to elementary teachers' mathematics anxiety. By taking a socio-cultural view of Vygotsky, this dissertation tried to consider not only individuals, but also the groups and the society they belong to. According to Vygotsky, social interactions are crucial for learners to explore knowledge domains with more capable peers and adults (e.g., teachers and parents) in and outside the classrooms. He asserted that "it is through others that we develop into ourselves" (1981, p. 161); which further accentuate the influences of others—capable peers, siblings, and adults—in the development of an individual. However, the process of learning is thrown into disarray when any hindrance, such as feelings of anxiety, is experienced by peers, siblings, and adults. This affects the individual's interaction with people in his/her environment (Wilder, 2012). Better understanding of the social and contextual causes and development of mathematics anxiety could be achieved with the examination of past experiences of individuals with mathematics anxiety, including their interactions with social group and other cultural factors.

METHOD

The study employed an explanatory sequential mixed methods design (Creswell & Plano Clark, 2011; Teddlie & Tashakkori, 2009) that consists of quantitative and qualitative phases. The quantitative data were collected through surveys and analyzed in the first phase, while the qualitative data were gathered through interviews in the second phase, as a follow up and to help explain or elaborate on the quantitative results that were obtained in the first phase. The quantitative and qualitative findings were integrated to draw conclusions.

PARTICIPANTS AND INSTRUMENTS

In the first phase of the study, 111 in-service elementary school teachers from grades 1 to 8 participated. Convenience sampling was used to select two school boards (Cohen, Manion, & Morrison, 2000; Nardi, 2005). Eighty-seven (78.4%) of the participants were females, 20 (18%) were males, and four (3.6%) of them did not respond to the survey's gender question. Data were gathered using three instruments. The first instrument was a questionnaire that was developed by the researcher and it contained questions relevant to demographic factors such as gender and race/ethnicity. The second instrument, the Revised Mathematics Anxiety Rating Scale (RMARS, developed by Alexander & Martray, 1989), was used to measure the elementary teachers' mathematics anxiety scores whilst the third instrument, the Mathematics Teaching Anxiety Survey (MATAS, developed by Peker, 2006), was used to measure the types of mathematics teaching anxiety they experienced. The RMARS is a 25-item survey comprising of three subscales with each of its item designed using a 5-point scale. Two-week test-retest reliability of RMARS obtained from a sample of 62 undergraduate students was reported as .86 (Alexander & Martray, 1989) and high reliability coefficient (.95) was also reported by Balogˇlu and Koçak (2006) for the entire RMARS scale. In my study, the reliability coefficient of RMARS scale was found to be reliable at .97.

MATAS is a 5-level Likert scale Turkish instrument that consists of 23 items with 5 responses to each item. Positive responses were coded from 1 to 5, while the negative ones were reverse-coded from 5 to 1. MATAS is a four-factor scale that measures mathematics teaching anxiety due to subject knowledge, self-confidence, attitude towards teaching mathematics, and subject teaching knowledge. The reliability coefficient of the entire scale was reported to be .91 (Peker, 2006). MATAS was translated into English language for the purpose of the study and the translated version referred to as MATAS-E has a .96 reliability coefficient.

In the second phase, data were collected through individual interviews with four (one male, three females) purposefully selected teachers. The interview protocol was developed by the researcher and contained 10 open-ended questions, which was used to elaborate on the results

from the quantitative part and also to explore the causes of mathematics anxiety among participants and how they overcame or coped with their anxieties. The interviews with each participant took about 45 minutes to complete and were recorded on a digital recorder.

PROCEDURE AND DATA ANALYSIS

The survey instruments, which took about 20 minutes to complete, were administered online and accessed by teachers through the in-service teachers' active email addresses. Scores for each participant were computed by adding the item values on the RMARS and MATAS-E. Data were analysed by calculating the means, standard deviations, and by conducting non-parametric test such as Mann-Whitney U test since the data collected with RMARS and MATAS-E were skewed and the variances for the groups (that is, females and males) that were compared could not be assumed equal. The data from the interviews were transcribed and the transcripts were analyzed using thematic analysis (Braun & Clarke, 2006; Ezzy, 2002).

RESULTS

ONLINE SURVEY RESULTS

The results from the online survey showed that

1. participants experienced different levels of mathematics anxiety with 19 (17.1%) of them having low level, 71 (64%) moderate level, and 21 (18.9%) high level;
2. female participants had higher mathematics anxiety scores ($M = 56.67$, $SD = 22.67$, $n = 87$) than their male counterparts ($M = 37.6$, $SD = 10.8$, $n = 20$);
3. there were no significant differences in mathematics anxiety based on participants' race/ethnicity, perceived socio-economic status, and mother's educational levels; and participants with fathers in the lowest educational level experienced higher mathematics anxiety than those whose fathers had higher levels of education, specifically graduate degrees;
4. a statistically significant, strong positive correlation ($r_s = .72$, $p < .01$, $n = 107$) exists between mathematics anxiety and mathematics teaching anxiety. The result indicated that as mathematics anxiety scores increased, mathematics teaching anxiety scores also increased. That is, participants with low levels of mathematics anxiety experienced lower levels of mathematics teaching anxiety while those with high levels of mathematics anxiety had much higher levels of mathematics teaching anxiety.
5. overall, participants' mathematics teaching anxiety due to lack of self-confidence was higher than the other three types of mathematics teaching anxiety (that is, subject knowledge, attitude towards teaching mathematics, and teaching knowledge). However, with the consideration of gender, mathematics teaching anxiety due to subject knowledge and self-confidence were significantly higher in female participants than their male counterparts and the two groups did not significantly differ in their mathematics teaching anxiety due to attitude towards mathematics and teaching knowledge.

INTERVIEWS RESULT

The participants attributed the causes of mathematics anxiety to their past teachers' teaching strategies and insensitive comments, mean behavior, as well as their own lack of understanding of mathematics concepts in developmental years. For instance, one of the participants stated that

> [The teachers] *assumed we knew everything and didn't, maybe, review everything. Sometimes I felt the teachers were just going at the speed of strong students, not at the speed of some other students and you didn't always want to ask questions because*

> *sometimes you didn't know what to ask* […] *they teach the top of the class and they leave other people behind.*

The negative effects of mathematics anxiety on the participants include lack of confidence in doing mathematics; negative attitude towards mathematics; feelings of inferiority and exclusion; and avoidance of mathematics courses and mathematics-related careers. However, the participants expressed that their own teachers and parents helped them overcome, reduce, or cope with mathematics anxiety.

DISCUSSION AND CONCLUSION

The findings from this study not only provided evidence on the existence of different levels of mathematics anxiety among elementary teachers but also supported the commonly acknowledged difference in mathematics anxiety between females and males, which is in favor of males (Balogˇlu & Kocak, 2006; Hadley & Dorward, 2011; Hembree, 1990; Ma & Cartwright, 2003). The findings also revealed a positive correlation between mathematics anxiety and mathematics teaching anxiety. A similar result was reported by Hadley and Dorward (2011) in their study with grades 1 to 6 teachers in the United States. A few studies have provided evidence in relation to gender differences in mathematics teaching anxiety (Yazıcı & Ertekin, 2010). This study is no exception, as classroom teachers' differential treatment plays an important role in the differences found in mathematics learning that favor male students (Geist, 2010). In addition, from the interviews, the findings showed that the experience of mathematics anxiety affected the participants both personally and professionally.

Essentially, findings from this study indicate mathematics anxiety and mathematics teaching anxiety as concerns that warrant further research. These findings emphasized the need to eradicate, or at least reduce, mathematics anxiety and mathematics teaching anxiety among in-service elementary school teachers. Elementary teachers need to acknowledge their fears, anxieties, and difficulties with mathematics, create a plan to get rid of them, and ask for assistance when they need it. Frequent discussions about mathematics are recommended as helpful, specifically for women, to prevail over their fear and negative feelings about mathematics. Arem (2003) suggested positive self-talk and journal writing about the past and everyday experiences of mathematics as progressive means of handling mathematics anxiety.

Mathematics is a compulsory subject that is required to be taught by all elementary school teachers. Therefore, implementing changes to the teachers' practices and professional development is paramount. School boards have a vital role to play by investigating the status of mathematics anxiety and mathematics teaching anxiety in their schools. All elementary in-service teachers need to be more informed about the causes, damaging consequences, and preventive measures of mathematics anxiety. School boards should provide such information through reading materials, videos clips on mathematics anxiety, and effective workshops and conferences for in-service elementary teachers. Such workshops should emphasize strategies, including better understanding of the mathematics curriculum, which teachers could use to address many aspects of mathematics anxiety and mathematics teaching anxiety that they may be experiencing. Progress towards breaking the re-occurring cycle of mathematics anxiety will be achieved when more efforts are expended to help teachers who suffer from it.

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TEACHING TEACHERS: A LOOK INSIDE PROFESSIONAL DEVELOPMENT[1]

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INTRODUCTION

As I was doing research regarding successful professional development, I found no descriptions about the lived experiences between professional developers and teachers. What I usually found were survey results from teachers' program evaluations, which provided information about teachers' individual perspectives regarding the effectiveness of professional development sessions, or statistical analysis of school performances after the application of a particular professional development opportunity. Not much if anything has been done to show what happens during those learning opportunities and how teachers' and professional developers' predispositions and the setting allow for a fruitful learning opportunity and when they will not (Garet, Porter, Desimone, Birman, & Yoon, 2001; Grammatikopoulos, Gregoriadis, & Zachopoulou, 2013; Linder & Simpson, 2014). This study presents a framework that attempts to provide an initial tool for a description and analysis of this experience.

FRAMEWORK

I made use of the phenomenological perspective, to analyze the 'lived experience' of professional development sessions and describe teachers' reactions to a variety of approaches and activities. This study explores and describes 'lifeworld' learning experiences, and looks for the meaning of a phenomenon by uncovering, as much as possible, the many layers that socially and culturally influence a person's experience in their lifeworld, where 'lifeworld' is defined by Van Manen (1997) as "the world of immediate experience", the world as "already there" (p. 182).

According to Smith (2013) phenomenologist practices consist of three different methods:

- rich descriptions of lived experience,
- use of relevant features in the context to interpret the experience,
- analysis of the form of a type of experience.

One can combine all three to analyze the description of the lived experience, and to interpret it by assessing and using the relevant features in the context and analyzing structures which resonate with our own experience—that which one can be conscious of. In this study, a tool

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[1] This is a summary of my PhD Thesis presented at Simon Fraser University under the direction of Peter Liljedahl.

was developed to analyze and provide a description of a professional development session as it occurred. As a result of my analysis of professional development sessions, I was able to bring forward some important elements to consider like mood, motives, wants, and who is carrying the flow of the conversation (flux) during the sessions. I also focused on engagement, and how, through engagement, teachers show their motivation, wants, moods, needs and learning. In this study, engagement is defined as different ways in which individuals and groups come into action, or become involved with an activity, where these different ways of acting/engaging are influenced by different levels of motivation, motives, and goals.

This framework, which is based on a modified version of Remillard's (2012) analytical perspective, connects positioning and engagement in order to analyse the various ways a professional developer tries to position teachers in an experience that is engaging to them. Professional developers want to position teachers in a way where interaction is possible, and furthermore this is something that they want to do of their own volition. Remillard classifies positioning and interactions by using the following terms, which I redefine as follows:

- mode of address (MoA): how a group of people are positioned in order to become engage in an activity;
- forms of address (FoA): artifacts/resources used in order to position people;
- modes of engagement (MoE): the way people react/engage;
- forms of engagement (FoE): what artifacts/resources, ideas result from the activity.

To represent the phenomenology of professional development in a way that was succinct and useful, I developed the idea of scenarios for my analysis. A scenario is defined as a unit of exchange, where the professional developer has a plan, and in accordance with it (mode of address), s/he introduces or presents an idea or task (form of address). This action is taken in by the teachers (mode of engagement), and the teachers then respond (form of engagement). Communication is not perfect, the original idea passes through the teachers' 'wants-motives-mood' filter, so what the teachers take out of what the professional developer presents to them is not necessarily what the professional developer expects, given that the response the professional developer receives also passes through her/his 'wants-motives-mood' filter. The unit is completed when the professional developer takes in the response and sees a need to re-direct.

By dividing activities into scenarios, and then focusing on each of its components, the analysis was considerably simplified. I was able to find units of meaning, and significant themes emerged from the analysis: how teachers use the teachers' guides, how previous practice can play a role for change, and how a teacher's resistance can interfere with the learning of others.

As I look at the data using a visual model of the framework and the idea of Scenario as a unit of exchange, six possible basic scenarios emerged from my analysis. I was able to analyze the professional development process by taking a look at each component (MoA, FoA, MoE, FoE, moods, wants, motives and flux) and the part it played within this unit of exchange. In my experience, these six scenarios seemed to cover most of what can take place during teachers' professional development.

The analysis is illustrated through a bowtie (for the first five scenarios) and a circular visual model (for the sixth scenario). The scenario starts when the professional developer plans and applies particular modes and forms of address during the professional development session. The particular forms and modes of address run within a particular background of goals, motives and wants from all the participants involved, including the professional developer. Particular modes of engagement follow with particular moods and levels of activity or flux, and forms of address are re-sourced into particular forms of engagement. As the professional developer

observes and reflects upon the resulting modes and forms of engagement, she responds accordingly either by continuing or modifying the current mode and form of address. Modifications depend on how much the professional developer is able to perceive a level of learning where hopefully the teachers' and professional developer's motives, wants, and goals are addressed and realized.

METHODOLOGY

Data regarding the actions of teachers in a professional development setting were gathered in order to describe their participation and engagement throughout the sessions. This was done by audio-recording most of the professional development sessions, surveys, and through notes made by the facilitator during and after each session about teachers' engagement. After every session, notes were made by the professional developer reflecting on the effectiveness of the session, which concerns were addressed, the level of engagement, her feeling and thoughts as well as questions that she wished she had asked during the session, and questions that she hoped to ask at the next session with the group. This reflective process served as a preliminary analysis of the data (Glesne, 1999). Most observations of teachers' behaviour were made during the professional development sessions.

For the data analysis, the following phenomenological protocol was used, which is a simplified version of Hycner's (1999) process used by Groenewald (2004) together with some steps delineated by Van Manen (1997). First, we investigated the experience as we lived-it, then we bracketed the assumptions being made, as we looked at the data we delineate units of meaning, then we clustered them in themes and we created a composite. The participants in this study were the teachers involved in professional development and the facilitator. The professional developer is also a participant because her reflections and responses to teachers' engagement as well as my motives and wants are part of the study.

ANALYSIS

What follows is a description, along with some illustrative examples, of the six possible scenarios.

SCENARIO 1

This type of scenario usually occurred when the professional developer came with a plan, put it in action, and the mode of engagement was such that the form of engagement turned out to be empty. That is, no apparent form of engagement was produced: there was simply no reaction. Either the teachers did not respond in any obvious way or their comments/answers added no additional information that would help the professional developer to follow up with a more suitable mode of address for them to actively engage. The teachers were not actively engaged and their mood can be one of confusion, boredom, resistance, passivity and/or fear. There was no conversational flux, and it was difficult for the professional developer to figure out their motives or wants without some previous information that could be linked to this behaviour.

Example 1

This is a general example that occurred with several of the groups I worked with, which fit Scenario 1 (Figure 1).
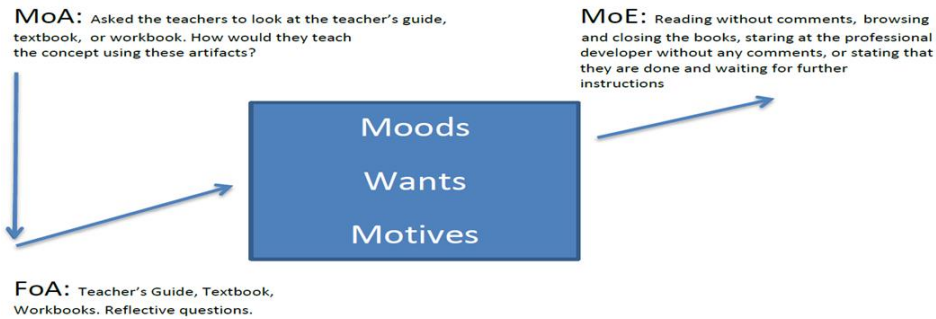
Figure 1. Example 1 diagram.

SCENARIO 2

Scenario 2 involves what one might call a lateral movement: a form of engagement is produced but what is being produced is an unexpected outcome, or a confusing response. Usually the professional developer will have to re-direct, further inquire or address what was produced. The form of engagement being produced is not useless; as a matter of fact, it can give us great insight into teachers' wants, preoccupations and mistaken assumptions, as we will see in the example in Figure 2. In this scenario, the teachers produced statements, but there was no conversational flux. Nevertheless, the form of engagement provides the professional developer with hints on how to proceed. The professional developer carries the flux of the conversation by redirecting or asking questions.
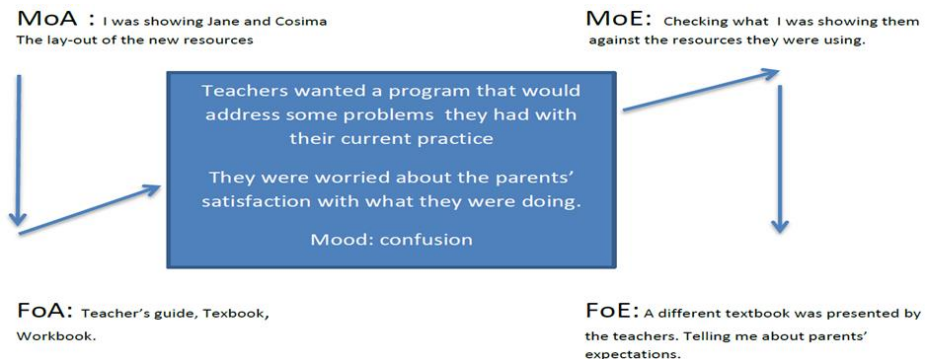
Figure 2. Example 2 diagram.

SCENARIO 3

In this scenario a form of engagement has been produced which indicates either a clear interest in or understanding of the professional developer's plans. The form of engagement being produced could be questions or comments that indicate interest in, or understanding of, the material the professional developer has introduced. However, the teachers are not yet able or willing to carry the flux of the discussion themselves.
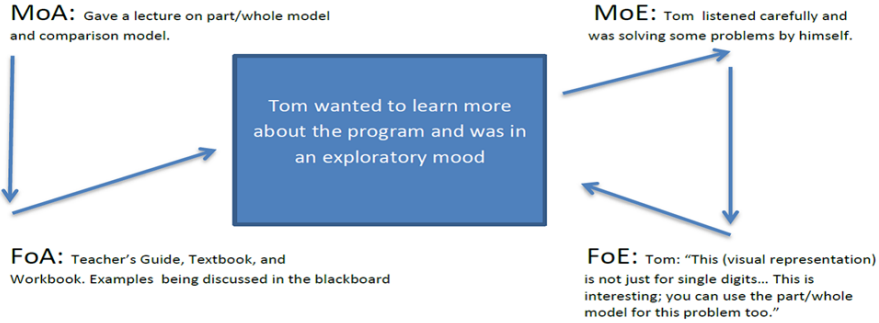
Figure 3. Example 3 diagram.

SCENARIO 4

In this diagram the facilitator starts the conversation but then the teachers mainly carry the flux of the conversation for the rest of the activity. The professional developer is a non-participant in the discussion, except perhaps for a few guiding comments (indicated by the dotted arrow in Figure 4).

Figure 4. Example 4.1 diagram.

The semicircular arrow indicates that the teachers discussed a variety of mathematical concepts in order create meaningful report cards (see Figure 4). After they got started, I was mainly ignored. They carried the flux of the conversation on their own.

SCENARIO 5

This scenario represents that there is a back-and-forth of ideas, where the flux of the conversation flows from professional developer to teachers and from teachers to the professional developer. The graph depicts a discussion between two groups or participants (see Figure 5). In this case there is a separation, between the professional developer and the teachers.

Figure 5. Scenario 5 diagram.

SCENARIO 6

In this scenario, positioning by the professional developer is no longer needed as everyone is engaged and is able to promote/trigger the conversation. Anyone participating in the session can trigger the form of address that will propel active engagement to continue. Resulting forms of engagement can become or are used as forms of address. Moods, motives and wants do not filter the interaction among the participants. This scenario was the least common in my sample. In my experience, one needs time to get a level of comfort amongst the participants for this scenario to occur. I find this scenario also occurred most often when the teachers felt more comfortable in their mathematical knowledge and practice. Sometimes this scenario evolves from Scenarios 4 and 5.
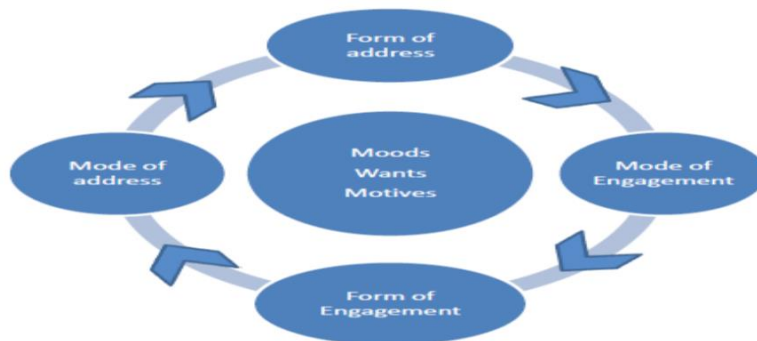
Figure 6. Scenario 6 diagram.

CONCLUSIONS

The visual representation of the six scenarios provided me with a way to deal with many of the elements that compose the live-experience of professional development. The first three scenarios can happen almost instantly as a facilitator starts working with a group for the first time. This instantaneity is not the case with the last three scenarios. There has to be a level of confidence, trust and/or knowledge for these scenarios to happen. Mood and wants are important factors that affect the predisposition towards learning. Mood is very much influenced by the confidence teachers have in their knowledge, but it is also influenced by working with someone unfamiliar. Also the mood changes as the lived-experience of professional development unfolds. However the moods, motives and wants are not always clear during the interaction. Sometimes one cannot determine them until the scenario has run its course. Here is where the professional developer's experience comes into play to inform how to proceed based on the information that s/he perceives coming from the teacher's mode and form of engagement.

Overall, I have identified six different types of scenario. Each type can be part of a learning activity or task. As each scenario runs its course, the professional developer should use what was learned from what took place in this interaction to answer the following questions: Has the goal for this task been accomplished? How should s/he guide the process through a mode and form of address in order to arrive at the desired goal? The forms of engagement being produced, the mood of the conversation, and who is carrying the flux of the conversation are important elements that had helped me evaluate whether I should continue with what I was doing or if I should redirect. It is my hope that research like the one initiated in this dissertation will be helpful in designing professional development workshops that are more meaningful and helpful to all teachers.

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BE INNOVATIVE BUT DON'T BE WRONG: ARE 21ST CENTURY STUDENTS EXPERIENCING 21ST CENTURY MATHEMATICS?

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ABSTRACT

Using a theoretical framework drawn from appraisal linguistics focussing on interpersonal instances of text, this presentation will report the findings of a study on the judgements made about teaching and learning mathematics and how those judgements reflect the values embedded in mathematics education. As judgement is a highly contextualized enterprise, my presentation will begin with an overview of the current socio-political climate within which teachers are expected to teach and within which students are attempting to succeed. I will also explore current theories that provide a means for researchers to investigate mathematics teaching in context. When outlining the findings, the discussion will be qualitative and interpretive, and I will argue that, despite the fact that the classes were well taught, the students continue a legacy of a narrow concept of the nature and purpose of mathematics and mathematics education.

INTRODUCTION

> *We reiterate our deep concern at the multiple and interrelated crises, including the financial and economic crisis, volatile energy and food prices and ongoing concerns over food security, as well as the increasing challenges posed by climate change and the loss of biodiversity, which have increased vulnerabilities and inequalities and have adversely affected development gains, in particular in developing countries.*
> (United Nations General Assembly, 2010, p. 6)

The 21st century has brought with it dramatic challenges, as formally declared by the United Nations in the quote above. There is significant research on economic (Organisation for Economic Cooperation and Development (OECD) Economics, 2013; United Nations, 2013) and environmental (Intergovernmental Panel on Climate Change, 2013) trends to support the concerns of the United Nations and encourage policy makers to take action to equip the population with the skills and knowledge to manage these changes. Globally, governments are responding by reviewing and reforming educational policy (Brochu, Deussing, Houme, & Chuy, 2013) and provinces have begun to implement strategies that focus on specific competencies such as creativity, innovation, collaboration, communication, character, culture and ethical citizenship, and computer and digital technologies (e.g., British Columbia Ministry of Education, 2013; Alberta Education, 2011; Gouvernement du Québec, 2007). This paper arises from the context of '21st century' education and rests in the underlying tensions related to the responsiveness of public education to the changing needs of society and asks the question: Are these goals realized in the classroom in the teaching and learning of mathematics?

DEFINING 21ST CENTURY EDUCATION

Throughout education-based literature, the terminology of 21st century education or 21st century skills, competencies, or learning, is used frequently. Many international jurisdictions (e.g., Australian Curriculum, Assessment and Reporting Authority, 2013; United Kingdom Secretary of State for Education and Skills, 2003) have produced comprehensive definitions of 21st century learning with accompanying frameworks and educational goals. In the United States, the American National Research Council (2012) produced a synthesis of these skills, grouped into three broad domains: cognitive (e.g., creativity, innovation, critical thinking), interpersonal (e.g., leadership, collaboration, service orientation), and intrapersonal (e.g., adaptability, curiosity, perseverance). In Canada, the Province of Ontario (Ontario Ministry of Education, 2016) has recently published a review and synthesis of the literature and international policies regarding the 21st century competencies discussion. Through their synthesis, the authors proposed a draft framework of six broad categories of 21st century characteristics which include: critical thinking and problem solving; innovation, creativity, and entrepreneurship; learning to learn/self-aware and self-directed learning; collaboration; communication; and, global citizenship.

Through an analysis of language use, this research looks at the judgements students make about mathematics and asks the question: How do the judgements students make about mathematics align with the goals of 21st century education?

LANGUAGE ANALYSIS

As participants in relationships in the context of schools and classrooms, students and teachers negotiate and act out their feelings about each other, the subject matter, their position within the classroom, and the institution of public education. These relationships are at the core of teaching and learning and affect achievement (Hattie, 2003; Hughes, Wu, Kwok, Villarreal, & Johnson, 2012), behaviour and motivation (Curby, Rimm-Kaufman, & Ponitz, 2009; Skinner & Belmont, 1993) and persistence (Knesting & Waldron, 2006; Montalvo, Mansfield, & Miller, 2008). Language reflects these dynamics and deconstructing language use can help people to see to what extent their language rests upon culturally shared assumptions and relations of power (Fairclough, 1989).

Systemic Functional Linguistics (SFL) is an analytical tool that acknowledges the social interaction and complexity of language by describing both the form and the function of language and recognizing that context is inseparable from understanding language (Halliday & Matthiessen, 2014). According to the theory, there are two basic functions of language: making sense of experiences and acting out social relationships (see Figure 1). When construing experiences, we are also enacting our personal and social relationships, thus language usage both represents a process (a doing or happening, saying or sensing, being or having), and a proposition or proposal (informing or questioning, giving an order or making an offer, expressing our appraisal of or attitude towards whomever and/or whatever is being addressed) (Halliday & Matthiessen, 2014).

Appraisal is a discourse system within the interpersonal metafunction and regards how humans act out social relationships. As such, it provides a method to uncover a voice, or a subjective presence, evident in the pattern of choices in text. These patterns reveal the speakers/writers' stances toward the material they are presenting and the person with whom they are communicating. They also reflect the community and/or the value system of the appraiser (Martin & Rose, 2003). The graphic in Figure 1 outlines the SFL and how appraisal fits within the model.
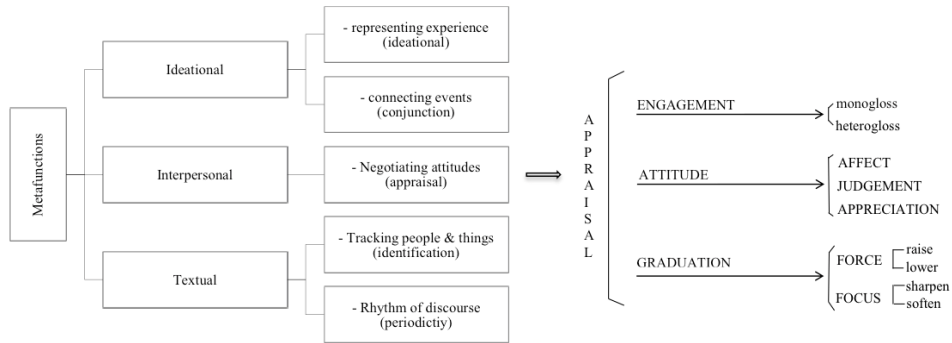
Figure 1. Appraisal as related to the metafunctions of language. Adapted from Martin and White (2005).

The analysis in this investigation focuses on attitude a subset of appraisal. Attitude provides a framework for mapping feelings and consists of three regions: expressing emotions ("affect"), judging personal and moral character ("judgement"), and valuing the worth of things ("appreciation").

Figure 2 provides examples of expressions of attitude through judgements and appreciations.
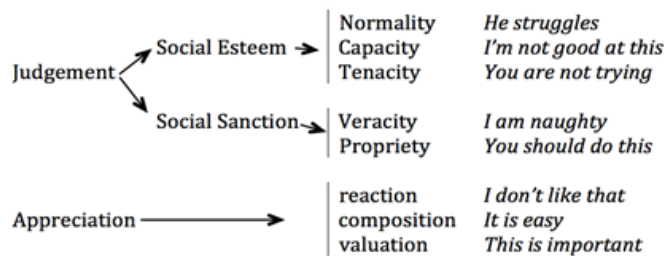
Figure 2. Examples of Judgement and Appreciation. Adapted from Martin and White (2005).

As outlined, appraisal is one aspect of the different social functions of language use and, identification and categorization of the instances of judgement and appreciation within a context, provides access to reveal how we enact social relationships. This investigation uses appraisal as the system of analysis to understand values held about mathematics through the judgements and appreciations made while teaching and learning.

RESEARCH DESIGN

This data set consisted of textual analysis drawn from different interactions in two grade 8 classrooms. The data included five recorded lessons from each classroom, journal entries from the teachers and students, teacher interviews, and student focus groups. The text was gathered over two school years. The teachers were experienced senior teachers and the schools were representative of typical Canadian schools. The notion of a 'typical school' is of course variable, however, where other high achieving countries tend to have a large achievement gap, Canadian schools comparably have a very narrow achievement gap, demonstrating a consistency from school to school. The schools were selected based on being close to the average Canadian social economics (Statistics Canada, 2011). The data included text from five recorded sessions of the regularly timetabled class for each school, journal writing from both the students and teachers, recorded focus group sessions of students, and interview with the teachers. The recordings took

place mid-way through a unit on graphing equations in one class, and mid-way through a unit on integers in the other class.

THE DATA

When analyzing a text using SFL, the clause is the basis of analysis and is what construes the meaning in text. The results analysis relies on clauses extracted from classroom conversation, in particular, clauses that emote attitude (affect, judgement, appreciation). I encountered many issues when trying to make the model fit to real classroom conversation, and I had to make a number of decisions to ensure consistency. Those decisions are elaborated on in another publication (Godfrey Anderson, 2016).

To begin the analysis of text, first I identified clauses that appeared to express an attitude (for example, when comparing two equations the teacher says, "*It is not very different is it?*"). Then I identified the subject of the attitude. After identifying judgement clauses, I analyzed each instance to identify the appraiser (teacher or student), the subject of the appraisal (e.g., teacher, student, mathematics as a subject, technology), whether the appraisal was positive or negative (e.g., "*that is acceptable*" or "*it is not very fun*"), and the type of appraisal. I repeated this process in different formats (electronic and printed versions) with the same text several times to ensure accuracy. This process continued through both the oral text (lesson transcripts, focus groups, and interviews), and the written text (journal writing). Decisions were made to maintain consistency and the information from the tables was then sorted according to instance of appraisal (for elaboration on methods and limitations see Godfrey Anderson, 2016). Table 1 is an example of this sorting.

	Affect	Judgement				Appreciation		
A2		Social Esteem		Social Sanction		Reaction		Composition	
Teacher		norm	6	ver		imp		bal	2,3,4,7
		cap	1,8,9,10,13,15, 16,19,21,22,18,29, 30,31,32,33,34, 35,36,40	prop	20,27	qual		comp	18
		ten	11,12,14,17, 22,24,25,26						
Student		norm		ver		imp		bal	
		cap	23,38,39,41	prop		qual	5	comp	
		ten							

Table 1. Instances of appraisal categorized: Class A, second recording.

FINDINGS

The distribution of judgements and appreciations as exemplified in Table 1 typified the classroom recordings. Table 2 summarizes classroom recordings and shows that there were more judgements of capacity than any other category. Judgements of capacity include teachers judging students' ability and students judging their own or another's ability. Examples of such judgements include "*you are correct*", "*good work*", et cetera. They also included clauses such as "*she is ahead of me*", "*you're getting there*", "*you're good*", or less explicit judgements such as "*for some of you that* [using the model] *is a long way off*", or "*I can't do this*". In both classes both teachers almost entirely judged capacity positively, with a few exceptions. In contrast, the students' judgements of capacity were for the most part negative (see Table 2).

		Teacher Class A		Teacher Class B		Students Class A		Students Class B	
		pos	neg	pos	neg	pos	neg	pos	neg
Social Esteem	Normality	1	-	4	2	-	-	2	1
	Capacity	53	3	14	2	2	3	4	14
	Tenacity	1	10	3	6	-	-	-	1
Social Sanction	Veracity	-	-	-	-	-	-	-	-
	Propriety	-	2	-	-	-	-	-	-
Reaction	Impact	-	-	-	-	-	-	-	-
	Quality	3	-	6	2	1	-	1	3
Composition	Balance	10	4	-	-	-	3	1	1
	Complexity	1	2	12	5	-	-	4	3
Valuation	Valuation	1	-	8	3	-	-	-	1

Table 2. Summary of textual instances of attitude (Classroom recordings).

The text analysis of the journals and focus groups showed distinct and consistent student attitudes towards the utility of mathematics. A journal prompt asked students why they thought they were learning the particular topic they were studying at the time of my visits. Class A was learning how to graph an equation. Again, the appraisal clauses were extracted and categorized from the response. With few exceptions, all student responses referred to practical and future-oriented purposes: for jobs, to complete assignments, or for future schoolwork. Class B were working with integers but had similar results. Where very few students in Class A discussed personal utility outside of employment and preparation for future school mathematics, many students in Class B related the purpose of the knowledge of integers to practical applications, most likely given as examples, such as working with banking, financing, and weather. Their responses included a recognition that "[math] *is important in every job*" and "*If we sell products or offer a service we need to be able to get a price that allows us to make profit*". Many students commented that the only thing they will use is the basic operations and even then it was noted by several that technology would do that for them. There were a number of responses referring to the futility of learning certain skills that, in their lives, have been (and will be) taken up by technology ("*by the time I need it there will be technology to do it for us and we won't need to do it then*"). It was surprising to note how many students discussed the need to know mathematics to help them in their generalized *daily lives*.

Perhaps the most significant trend in the findings was student judgements about the subject of mathematics. Noticing that the only references to mathematics content were number related (except when discussing graphing a line which was the subject of study in Class A), the first question I asked the focus groups was "*what is mathematics?*" They repeated their previous judgements, making statements such as "*a lot of numbers*". I did not want to direct the conversation aside from asking the question, and this particular question "*what is mathematics*" quickly moved from "*a lot of numbers*" to the utility of mathematics: "*it is important because you need math for pretty much every job, and it's like for everyday lives and like time and weights and buying things*" and "*school and work and to pay your taxes and stuff*". When pressed to find meaning of what they were learning the response was "*I don't really think we will use that outside of school but I think that we need it here for school if you build on top of it*". In response to this statement, a student responded by saying that they would need "*adding and multiplying*", again, referencing only the number strand of mathematics. Outside of measurement for cooking, temperatures, and carpentry (and the graphing a line as a topic rather than a skill), all reference to the content of mathematics included only calculations (paying for things, taxes, etc.) and "*for counting*". When I reiterated the question, "*what is math to you*"

again, I received the same comments. Mathematics is number, you need mathematics "*for jobs*" and "*paying taxes*", and "*it is used in everyday life*".

Out of 46 students, not one student in either class referred to the aesthetics of mathematics, and only one student spoke of learning for personal development (outside of utilitarian purposes) when she wrote: "*Any bit of knowledge someone can teach you, learn it. It's not going to hurt you. Even if it's not used in everyday life, it will just make you more intelligent than someone who has not learned it. And being intelligent is a good thing to be*" (Student 18 Class B).

DISCUSSION

The social impact of the challenges of the 21st century require a significant shift in thinking and acting. Education, being at the heart of our communities and a foundation for our children, is a central player in this shift. The results show that students have values that reflect a limited understanding of mathematics as a compartmentalized and static subject, with traditional and limited utility. Students regarded mathematics as little more than the number strand. According to Devlin (1997) these are typical responses. Students regard mathematics as a subject of calculations, procedures, or rules. There are hints that changes are occurring in their instruction. Students did talk about the need for mathematics in their 'daily lives' and the 'real world' but they said them like a cliché—they could not explain the meaning (and what meanings they did have were almost entirely financial such as "*paying for things*" and "*taxes*").

The other striking finding was students' attitudes towards the purpose of mathematics. With few exceptions, students judged that mathematics only has three purposes: to find employment, for future study, or for the vaguely understood notion of 'daily lives'. While there was some discussion as to how mathematics is different from other subjects, there was no mention of the use of mathematics in other subjects. Article 83 of the United Nation's (2016) *Framework on climate change adoption of the Paris Agreement draft*, "calls upon all Parties to ensure that education, training and public awareness, as reflected in Article 6 of the Convention and in Article 12 of the Agreement are adequately considered in their contribution to capacity-building" (p. 11). Education, and particularly scientific education, is central to combating climate change yet there is no evidence that this is on the minds of our students. Inquiry-based experiences wherein students are able to investigate, make connections and conjectures, discuss and share solutions (Boaler, 2016; Devlin, 1997; Suurtamm & Vézina, 2010) are essential in making these connections. Teaching holistically, experientially, may be a method of integrating the subjects and reconnecting students to the entire, beautiful subject of mathematics.

CONCLUSION

To return to the opening quote, we are indeed living in a challenging time. Within this context, these findings show that there is a need to connect students to the breadth and possibilities that lie within the study of mathematics. I also discovered that formative and normative judgements of capacity are characteristic of mathematics classrooms. I believe this is a key finding and further investigation into other formative ways of interacting while learning is required to discover how to better foster curiosity about mathematics and how it can be used to better understand and live in the world.

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COLLEGE FOUNDATIONAL MATHEMATICS: CAN THE AFFORDANCES OF ICT ENHANCE SELF-REGULATION SKILL OF STUDENTS?

MATHÉMATIQUES FONDAMENTALES DU COLLÈGE : EST-CE QUE LES APPORTS DES « TIC » PEUVENT AMÉLIORER L'AUTORÉGULATION DES COMPÉTENCES DES ÉLÈVES?

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This investigation examined the learning gains of an intervention that employed the affordances of information and communication technologies (ICT) to enhance the self-regulation skills of 17 students taking a foundational mathematics course at an Ontario community college. The learning intervention consisted of a) surveys on demographics and perceived ability to self-regulate; b) materials delivered in real time using interactive software and pen-based computing or asynchronously distributed via the learning management system (LMS); and c) student design of studynote (stylus-written MS Word document) or screencast (audio-visual recording) artefacts to demonstrate mathematical solutions. Semi-structured interview responses revealed that the creation of these artefacts required goal setting, environment structuring, task strategy planning, and time management skills. When artefacts were viewed using the LMS, individuals compared their work to others (self-evaluation) and sought help if required. These findings indicated that the affordances of the learning intervention contributed to a transformation in self-regulation skill.

Cette étude a examiné les apprentissages acquis suite à une séquence d'enseignement qui employait les technologies d'information et de communication (TIC) afin d'améliorer les compétences d'autorégulation de dix-sept étudiants suivant un cours de mathématiques fondamentales dans un collège communautaire. La séquence d'enseignement comprenait : a) des questionnaires sur la démographie et la capacité de s'autoréguler, b) du matériel transmis en temps réel à l'aide de logiciels interactifs et basés sur l'utilisation d'un crayon ou encore distribué de façon asynchrone via le « learning management system » (LMS) et c) la conception par les étudiants de « studynote » (un document MS Word écrit au crayon) ou d'objets de type « screencast » (enregistrement audiovisuel) pour montrer une solution mathématique. Des réponses à des entrevues semi-dirigées ont révélé que la création de ces artéfacts nécessite la détermination d'un objectif, un environnement structurant, la planification de stratégies en lien avec une tâche et des compétences en gestion de temps. Lorsque les artéfacts ont été affichés à l'aide du LMS, les participants ont comparé leur travail à d'autres (autoévaluation) et ont demandé de l'aide au besoin. Ces résultats indiquent

que la séquence d'enseignement a contribué à une transformation vis-à-vis la compétence d'autorégulation.

> *Perhaps one of the greatest challenges facing education is how we engage individuals with the learning process and design and tailor instructional methods to sustain positive changes that are feasible, accessible and realistic.* (Underwood & Farrington-Flint, 2015, p. 31)

BACKGROUND

During the mid-1960's, the community college system was created to meet the changing requirements of the Ontario workforce. Currently, colleges are "the higher education pathway choice for many students with a lower level of academic preparedness" (Dziwak, 2014, para. 2). These students may be less motivated (Grimes & David, 1999), have poor study habits (Jairam & Kiewra, 2010), and/or lack the ability to self-regulate (Ley & Young, 1998). My interests focussed on the rather less studied aspects of Ontario student preparedness for taking college-level mathematics courses (Dion, 2014; Maciejewski, 2012; Orpwood & Brown, 2013). Calling for changes to both mathematics curriculum and pedagogy, Dion (2014) underscored the lack of numeracy skills among college students—especially evident in developmental/foundational mathematics courses. Placement into developmental courses may be indicated by demographic factors (Reason, 2009) and/or affective traits like learning attitude, willingness to seek or accept help, and personal effort (Boylan, 2009; Garrett, 2010). Indicating a lack of ability to self-regulate, Ley and Young (1998) stated "developmental students may differ from regular admission students in the way they plan, organize, monitor, evaluate and even think about the learning process" (p. 47). As this ability is a learned response, research has demonstrated that students can benefit from a learning design that instructs and practices self-regulation (Ley & Young, 2001). In my opinion, educators should utilize the learning affordances provided by information and communication technology (ICT) to encourage students to explore, analyze, and develop their self-regulation skills.

PURPOSE OF THE STUDY

At my college, students in technology programs are required to take a placement test and based on score, are enrolled in a Mathematics Foundations for Technology (MFT) course. With a desired outcome to improve student success, the MFT course was designed to strengthen and correct the previously learned mathematics concepts using a methodology requiring an increased use of ICT. After several years of practical experience teaching MFT, and a genuine curiosity to understand student preference in a multimedia environment, I questioned the effectiveness of an ICT-enhanced approach (Carruthers, 2010; 2012). Research that connected self-regulation, the affordances of ICT, and college mathematics courses taken by foundational students was limited. The purpose of this study was to evaluate the effectiveness of an instructor designed learning intervention in which college students taking a foundational mathematics course utilized the affordances of ICT to obtain skills in self-awareness and to better develop self-regulatory ability.

RESEARCH QUESTION

This presentation focussed on one of my research questions: How do the affordances provided by the learning intervention, that are perceived to support metacognition (i.e., self-awareness and self-regulation) enhance self-regulation of students in a foundational mathematics college course?

PARTICIPANTS

Of the 17 students enrolled in a Mathematics Foundations for Technology (MFT) course, more than half were female (n = 10, 58.8%), the majority of participants were 25 years of age or younger (n = 12, 70.6%), and 58.8% identified their first language as other than English or French. Nine participants (53.0%) self-identified as Asian/South Asian/Southeast Asian and eight as other (Caucasian, Caribbean/West Indies, and Black). Ten participants either experienced a delay or had taken another post-secondary course, while seven entered directly from high school. Twelve students disclosed that the highest course they had taken was at the high school level. Some students had not taken a mathematics course in over five years, while 82.4% had taken one more recently.

METHODOLOGY

In this mixed-methods research study, the learning intervention consisted of students completing surveys, creating studynotes and screencasts, and using scaffolded learning materials delivered either face-to-face (through a classroom learning system [CLS] DyKnow software utilizing pen-based tablet PC computing) or asynchronously (through the learning management system [LMS]) (Carruthers, 2016). It was conducted in three phases: 1) pre-intervention, 2) intervention, and 3) post-intervention. Though primarily a quantitative study, qualitative data collection was used to illustrate the experience of the participants. In the pre-intervention phase, quantitative data was collected by administering a survey package (see Table 1), and the Online Self-Regulatory Learning Questionnaire (OSLQ) survey was re-taken in the post-intervention phase. Qualitative data was gathered in both the intervention and post-intervention phase through the use of activity surveys and semi-structured interviews. Thematic analysis (Braun & Clarke, 2006) was used to categorize student responses. Integration of quantitative and qualitative data in a non-sequential way validated findings.

Phase	Item	Description
Pre-Intervention	Survey Package	Demographic Survey Learner Demographic and Characteristics Adapted from (Colleges Ontario, 2014)
		Online Self-Regulatory Learning Questionnaire (OSLQ) (Barnard, Lan, To, Paton, & Lai, 2009; Barnard, Paton, & Lan, 2008)
		Index of Learning Styles (ILS) (Felder & Soloman, n.d.)
		Mathematics and Technology Attitudes Scale (MTAS) (Pierce, Stacey, & Barkatsas, 2007)
Intervention	Activity A	Students viewed and designed a studynote artefact
	Activity B	Students viewed and created a screencast artefact
	Activity C	Students designed a studynote and created a screencast artefact
	Activity D	Students designed a studynote or created a screencast artefact
Post-Intervention	Survey	OSLQ survey re-administered
	Semi-structured Interviews	Conducted with six individual students

Table 1. Study outline.

ACTIVITIES

In Activity A, students viewed an instructor-written studynote. A studynote was a one page MS Word Document written using the stylus of the MS Surface Pro 3 tablet PC. Participants responded to open-ended survey questions related to their learning gains. Students were then

instructed on how to design their own version of a studynote, both in its construction and the use of the handwritten annotation features of the tablet PC. This task required students to plan, organize and execute a one-page summary document reviewing basic concepts before giving a step-by-step solution to a problem. Studynotes were submitted, checked by the instructor, and posted to the course website. Participants were given access to these student-created artefacts for 31 days after posting.

In Activity B, the same procedure was used, except that students viewed an instructor-produced screencast. A screencast was an audio-visual recording of the computer screen capturing the pen stroke motion simultaneously with an audio explanation of a problem solution. Students then created and submitted their own to be posted to the course website by the instructor.

In Activity C, students were asked to create both a studynote and a screencast, while in Activity D, students had their choice of which artefact they preferred. As timing was limited to produce each of these artefacts, I felt that students would be challenged to activate their newly learned self-regulatory skills, thus furthering their development.

FINDINGS

The learning intervention afforded students a multimedia environment through the enhanced use of ICT, and for this study, development of self-regulation skills was measured. Student perception of their self-regulatory ability was evaluated pre- and post-intervention using the OSLQ survey. A paired sample *t*-test was used to compare the subscales of goal setting, task strategy, time management, environment structuring, help seeking, and self-evaluation across two matched groups. No statistically significant difference was found for a change in either the overall self-regulation ability, or each of the subscales noted above.

Valuable evidence was gained by the analysis of student responses to the activity surveys and semi-structured interviews. For example, when explaining what self-regulation meant to them, participants illustrated many of the subscales of the OSLQ. In general, their definition saw self-regulation as goal setting and using schedules to be productive. They found building an environment using an organized strategy to concentrate and self-evaluate was beneficial for their learning.

A comparison was made of student awareness and self-regulation gains when presented with either an instructor provided studynote or screeencast.

1. *Students viewing an instructor designed studynote*
   They demonstrated a greater knowledge of themselves as mathematics learners with comments such as "it gives me a better understanding of my ability to study," and "*it makes me aware of my learning*". They felt they could learn "[anytime] *we want*" and "*whenever* [we] *need*", identifying they found the material to be accessible and malleable. Efficiency was also noted, as they realized that a studynote provided "*a quick review*" with "*no need to read again and again other notes*."
2. *Students viewing an instructor designed screencast*
   The benefit of screencasts was summarized with "*it enhances the learning process with both audio and visual* [modes]". The response "[it] *made it feel real, like I was in the class again*", indicated that a screencast could be used to remind them of previously learned material. They stated "*the screencast just answered it all*" and that they could "*watch the video more than once if I don't understand*", implying its functionality.
3. *Instructor designed studynote versus screencast*
   Although students described the studynote as an efficient summary tool for review and study, most found the screencast recreated the classroom experience at a time more

suitable to their personal learning needs. Present throughout was evident understanding of the connection between the affordances of the ICT and the growth of self-regulation skills.

A comparison was made of student awareness and self-regulation gains when creating their own studynotes and screencasts.

1. *Students creating their own studynote*
   When choosing to solve a mathematics question related to order of operations, fractions, and proportions, designing a studynote helped students to "*think through everything*" and "*learn, understand, and concentrate*". "*Learn*[ing] *in a calm and comfortable environment*" allowed them to structure their environment and manage their time "*wisely*". I was interested in the finding that although students were familiar with writing studynotes, many expressed they had not considered this practice for reviewing mathematics concepts. Conversation with participants revealed using a stylus to write a studynote in a Word Document was unique, and further, that they had never previously shared mathematics work with their peers via a LMS.

2. *Students creating their own screencast*
   While solving a mathematics problem related to proportions, metric measurements, and conversions, creating a screencast taught them "*how to study properly*" and was "*self-explanatory for avoiding mistakes*". One commented they "*felt like the teacher where I was going to lecture the students*". As the artefact was always accessible it "*help*[ed] *to study and understand when not in the class*". In conversation, some mentioned they were familiar with video creation, but none had used a stylus on screen while recording their voice, nor had they considered this technique for solving mathematics problems.

3. *Students creating their own studynote versus screencast*
   The requirement of the increased effort in having to explain mathematical operation in real-time was viewed differently by individuals. Some felt it was more labour intensive, as they had to "*input step by step so the user can understand*", it was "*hard to explain what I know*", and felt they were "*not good at explaining clearly*". Although some thought this presented a challenge, others recognized it as a learning benefit as they were able to "[find] *my mistakes*" through this practice.

A comparison made of the affordances of the CLS and the LMS in their potential to develop self-regulation abilities.

1. *The Classroom Learning System*
   The CLS consisted of using DyKnow Vision software to push instructor designed content to individual student pen-based tablet PC's. This software provided an interactive and collaborative workspace for both instructor and students to share information. In terms of self-regulation, comments more closely related to the environment structuring and task strategy development subscales. They felt they "*could take ownership of* [the work]". As instructor annotations were saved, they did not "*need to copy all things on paper*", and thus had more time to be "*focused on one thing* [… because] *it is in front of me*". Within the software, students could move at their own pace, thereby taking responsibility for their learning.

2. The Learning Management System (BlackBoard)
   The LMS allowed students to log into the course website at their convenience, in particular, outside of class time. The MFT course was replete with calendars and guidelines, subject outlines, lecture posts, and textbook assignments. Further, due to the study design, students were able to access their peers' studynotes and screencasts. Related to the use of LMS, their comments focused on the self-regulation subscales of goal setting and time management. Participants were able to "*prepare* [themselves] *for*

when class was going to start", "*that way I know what I need to do*". Throughout the study, a total of 53 student-created artefacts (studynotes and screencasts) were made available, which they reviewed a total of 465 times within the 31 day time period. They mentioned using these resources for help-seeking and self-evaluation, such as "[I] *go to answers and check what I did wrong*".

CONCLUSION

This work connected some of the pre-existing gaps in the literature related to how a learning intervention using the affordances of an enhanced ICT environment could improve the self-awareness and self-regulatory abilities of students taking a foundational mathematics course within the Ontario community college system.

A noteworthy perspective derived from this study is that a learning intervention which instructs and practices self-regulatory skills is beneficial for all learners. Based on response, a contrast was evident when viewing an instructor-designed artefact versus students creating their own. When viewing an artefact, their comments related to task strategy thinking and how it might be used for study and review. When creating their own artefact, responses related to setting goals for task completion within the prescribed time and structuring their environment for optimal success. This challenged students to develop and practice a new self-regulation skill set. When given the opportunity to view the work of their peers (through the LMS), students were able to draw comparisons and seek-help to enhance their understanding. Through this experience, I have recognized that the self-regulation skills needed in a classroom learning system (CLS) are different than those used through the learning management system (LMS). Typically the classroom is organized by the instructor, and the need for self-regulation skill is dependent on the provided environment (e.g., student-centered, teacher-centered, or technology-centered). Outside of the classroom, the student assumes all responsibility for learning, and must put their self-regulation skills into practice. For instructors, an awareness of the differing benefits to self-regulation provided by the ICT affordances should be taken into consideration when designing student activities.

The findings were restricted due to the small sample size, utilization of only one cohort, and a single course (Mathematics Foundation for Technology). Though relevant as a direction for further study, results may not be generalizable to other situations.

Future research should include using ICT to understand how students can develop specific self-regulation abilities through the creation of their own study artefacts. As education moves towards more flexible learning environments, straying from traditional face-to-face to the convenience of being fully online, course design needs to become more relevant. For students struggling in mathematics, understanding how they prefer to learn and self-regulate should guide learning interventions utilizing ICT.

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INFLUENCE OF THE LEARNING ENVIRONMENT ON STUDENT TEST PERFORMANCE IN UNDERGRADUATE MATHEMATICS COURSES

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It is common practice during an examination to divide students in the same undergraduate class into various locations. Often times, one group of students writes their exam in the lecturing room in which they learned the material, while the remaining students write elsewhere. Due to the familiarity of the learning environment, students writing a test in their lecturing room may be at an advantage over their peers writing the same test but in a different classroom. This raises concerns about academic fairness. Test scores of engineering students in an undergraduate mathematics course were collected. These results lend insights for training students to use their classroom environment as a mechanism for learning.

INTRODUCTION

At the undergraduate level, a common scenario in one class is for a group of students to write their exam in the lecturing room in which they learned the material, while the remaining students write elsewhere. This scenario is depicted in Figure 1. The lecturing room is defined as the room where the instructor taught the course; while the non-lecturing room is a room the instructor did not teach in.
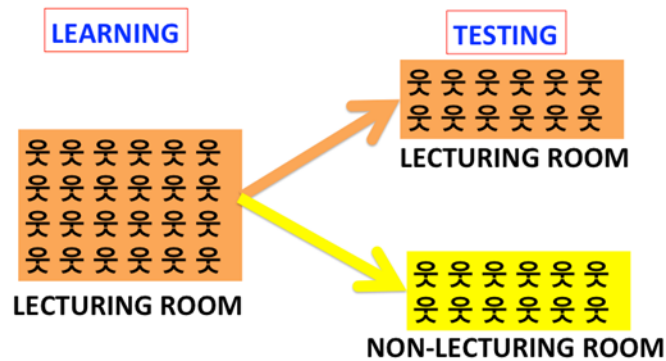

Figure 1. Setup of study

Due to the familiarity of the learning environment, students who are writing a test in their lecturing room may be at an advantage compared to their peers who are writing the same test

but in a non-lecturing room. This raises questions about academic fairness, especially during high stakes evaluations, such as a midterm or final, and choosing appropriate testing environments. There is also the deeper question of how the learning environment is influencing students' test performance and whether it can be used a mechanism for learning.

The theory that an individual's memory is easier to retrieve when considered in the environment it originated in is a psychology concept known as environmental context-dependent memory (Brinegar, Lehman, & Malmberg, 2013; Smith, 1979; Smith & Vela, 2001; Unsworth, Spillers, & Brewer, 2012). Environmental context-dependent memory has been explored in controlled experiments (Godden & Baddeley, 1975; Pointer & Bond, 1998; Smith & Guthrie, 1921), and for pedagogical explorations in the following disciplines:

a) music (Mirshra & Backlin, 2007),
b) psychology (Abernethy, 1940; Fansworth, 1934), and
c) medicine (Coveney, Timothy, Corrigan, & Redmond, 2013; Koens, Cat, & Custers, 2003; Koens, Mann, Custers, & Cate, 2005).

There is little literature available about environmental context-dependent memory in a mathematics discipline. Consequently, to address this, test results were collected from engineering students enrolled in several undergraduate math courses. Tests scores are based on students writing their examination in either their lecturing room or a non-lecturing room. This study was discussed at a talk at CMESG 2017, and this document summarizes that presentation. A full manuscript has been submitted for peer-review to a journal.

RESULTS

The math courses involved in this study ranged from 100 to 300 level, and course content focused on common areas of applied mathematics, such as calculus and differential equations. Class sizes ranged from approximately 75 to 140 students. Students were taught by one instructor per course in one lecturing room during either three 50 minute lectures or two 80 minute lectures each week. Therefore, for each math course, students' exposure to their lecturing room was between 150 to 160 minutes per week. Classrooms were standard rooms common to most universities.

Two types of test scores were collected. The first type was aggregated data in the form of averages based on evaluations such as quizzes, in-class tests and final exams from several mathematics courses. In total, 20 averages for the lecturing room, and 20 averages for the non-lecturing room were collected. This is displayed in Table 1, and demonstrates the sample mean between the lecturing room and non-lecturing room are essentially identical.

	Sample sizes	Sample means (%)	Sample standard deviation (%)
Lecturing Room	20	78.31	7.412
Non-lecturing Room	20	78.41	7.676

Table 1. Summary statistics for aggregated test scores.

The second type of data was individual test scores from 77 students in one particular undergraduate math course, which was selected randomly. Each of the 77 students wrote a midterm and a final exam with one of these tests written in the lecturing room and the other in the non-lecturing room. These results are displayed in Table 2, where the difference is defined as the test score in the lecturing room minus the test score in the non-lecturing room. From the table, the difference is positive. In addition, statistical analysis suggests test scores in the lecturing room are (statistically) higher than in the non-lecturing room. However, since the midterm test and final exam are not identical, this may be an additional influencing factor. To

reduce this disparity, the data was standardized by taking each students' test score and subtracting by the class average of the test. A summary of this standardized data is presented in Table 3, and shows similar results to the non-standardized test scores in Table 2.

Number of paired data values	77
Sample mean of the differences	2.32
Sample standard deviation of the differences	11.62

Table 2. Summary statistics for individual test scores.

Number of paired data values	77
Sample mean of the standardized differences	2.56
Sample standard deviation of the standardized differences	11.24

Table 3. Summary statistics for individual tests scores that has been standardized.

SUMMARY

The learning environment is explored as an influencing factor on student test performance during mathematical examinations. Aggregated data based on test averages of quizzes, in-class tests and final exams in several mathematics courses indicate the learning environment does not influence students' test performance. Test scores based on 77 students in the same (randomly selected) mathematics course indicate the learning environment does affect students' test performance. A definitive conclusion remains open. Future considerations could be an experimental study in which potential influencing factors are controlled by researchers. This will help to pinpoint factors in the learning environment that affects students' recollection of memory and in effect their test performance.

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THE ROLE OF ORAL COMMUNICATION STRATEGIES IN ACCESSING ANDASSESSING MATHEMATICAL UNDERSTANDING: CASE STUDIES OF PRIMARY SCHOOL TEACHERS' PERCEPTIONS OF TEACHING MATHEMATICS AND TEACHING LITERACY

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The study investigated primary teachers' perspectives on teaching mathematics and teaching literacy. The focus was on oral communication strategies to see if perspectives could be harmonized by building on teachers' greater comfort with teaching literacy.

Case studies in a small suburban GTA school provided qualitative data through classroom observation, on-going conversations about observed teaching episodes, semi- formal interviews with the teachers and principal at the beginning and end of the study, and one participant's blog and research report.

Participants' teaching experience ranged from 10 to 25+ years from kindergarten to grade 7. During the study (2013-2014), members of the school staff were organized into teaching partners by grade level to teach math through inquiry with an emphasis on communicating mathematical ideas. Evidence collected from grades 1 and 3 lead to the following findings:

1. *Essential resources for teaching math effectively that teachers need and want, are available; however, teachers are unaware of their existence;*
2. *Teachers who try to implement reform strategies without understanding how they work do not achieve the desired result;*
3. *A teacher with well developed processes for making sense of mathematics, can identify gaps in student understanding by relating student behavior to her own processes;*
4. *When a teacher contrasts his/her sense-making teaching strategies in literacy and mathematics, he/she can better identify areas of dissatisfaction in his/her math instruction and possible strategies to try;*
5. *A teacher who is able to think about teaching goals in more general terms may come to see parallel objectives between certain teaching strategies in math and non-math subjects;*
6. *Teaching mathematics through inquiry requires students to have a solid grounding in early literacy as well as early mathematics; and*
7. *Reporting requirements that ask for student achievement in mathematics on a strand by strand basis encourage teachers to teach the subject strand by strand.*

> *The study has implications for effective professional development, teachers learning math content and developing teaching materials, improving teacher confidence and the development of mindful reform practice. Suggestions for stakeholders to facilitate teachers' reform practice are included at the end of the study.*

RESEARCH QUESTIONS

A case study approach is used to pursue an understanding of the complex phenomena surrounding the implementation of oral communication strategies for deepening mathematical understanding and fostering mathematical reasoning in the elementary classroom. Teachers' perspectives on the communication strategies they use in teaching mathematics and the communication strategies they use in teaching literacy will be explored, with a view to helping teachers identify areas where they feel that their instruction in the less favored subject is lacking. The thick, rich description characteristic of a qualitative approach is necessary for capturing the meaning perspectives of participants and their culture as they construct their world around them. The following questions have been developed to guide the study:

1. What teaching practices are used to strengthen the connections between spoken language and mathematical ideas?
2. How is students' ability to reason facilitated through dialogue?
3. What kinds of challenges do teachers encounter as they facilitate classroom dialogue for deepening mathematical understanding?
4. What support and assistance do teachers need for facilitating oral communication?
5. What are teachers' perceived needs for doing mathematics in order to teach math well through inquiry?

These five questions form the basis for the development of this study and provide insight into (1) the effective facilitation of oral communication for voicing intuitive mathematical understandings, (2) the facilitation of classroom dialogue as a tool for developing students' ability to reason, (3) the kinds of problems that arise as teachers work at implementing these practices, (4) where teachers find support for their developing practices, and (5) how self-perceived teacher efficacy is affected by teachers' own competence in doing mathematics.

CROSS-CASE ANALYSIS

To elicit as much convergent information as possible from the contributing cases, it seems reasonable to look at possible patterns in what the participants have to tell us about their efforts around the central theme the school adopted during this study. The theme was 'collaborative inquiry in mathematics' with a strong, embedded emphasis on facilitating student communication of ideas. To tease out the contributing components of teachers' efforts, these two teaching objectives are considered individually, as well as how they are connected. Therefore, I examine teachers' efforts to teach students to communicate their understandings, their efforts to teach mathematics as inquiry, and how these efforts might be connected.

All participants made a big effort to teach their students to communicate; however, the bulk of their efforts were not based in mathematics. Even Sydney and Tom, who were concerned that they would have trouble eliciting students' ideas during the debriefing part of the mathematics lesson, taught their students conversational moves in a language setting. To ensure students would be able to carry on extended, independent, topic based conversations, they practiced on stories they read. The teachers assumed the skills would transfer to conversations about mathematics.

Though students eventually learned to carry on topic based conversations for 20 to 30 minutes in small groups, Sydney and Tom found that the skill did not automatically transfer to conversations about mathematics. According to the literature students' ability to use mathematical jargon meaningfully, referred to as mathematics register, needs to be developed as a part of regular conversation in the classroom (Forman, 1996). For students to develop an adequate basis for mathematical communication, they must negotiate shared understandings, a common language, and a relatively symmetrical relationship in which none of the students is seen as a mathematics authority (Cobb, 1995; Lampert & Cobb, 2003). Furthermore, they must be taught to participate in mathematical reasoning practices, which form the core of mathematical conversations (Ball, Lewis, & Thames, 2008; Lampert, 2001).

As the dates of the references show, none of this research is brand new. Teachers' intensions were highly commendable. However, the lack of awareness of available resources that would have tailored their efforts to ensure better success sheds light on a serious problem in communication between curricular requirements and curricular support. If teachers are expected to teach students to communicate mathematical ideas effectively, they need structured, long-term instruction in how to do it.

Teachers need to learn the language (mathematical terms) and the grammar (mathematical reasoning), and become comfortable with it, before they can be reasonably expected to facilitate such communication among their students. They must also teach their students how to talk about mathematics. They must develop shared meanings with their students about what it means to give a mathematical explanation, what a reasonable justification is, what it means to understand someone else's explanation or justification, what it means to collaborate when learning mathematics, and so on (Cobb, Yackel, & Wood, 1989). When the communication of mathematical ideas is seen as an aspect of students participating in the activities of a mathematical community, "learning to communicate as a goal of instruction cannot be cleanly separated from communication as a means by which students develop mathematical understandings" (Lampert & Cobb, 2003, p. 237).

The mathematics lesson excerpt from Anne's class on the word problem about "feet living at your house"; and the one from Tom's class on describing geometric shapes both show the teachers' efforts to teach the proper use of notation and mathematical vocabulary. Both teachers ask students for input, but the dialogue is very traditional in that the teacher does most of the talking in both cases. Students give short answers to specific questions with little or no reasoning required on their part. In addition, the teacher, as authority, automatically shuts down student reasoning because in the presence of authority, there is no need for argumentation. The authority's word is accepted as known (Cobb, 1995).

Teachers in this study seemed to do a better job of facilitating student communication and reasoning in non-mathematical subjects. Tom's social studies lesson on the lifestyle of pioneers was conducted as inquiry to the point of students asking and answering their own research questions. What made it easier for him to facilitate students' reasoning in this type of lesson? Was he more comfortable facilitating student communication and reasoning because he was less intent on drilling students in new vocabulary and new concepts? Was it easier for him to facilitate reasoning in social studies because he himself was more familiar with the language and felt more knowledgeable about the topic he was teaching? If we draw the analogy that learning to participate in mathematical reasoning is like learning to speak a new language, these kinds of explanations for teachers' difficulties would seem reasonable.

Roma was also more at ease facilitating reasoning in non-mathematical subjects. In her case, it was reading and writing. Her need for facilitating reasoning and making connections has to do with her teaching goals. Two of Roma's top priorities for student learning focus on students

making sense of what she is teaching them and students seeing the reason for learning it. When she teaches students to read, the focus is on the story, so they can see the value of reading. She teaches reasoning strategies that enable students to better anticipate the story-line and thus make reading easier. In the same way, she believes that the math they are doing has to matter to students and the problems have to be real for them if they are to engage with mathematics in a meaningful way.

In her opinion, the best way to engage students in meaningful conversation about mathematical ideas is to have them work together on problems they want to solve. Helping each other solve a real problem they care about will naturally result in students articulating their ideas. She believes that students need to talk about math in order to make sense of it because her own mathematical understanding is built on an on-going conversation about mathematics with her sister all the way through school. The literature supports the idea that mathematical meaning evolves through the repeated discussion of actions taken during engagement with mathematical tasks, particularly when students articulate their mathematical reasoning and connect their solutions to key mathematical ideas (Franke, Kazemi, & Battey, 2007; Hiebert et al., 1997).

Roma's contention that the math has to be real, even at the earliest stages of instruction, is supported by research that demonstrates that people are capable of figuring out mathematics they need without formal mathematical training. For example, a study of housewives in California found that the women could solve mathematical problems when comparison shopping, that they could not solve when the same problems were posed as formal mathematics (Lave, 1988; Sternberg, 1999). Similarly, Brazilian street urchins could do mathematics when selling merchandise in the streets, but they could not solve similar problems when presented in a school setting (Carraher, 1986; Carraher, Carraher, & Schliemann, 1985).

Anne's perspective on the idea that the math has to be real is her belief that students can see the value of math in organizing and making things more efficient from a very young age. The emphasis in her teaching is on having students work through concepts because in her experience, young students need to be exposed to a variety of representations of a math concept in order to understand it. She therefore plants the conceptual seeds before she adds vocabulary, which is completely consistent with the literature that advocates manipulative use for presenting ideas before adding mathematical language (National Council of Teachers of Mathematics, 2000; Ontario Ministry of Education, 2003). She and Roma even introduce problem solving to their grade 1 students based on composing and decomposing numbers even before they cover formal operations. So they are focusing on teaching math through inquiry; when they have children engage with manipulative based activities in small groups, their objective is to teach mathematics through collaborative inquiry. However, they still have to work on adding the language, the back and forth discussions, that will make the activities mathematically meaningful.

Anne has specifically stated that she wants and needs professional development on how to elicit students' ideas. Her stated needs in this area, and her related actions helped shed light on some teacher challenges associated with mathematically meaningful dialogue in the classroom. Anne's tendency to run out of time before she elicits student ideas orally limits the effectiveness of her manipulative-based lessons in generating mathematical meaning. Her habit of stating the main ideas herself instead of making students articulate them does little to support the emergence of meaning, according to the literature. Research shows that meaning emerges through the repeated back and forth between engagement with mathematical tasks and discussions of these actions between students and teacher. Specifically, the emergence of meaning is connected with students' articulation of their own mathematical ideas about the tasks they are engaged with (Jackson, Garrison, Wilson, Gibbons, & Shahan, 2013; Moschkovich, 1999; Walkerdine, 1988).

All participants realized that they wanted and needed more professional development to be able to facilitate classroom dialogue for the purpose of eliciting student ideas and engaging students in mathematical reasoning. Tom said he wanted to see accomplished practitioners demonstrate how to teach math through inquiry in a real classroom. He and Sydney searched to find classrooms in their school district where student communication of mathematical ideas was successfully facilitated. Tom said it was hard to find any, while Sydney found one practitioner on-line who worked in another urban center hundreds of miles away.

Anne and Roma were frustrated with the professional development activity they participated in because it stopped short of addressing their specific interests/concerns related to the facilitation of classroom dialogue. They wanted to talk about the details of eliciting student thinking: what language to use, and the specifics of questioning techniques that result in the student articulation of mathematical ideas. According to Anne, they had just gotten started on that conversation when time ran out.

Three of the four participants either showed in their actions, or specifically stated, that they were more comfortable and more confident in facilitating student reasoning in non-math subjects. When asked to reflect upon the reason for this, two of them ascribed it to having a much clearer sense of the stages of student progress in literacy than in mathematics. They all said they needed more professional development to understand what was required of them and how to facilitate student reasoning and elicit student thinking in mathematics. To help participants see that they already use some working strategies in this area, I encouraged them to reflect on the strategies they employ in facilitating student reasoning and eliciting student thinking in non-math subjects. When asked to contrast their use of those strategies in mathematics, they seemed to get a clearer picture of what is missing or unsatisfactory to them in their mathematics practice.

SUMMARY OF THE RESEARCH FINDINGS

Based on the data collected in this study, the following findings emerged:

1. A description of the stages of student development in mathematics was identified by teachers as crucial pedagogical content knowledge they needed to have to teach mathematics with confidence. Two of the participants specifically attributed their lesser comfort level with teaching mathematics as opposed to teaching literacy, in large part, to the fact that they were much less clear on the stages of student development in mathematics than in literacy. Therefore the prompts for advancing children's understanding available to them were much less accurate, and consequently, less targeted. There is a resource available that addresses this need: PRIME (Professional Resources and Instruction for Math Educators), offered by Nelson, Canada, is a series of courses on developmental learning in elementary mathematics developed by Dr. Marion Small. The course package offers maps of the stages of student development (referred to as phases), examples of teaching strategies and phase appropriate problems. There is also a pdf available on-line from Nelson which summarizes the research, theories and best practices in Math Education as of August 2002; however, participants were unaware of these resources.

2. It is well documented in the literature that the articulation of mathematical ideas by students is crucial in the emergence of mathematical meaning (Boaler, 2002; Jackson et al., 2013; Moschkovich, 1999; Walkerdine, 1988). Tom and Anne both know that it is important to elicit students' ideas during class discussions in math, and they believe they are working to facilitate the student articulation of ideas. Observation of their classes reveals, however, that their students rarely get a chance to articulate more than short responses to targeted teacher prompts. Anne routinely runs out of time for

eliciting students' ideas and summarizes the important points of the lesson herself to ensure that they are voiced. She expresses feelings of guilt about not doing what she is supposed to but continues to run into the same problem. Tom feels the debriefing part of his lesson always goes on too long, and students lose interest. Observation of his classes reveals that students' reasoning is only minimally engaged during whole-class discussions. He usually requires students to give short answers to specific questions and proceeds to analyze and connect ideas in his own words.

In trying to elicit student ideas through specific prompts or stating the main points instead of getting students to articulate them, these teachers show that they are committed to communicating the important ideas in their lessons. They apply the recommended strategies for eliciting students' ideas and engaging students in reasoning practices as they understand them. However, both teachers have expressed the need for additional professional development to practice these strategies more successfully. In allowing their classrooms to be observed, they have helped to identify the kinds of help they need. They need to understand (i) the crucial importance of having students articulate their understandings so that meaning will emerge, and (ii) the importance of having students engage in reasoning so they will remain engaged in the discussions.

3. Roma's solid understanding of math, and her well developed processes for making sense of mathematics allowed her to identify gaps in her student's understanding. She was able to do this despite her inexperience with reform methods and her lack of familiarity with the stages of student development in mathematics. By reflecting on her own processes and contrasting them with her student's observed behavior, she was able to identify the gap in the student's process that hindered his understanding. Roma's experience would suggest that having a good understanding of the mathematics he/she teaches allows a teacher to improve his/her teaching of mathematics because he/she can analyze his/her own processes in trying to understand students' observed struggles. Through this type of reflection and contrast, there is a good chance the teacher can identify the problem and subsequently find ways to help the student over the conceptual hurdle.

4. Three of the participants reported that they wanted help in building their practice for promoting student reasoning and eliciting students' ideas in mathematics. In observing their classes, I noticed that they routinely promoted student reasoning and the articulation of student ideas in reading and social studies. Therefore, I encouraged them to reflect on the teaching strategies they used in these non-math subjects and contrast them with strategies they used in teaching mathematics. This technique of contrasting helped them better identify areas of dissatisfaction with their math instruction, as well as, suggest possible teaching strategies to try in mathematics that they were already using in non-math subjects.

5. Contrasting the teaching strategies used in literacy with teaching strategies used in mathematics can prompt teachers to think about their goals in teaching the different subjects in more general terms. This kind of analysis may help to narrow the gap between their divergent perceptions on teaching these subjects. When Roma reflected on why she wanted students to know the word wall words without thinking about them, she was able to reframe her concept of them as letter patterns that are recurring, widely applicable units of literacy. Once she described them in this general way, it was not difficult for her to see a parallel between the role of word wall words in literacy and the role of number facts in mathematics.

Teachers who are able to find this kind of bridging perspective between their practices have a better chance of lessening their fear of teaching mathematics.

6. Teaching mathematics through inquiry requires students to have a solid grounding in early literacy as well as early mathematics. Knowledge gaps in oral communication skills, for example, seriously hamper students in paying attention, understanding what

is being said and expressing their own ideas. All participants commented on the knowledge gaps students arrive with. By grade 3, the gaps can be so severe that learning math through inquiry is overwhelming for many students, and consequently teaching math through inquiry becomes very challenging for the teacher as well as students.

7. Two participants felt that reporting on student achievement strand by strand was counter- productive to teaching mathematics well. As experienced teachers, they felt that the strands should be integrated in teaching the curriculum and that would be the best way to achieve a unified knowledge structure in student understanding. However in order to comply with reporting requirements, they felt compelled to teach strand by strand. It can, of course, be argued that strand by strand reporting requirements ensure that teachers cover all the strands. Still, it has also been shown that an integrated strand approach is more efficient and builds better student understanding. Therefore, the work of teaching mathematics well would be better supported by reporting requirements that reflect student achievement when strands are integrated.

CONCLUSION

1. *Interviews that contrast* sense-making teaching strategies can *probe math reform efforts more deeply*.
2. The *realization* that sense-making teaching strategies serve well in all subjects can *improve teachers' perceptions* of teaching math and *align perspectives*.
3. *Facilitated analysis* of teaching strategies *to identify parallel teaching objectives* may *help integrate math* into the fabric of teachers' practice.
4. Three *barriers to* teachers' work in *advancing reform practice* must be addressed:

   a) Teachers need ready access to stages of student development in math and associated teaching prompts as an integral part of pre-service curriculum and in-service professional development.
   b) When teachers use reform strategies in a mechanical way to train students to do certain things rather than engage students in reasoning practices, they are not developing mathematical understanding. Both students and teachers have to engage in reasoning practices to achieve mathematical understanding. Teachers who have a solid understanding of the mathematics they teach are in a better position to engage their own reasoning to help their students learn.
   c) Even experienced participants feel compelled to teach strand by strand due to reporting requirements. Therefore, the modification of reporting requirements to reflect student achievement in an integrated fashion would encourage teachers to deliver the mathematics curriculum more effectively.

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RECIPROCAL PARTNERSHIP: AN INTERVENTION TO ENHANCE MATHEMATICS SELF-EFFICACY AND ACHIEVEMENT OF FIRST- AND SECOND-SEMESTER COLLEGE STUDENTS

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In response to the call to support the growing number of college students who do not have the mathematical groundings to pursue math-related careers, I have conceptualized a mathematics intervention program called Reciprocal Partnership. It is proposed to enhance students' mathematical learning while supporting their social development during their secondary-tertiary transition. This study investigates the impact of Reciprocal Partnership on the mathematics self-efficacy and achievement of first- and second-semester college students. It also examines the structure of Reciprocal Partnership to inform the design of effective intervention programs for mathematical learning.

Quantitative results revealed significant effect of Reciprocal Partnership on the mathematics self-efficacy of only the first-semester college students but no significant effect on mathematics achievement for first- and second-semester students. However, qualitative results identified learning benefits for both groups of students such as gains in mathematical knowledge and skill, confidence, motivation, social connection, and comfort. Results from this study suggest that mathematics intervention programs should not focus mainly on the explanatory situation (tutoring) but also on the exploration situation (problem solving) and the extensional situation (comparing solution methods) to maximize learning outcomes.

INTRODUCTION

I define Reciprocal Partnership as the collaboration among dyads to engage in reciprocal learning and teaching under the influence of constructive and collaborative environments that are structured by the Three Learning Situations framework. Students are paired together to reciprocally ask one another questions on which they need assistance, and questions on which they can provide assistance. By raising these two types of questions, it is possible to create a learning situation where both students of the dyads have no understanding of the raised question (exploratory), where only one of the students has knowledge of the raised question (explanatory), and where both students have knowledge of the raised question (extensional). Reciprocal Partnership, therefore, follows the Three Learning Situations framework to generate all these situations and to engage the mental activities introduced by Carpenter and Lehrer (1999) as conducive to mathematical understanding

The Three Learning Situations framework, as I have outlined, illustrates that three learning situations can be created by the different states of understanding of a student dyad (one student possesses understanding, both possess understanding or both do not possess understanding). In each learning situation (exploratory, explanatory, and extensional), the student dyads engage in distinctive learning activities such as problem solving, tutoring, or comparing solution methods. These learning activities are proposed to capitalize on the social and cognitive benefits of collaborative learning (Johnson, Johnson, & Smith, 2007) and adhere to the tenets of constructivism (Thompson, 2013). They also engage some form or degree of the five mental activities introduced by Carpenter and Lehrer (1999): construct relationships; extend and apply mathematical knowledge; reflect about experiences; articulate what one knows; and make mathematical knowledge one's own.
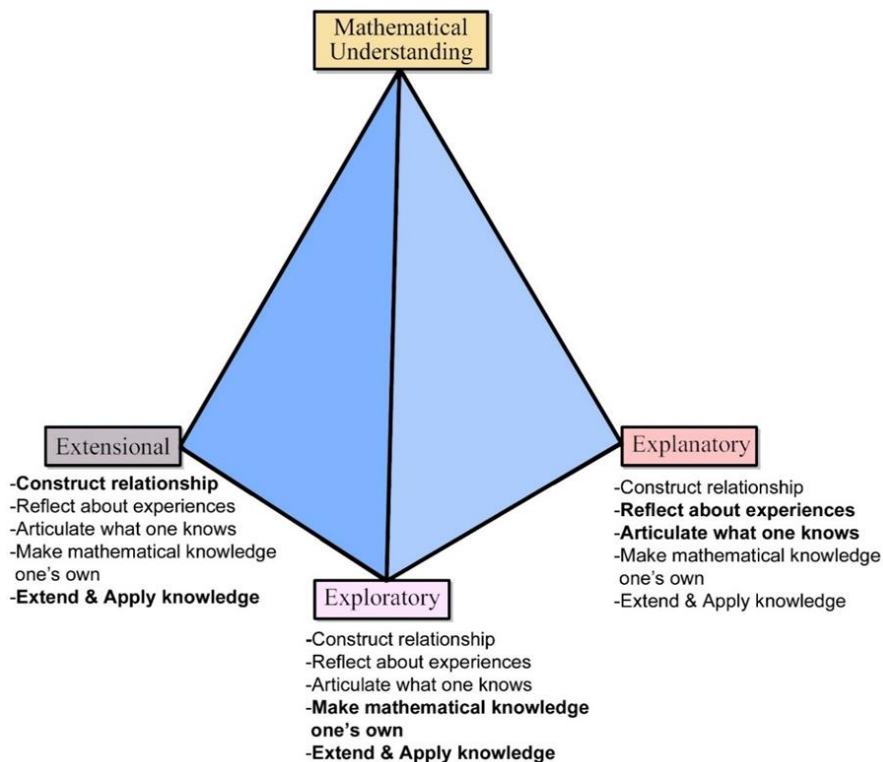
Figure 1. The Three Learning Situations framework.
Note. This figure illustrates the five mental activities as they fit into the Three Learning Situations framework. It displays "mathematical understanding" at the peak of the pyramid where it is supported by the three learning situations: extensional, exploratory, and extensional. The five mental activities are listed in all three learning situations and the focused mental activities of each situation are bolded.

First, in the exploratory situation, problem solving allows learners to articulate their method approaches, justify their conjectures and describe their viewpoints. Through articulation, the learners are reflecting on relations among existing knowledge and the parameters of the mathematical problem. By analyzing, comparing and evaluating the parameters of the problem to arrive at the solution, the learners are applying and extending their mathematical knowledge (Carpenter & Lehrer, 1999). As the learners derive their own solution methods, they make mathematical learning their own.

Second, in the explanatory situation, when learners are providing explanations, they are generating explicit links among ideas in ways that make sense to others. In the process, the learners are also extending and applying knowledge by relating their existing knowledge to new knowledge, examining and re-constructing the relations (Carpenter & Lehrer, 1999). Carpenter and Lehrer (1999) describe articulation as an overt reflection and state that reflection "is inherently personal, and encouraging reflection is critical in helping students develop a sense of ownership of their knowledge" (p. 30).

Finally, in the extensional situation, learners are acquiring a repertoire of strategies by comparing one another's solution methods. This is a form of extending and applying knowledge because when learners are comparing solution methods they are constructing multiple connections among ideas and relating new ideas to what they already know. By articulating the similarities and differences among solution methods, they are also reflecting more on their own solutions (Carpenter et al., 1999). In particular, when learners depend on one another to compare solution methods, they become more accountable for their own and other's learning (Cockrell, Caplow, & Donaldson, 2000).

It is an oversimplification to think that any one of the three learning situations alone can support all five mental activities introduced by Carpenter and Lehrer (1999). Likewise, it is improbable for any one learning situation to engage all five mental activities as each learning situation emphasizes some mental activities more than others. For example, an explanatory situation focuses more on reflection and articulation, but less on application. Therefore, when all three learning situations are applied, what is less emphasized in one learning situation can be complemented by another learning situation. Together the three learning situations can provide a complete support of mathematical understanding by complementarily engaging all five mental activities. Figure 1 is a visual representation of the five mental activities as they fit into the Three Learning Situations framework.

STRUCTURE OF RECIPROCAL PARTNERSHIP
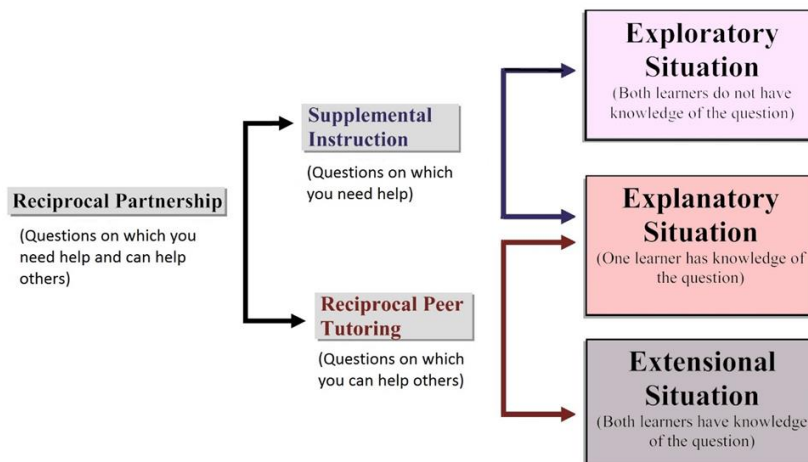
Figure 2. Structure of Reciprocal Partnership.
Note. This figure illustrates the structure of Reciprocal Partnership that combines both supplemental instruction and reciprocal peer tutoring to maximize the occurrences of all three learning situations. As displayed in the figure, supplemental instruction creates only the exploratory situation and explanatory situation, and reciprocal peer tutoring creates only the explanatory situation and extensional situation. Reciprocal Partnership is, therefore, a combination of both tutoring methods to create all three learning situations.

Since the five mental activities are best achieved through a combination of the three learning situations, Reciprocal Partnership is structured to make it possible for all three learning situation to occur. Hence, Reciprocal Partnership combines features of supplemental instruction and reciprocal peer tutoring. Supplemental instruction is a form of group tutoring in which students who are enrolled in 'high risk courses' raise questions to upper level students on areas that they need assistance (Arendale, 1998). Raising these types of questions creates only the explanatory and exploratory situations. On the other hand, reciprocal peer tutoring affords students the opportunity to enact the role of the tutor by raising questions on areas that they can provide assistance (Pigott, Fantuzzo, & Clement, 1986). Raising this type of questions creates only the explanatory and extensional situations.

Reciprocal Partnership, therefore, allows student dyads to choose questions on areas that they need assistance (e.g., supplemental instruction) and generate questions to which they have full solutions as required in reciprocal peer tutoring. By allowing each member of the dyad to administer these two types of questions (without and with understanding) to his or her partner, all three learning situations can be created. Figure 2 is a visual representation of the Reciprocal Partnership structure that combines both supplemental instruction and reciprocal peer tutoring to maximize the occurrences of all three learning situations.

METHODOLOGY

The purpose of this study is to investigate the effects of Reciprocal Partnership on students' mathematical achievement and self-efficacy during their secondary-tertiary transition. A mixed methods approach was employed to examine the following research questions: 1) What effects does Reciprocal Partnership have on the mathematics self-efficacy of first and second-semester college students? 2) What effects does Reciprocal Partnership have on the mathematics achievement of first and second-semester college students? 3) What insights does Reciprocal Partnership have to inform the design of effective intervention programs for mathematical learning?

Quantitative and qualitative data were collected within the same timeframe. Final examination grades and semi-structured interviews were used to examine the academic impacts of Reciprocal Partnership on the participants; whereas, pre-and post-surveys on mathematics self-efficacy and semi-structured interviews were used to examine impact of Reciprocal Partnership on participants' self-efficacy. Although both qualitative and quantitative data were analyzed separately, they were converged at the discussion stage for the purpose of cross-validation. The intent was to combine the non-overlapping strength of the quantitative methods (generalization) and the qualitative methods (small group of participants, in-depth) to produce well-substantiated conclusions.

DISCUSSION

All students revealed in their interview responses that they had gained confidence after participating in the Reciprocal Partnership workshops. The reasons for the gain included extra practice. The workshops allowed the students to spend time outside of the classroom to practice on their course materials. The extra time and effort spent on practicing the mathematical questions bolstered students' confidence in their mathematical skills. The opportunity to teach one another also increased students' confidence as they felt rewarded and happy helping one another. Especially during the occasion when first-semester students were able to teach the second-semester students, they became much more confident in their own mathematical knowledge. In addition, by working in pairs, students received help, encouragement, and alternative perspectives from their partner which they believed had helped them to solve

difficult questions. By overcoming these challenging questions, students experienced greater self-efficacy.

Furthermore, students mentioned an increase in motivation, social connection and comfort in their learning. Students were more motivated to work on the mathematical problems under the positive influence of other students. The workshops created opportunities for the students to meet other people and establish friendships with one another as they could reciprocate the help that they had received from their partner. When students felt more accepted and respected by other students through the friendship they had established with their partner, they had more comfort in their learning, and mathematical learning became enjoyable. In general, the students benefited from the inclusive and positive learning environment of the Reciprocal Partnership workshops.

With regards to the academic impacts, students perceived achievement as gains in mathematical knowledge and skills rather than improved academic performance on tests and exams. They reported that they gained mathematical knowledge and skills through acquiring different method approaches because it helped them to use more effective methods to arrive at the solution and to identify their own mistakes. Solving problems with a partner was also an important avenue through which they gained mathematical knowledge and skills. They found it more valuable to work through the questions with a partner than having all the steps presented to them in a lecture. Another important avenue was the opportunity to reciprocally tutor one another. For example, second-semester students were able to review the mathematical materials from Math 1 by tutoring the first-semester students, and vice versa; the first-semester students consolidated their understanding of the course materials by providing explanations to the second-semester students.

In contradiction, quantitative data revealed significant increase of mathematics self-efficacy only for students in the first semester. The increase in mathematics self-efficacy for students in the second semester was found to be insignificant, and this might be affected by a type II error such as ceiling effect. Ceiling effect was created as the second-semester students rated their pre-survey ratings very high which made it unlikely for significant increase in the post-survey ratings. In addition, quantitative data revealed insignificant gains in mathematics achievement for both first- and second-semester students even when all students' interview responses revealed positive gains in their mathematical knowledge and skills. Reasons for this discrepancy might involve ineffectiveness of the final examination to detect learning gains of relevant incidental information (Rittschof & Griffin, 2001) and the insufficient amount of time for students to improve their mathematical skills and knowledge throughout the short duration of the intervention (Falchikov, 2001).

Results from this research also revealed a noteworthy insight for effective mathematical interventions. Students reported gains in mathematical knowledge and skills during all three learning situations and most students preferred the exploratory and extensional situations over the explanatory situation in terms of learning gains. However, the explanatory situation is often emphasized in most if not all mathematical interventions. Therefore, this research suggests that effective mathematics intervention should emphasize all three learning situations and not only the explanatory (tutoring) situation since any one learning situation alone cannot completely support all mental activities that are conducive to mathematical understanding.

CONCLUSION

The challenge to mediate students' secondary-tertiary transition and to enhance students' mathematical learning has been growing with increasing diversity of college students as well

as widening gaps in their mathematical knowledge. This research supports the need for an effective intervention that responds to both academic and social needs of college students in their mathematical learning. Not only should it afford opportunities for engaging the five mental activities that support mathematical understanding (Carpenter & Lehrer, 1999), it should also provide social interactions for students to assimilate into their college community (Tinto, 2006). Reciprocal Partnership appears to be a viable intervention to address both academic and social needs of students in their mathematical learning.

Reciprocal Partnership has demonstrated the potential to increase self-efficacy, improve mathematical knowledge and skills, create positive learning environments and motivate learners. These impacts of Reciprocal Partnership may have positive influence in diminishing mathematics anxiety, high failure rates, and large academic gaps in performance. It is hoped that further research can be conducted on Reciprocal Partnership to verify its potential to combat negative perceptions and academic difficulties associated with the learning of mathematics.

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UNDERSTANDING THE INTERACTIONS WITHIN A NEW TEACHER LEARNING COMMUNITY COMPOSED OF FIRST TIME PARTICIPANTS AND A NOVICE FACILITATOR

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SUMMARY

Recent efforts in improving the quality of mathematics instruction have oriented educators to teaching practices that are labeled as 'ambitious' because they aim to ensure accessibility of content to all students and to support students' development of solid and durable understanding of mathematical ideas. The notion of a professional learning community (PLC) has been increasingly promoted as a structure to support changes in teaching practices. Most studies present well-established professional communities led by expert facilitators who are most often the researchers themselves. It was unclear what opportunities for learning ambitious teaching practices exist in new communities composed of first time participants and supported by facilitators who are novices in working in this context.

This study sought to answer these two research questions

1. How do opportunities for teacher learning emerge from interactions within a new professional learning community composed of first time participants and led by a novice facilitator?
2. How and why do these interactions afford or constrain opportunities for developing ambitious mathematics teaching practices?

Practices grounded in the cultural, historical, and institutional moors are tacit and stable and so provide challenges to researchers looking into improving teaching practices. Using the sociocultural constructs of cultural models and the figured world, I analyzed the dynamics shaping the interactions within a new PLC composed of six volunteer teachers and one facilitator during 22 sessions over the course of one school year.

The results showed that a wide range of openings for opportunities to learn ambitious teaching practices arose from the conversations and that these provided windows into the teachers' practice. However, most interactions following the openings, with the exception of a few episodes, constrained opportunities for learning because ambitious practices were raised but were never unpacked or reflected upon critically by the group.

These constraining interactions were shaped by the participants' cultural models relating to 1) their participation within the PLC and the purpose of the PLC, 2) their conception about teaching and learning mathematics, and 3) the pressure of teaching a program in a high-stakes

educational system. The facilitator's and the teachers' conceptions of these cultural models diverged significantly, thus creating tension and frustration within the PLC. How the teachers and the facilitator positioned themselves and each other influenced the ways in which they interacted within this context. The teachers' perceptions on teaching and learning mathematics, especially at the grade 10 level where students are required to pass a standardized examination, strongly determined the value of some interventions in the conversations.

The overall findings also supported the need to identify facilitation moves and to support the facilitators in understanding the teacher participants' positions at the start of their work together. Skillful facilitation would be necessary to leverage the many openings to conversations about teaching practices and to encourage teachers to develop new patterns of interactions that would afford more productive conversations about these practices. Important aspects of effective facilitation are not only to understand the cultural models that powerfully shape how teachers participate in the PLC, but also to learn how to use these as a starting point to support their learning.

RÉSUMÉ

Les récents efforts pour améliorer la qualité de l'enseignement des mathématiques ont orienté les éducateurs aux pratiques d'enseignement décrites comme « ambitieuses ». Ces pratiques visent à assurer l'accessibilité du contenu à tous les élèves et de soutenir le développement d'une compréhension solide et durable des idées mathématiques chez les élèves. Plusieurs études ont montré le potentiel des communautés d'apprentissage professionnelles (CAP) pour soutenir les changements dans les pratiques d'enseignement. La plupart des études menées présente des communautés professionnelles bien établies et soutenues par des facilitateurs d'expérience qui sont souvent les chercheurs eux-mêmes. Cependant, la communauté scientifique en connait encore très peu sur le potentiel des pratiques ambitieuses dans une CAP composée d'enseignants et d'un facilitateur qui en sont à leur première participation.

Cette étude cherchait à répondre aux questions de recherche suivantes

1. Comment les opportunités d'apprentissage pour les enseignants émergent-elles des interactions dans une nouvelle CAP composée d'enseignants et d'un facilitateur qui en sont à leur première participation ?
2. Comment et pourquoi les interactions offrent ou contraignent les opportunités de développer des pratiques d'enseignement qualifiées d'ambitieuses ?

Les théories socioculturelles comme les modèles culturels et les mondes figurés ont guidé mon analyse des dynamiques qui façonnaient les interactions au sein d'une nouvelle CAP. Cette CAP était composée de six enseignants volontaires et d'une facilitatrice qui se sont rencontrés 22 fois pendant une année scolaire. Cette perspective a été prise afin de comprendre les points de départ des enseignants et de la facilitatrice, afin d'informer les chercheurs et les dirigeants du perfectionnement professionnel sur les considérations importantes pour soutenir l'apprentissage des enseignants dans les CAP.

Les résultats ont montré que de nombreuses possibilités ressortent des conversations. Une vaste gamme d'ouvertures au sein des interactions servait de fenêtres pour examiner les pratiques d'enseignants. Cependant, la plupart des interactions limitaient les possibilités d'apprentissage, car les pratiques d'enseignement ambitieuses y étaient mentionnées, mais jamais examinées de façon critique ou en tant qu'objet de réflexion par tout le groupe.

Les résultats ont montré que ce monde figuré a été façonné par des modèles culturels conçus par les participants sur 1) la participation et le but de la CAP, 2) leurs conceptions de

l'enseignement et de l'apprentissage des mathématiques, et 3) la pression ressentie lorsqu'ils enseignent un programme dans un système éducatif aux enjeux élevés. Comment les enseignants et la facilitatrice se positionnaient eux-mêmes et positionnaient les autres a influencé la façon dont ils interagissaient dans ce contexte. Les perceptions des enseignants sur l'enseignement et l'apprentissage des mathématiques, en particulier au niveau de la 10e année où les étudiants sont tenus de passer un examen uniforme, ont fortement déterminé le mérite de certaines interventions dans les conversations.

Les résultats généraux soutiennent aussi le besoin d'identifier les pratiques de facilitation et d'aider les facilitateurs à comprendre les positions des enseignants au tout début de leur travail ensemble. La facilitation des discussions demande une habileté particulière pour pouvoir tirer parti des nombreuses ouvertures lors des conversations, en plus de pouvoir amener les enseignants à développer de nouveaux modes d'interactions qui offriront des possibilités de conversations plus productives au sujet de ces pratiques. Comprendre les modèles culturels qui façonnent les façons dont les enseignants participent à une CAP est un des aspects importants d'une facilitation efficace, et apprendre à les utiliser comme point de départ pour soutenir leur apprentissage l'est tout autant.

VALUED KINDS OF KNOWLEDGE AND WAYS OF KNOWING INMATHEMATICS AND THE TEACHING AND LEARNING OF MATHEMATICS: A WORLDVIEW ANALYSIS

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After almost a dozen years of working as the K-12 Mathematics Consultant in the Saskatchewan Ministry of Education (a.k.a SaskEd and Sask Learning), while taking one of the first courses in my PhD program (Decolonizing Aboriginal Education with Dr. Marie Battiste), I read Leroy Little Bear's (2000) *Jagged Worldviews Colliding* and a whole new world of thinking and understanding started to emerge around me. It was in Little Bear's work that I finally, after being asked to infuse Saskatchewan's mathematics curricula with First Nations, Métis, and Inuit (FNMI) content, perspectives, and ways of knowing for a number of years prior, started to understand what this could mean beyond tokenism at best, and cultural appropriation, knowledge denial, and destruction at worst. After years of asking my Indigenous colleagues "what are FNMI ways of knowing mathematics" and receiving answers such as "I don't know myself" or "that is your journey; you need to find the way", I started to realize that I had been asking the wrong question and should have been asking instead "what are Indigenous ways of knowing" or more explicitly "what are the kinds of knowledge and ways of knowing that are valued within an Indigenous worldview?" It is very likely that over all those years, people had, in fact, provided me with the answer to this 'new to me' question but because it was not what I had asked I did not recognize the significance of their words. Why the stars aligned on that particular day as I read Little Bear's chapter I do not know, but I do know that I have grown intellectually, emotionally, and spiritually because they did, and my journey still continues, although with renewed focus, enthusiasm, wonder, and hope. I am no longer focused on the singular question of infusing content, perspectives, and ways of knowing, but on the broader and complex question of how teaching, learning, and mathematics itself are informed by the kinds of knowledge and ways of knowing that are valued, and in so doing, how revisioning and reclaiming mathematics and the teaching and learning of mathematics can meaningfully happen.

To set the backdrop for the results of my research, this paper will first give a description of what kinds of knowledge and ways of knowing are valued within what I have come to call 'the Traditional Western worldview' and 'an Indigenous worldview', the theoretical framework for my work, followed by a brief explanation of the methodological collage that I used within my analysis. The rest of the paper will focus on the results of analyses done on various sets of data and end with the proposal of a new theory and related philosophy of mathematics.

THEORETICAL FRAMEWORK: TWO WORLDVIEW LENSES

Before entering into the specifics about the two worldviews that are central to my doctoral work, it is important to emphasize that although the worldviews have names that reference a particular culture or set of cultures, it should not be concluded that membership within either of these cultures implies a person holds that particular worldview. These worldviews are named in response to perceived trends and tendencies, but not expected norms. Moreover, these are only two of undoubtedly an infinite number of worldviews, and I have come to find, at least for myself, a kind of fluidity in my 'worldview status', moving back and forth between the worldviews. One might then question the validity or usefulness of such a framework; however, I argue that it allows us to consider what else might be possible and why we have or have not made certain changes or decisions. This framework is not rigid; rather, it is flexible and accepting and brings forward questions and ideas that might otherwise not have been considered.

Given the limited number of pages allotted to this paper and the abundance of Indigenous and non-Indigenous scholars whose work has influenced my thinking on these worldviews I have chosen to reference all at once all those scholars whose work has influenced my thinking about these two worldviews. I do so now, with the understanding that what follows below is a composite of their wisdom, understanding, and thinking as I understand it: Aitken and Bruised Head (2008); Allen and Crawley (1998); Arviso Alvord and Cohen Van Peet (1999); Barnhart and Kawagley (2005); Battiste (2002); Ermine (1995); Graham (1988); Henderson (2000); Howard and Perry (2005); Irzik and Nola (2007); Kawasaki (2006); Kovach (2009); Malcolm, Sutherland, and Keane (2008); Mercer, Dominey-Howes, Kelman, and Lloyd (2007); Meyer (1998, 2003a, 2003b); Michell (2005); Nisbett (2003); Roy and Morgan (2008); Russell and Chernoff (2013); Schelbert (2003); Smith (1999); Snively and Corsiglia (2001); Sterenberg, Barrett, Blood, Glanfield, Lunney Borden, McDonnell, Nicol, and Weston (2010); and, Van Eiijck and Roth (2007). With the aforementioned in mind, I now provide a brief description of the Traditional Western worldview.

THE TRADITIONAL WESTERN WORLDVIEW

Central to the Traditional Western worldview is the belief in the existence and need for ultimate truths. Knowledge of value within this worldview is singular, linear in its development and progression, static in its existence, and above all abstract, compartmentalized, and isolated from human existence or interference (in the forms of number facts, formulas, procedures, proofs, and so on). These characteristics of knowledge of value naturally lead to a definitive dichotomization of knowledge: good and evil, true and false, black and white…. This linearity and dichotomization of knowledge ultimately leads to hierarchies of knowledge in which certain knowledges are considered higher-level, and thus more important, such as seen in the comparison of arithmetic to algebra and algebra to calculus. As the knowledge valued in the Traditional Western worldview falls into such hierarchies, then so to do the knowers of that knowledge become part of a hierarchy of knowing and specialization. Since more knowledge aligns with more authority and power within this worldview, the seeking of new knowledge is frequently done for the purpose of having more knowledge, and thereby moving up the hierarchies of knowledge and specialization. In the Traditional Western worldview, knowledge of value comes from rational ways of knowing with an emphasis on processes that are held to be objective, replicable and measureable, such as the scientific method and deductive proofs. Finally, because knowledge of value within this worldview is absolute and always true, written records of all knowledge are sought and valued because they preserve the truth of the knowledge for all times.

AN INDIGENEOUS WORLDVIEW

Unlike the Traditional Western worldview's emphasis upon facts and objectivity, central to an Indigenous worldview is relationships—relationship creation, strengthening, maintaining, and expanding drive the seeking of all knowledge. Because of the centrality of relationships to this worldview, both objective and subjective knowledge are perceived as valuable. Furthermore, since relationships go beyond the intellectual domain, diverse sources of knowledge are not only valued but sought. For example, within an Indigenous worldview emotional, experiential, spiritual, intuitional, traditional, and cultural knowledges are all considered equally valid sources of valuable knowledge. Moreover, since relationships are inherently grounded within the place and time of the relationship, place and time are also important parts of the knowledge. Thus, although abstract knowledge is not rejected within an Indigenous worldview as it would be recognized as being valuable in some circumstances, knowledge housed within place and context, with story embedded, is considered highly desirable. Diversity in ways of knowing and in the kinds of knowledge sought, shared, and produced are highly valued within an Indigenous worldview. Finally, since the value of knowledge is determined by the place and time in which it is to be used, oral knowledge is highly valued because it can be easily adjusted to meet the needs of the knowledge seeker in relation to their place and time.

With these understandings of the theoretical framework comprised of these two worldviews, I next briefly introduce the methodological collage within which this framework was used as a tool for analysis. I choose to call this a collage, rather than a bricolage, as the latter implies a random collection of what is on hand; whereas, my methodological choices were both deliberate to the kind of analysis I wished to do and supportive of each other.

METHODOLOGY

The collage of methodologies I have alluded to is comprised of auto/ethnography (see, for example Roth, 2005), Gademarian hermeneutics (for more information see Gadamer, 1989), and grounded theory (see Corbin & Strauss, 1990 for further explanation). Due to the complexity of each of these methodologies, I have chosen to delineate my reasons for choosing each of the three within the collage and then explaining how I blended them throughout my analysis of the data.

In auto/ethnography, the goal is for the author (and researcher) to reflect upon their personal experiences in relation to a particular culture in order to bring greater understanding of the culture in general. Within my research, there are two cultures that I am reflecting upon: mathematics and the teaching and learning of mathematics. In using this first methodological approach, I am attempting to both self-identify in relation to these two cultures and to highlight tensions and other points of interest that my past experiences draw my attention towards.

In the analysis of my story, I wanted to analyze how my various experiences reflected what I valued in relation to kinds of mathematical knowledge and ways of knowing. Thus, I chose Gadamerian hermeneutics to engage in a discussion between my story and the two worldviews. I used the framework of the two worldviews to engage in metaphorical discussions between a person grounded within each of the worldviews and my story in order to suss out what kinds of knowledge and ways of knowing I was valuing during the various epiphanies of my story and how these two fictitious people, as representatives of the two worldviews, would respond to my story and what I was valuing within it.

As expected, a number of points of tension and interest emerged from my analysis of the story, and using grounded theory to provide direction on my choices, I pursued each in turn through the collection of additional data and the resulting analysis. Grounded theory also guided my

identification of concepts throughout all of the data sets and helped me track which concepts continued to appear in my analysis of the data and whether those concepts were merging into a larger conceptual category and ultimately on to the proposal of a new theory.

Thus, my data collection and analysis began with a personal focus which auto/ethnography supports. The Gademarian hermeneutics provided a method of engaging with my story from the perspective of the two worldviews, and grounded theory provided me the tools to identify conceptual labels emerging through the dialogues between the two worldviews and my story. In addition, grounded theory aided in the identification of particular points of tension which needed to be moved beyond my personal story for data collection, examination, and analysis. At that point, auto/ethnography moves to the side, as my research moves from the perspective of my personal experience to those bound within the broader contexts of the cultures of mathematics and the teaching and learning of mathematics.

Explicitly, the analysis of data in my dissertation looped through the following steps:

1. presentation of the data,
2. identification of epiphanies within the data (points in the data in which conflict, tension, or even possibly significant change has occurred),
3. reflections upon and analysis of the epiphanies through the distinct lenses of the two worldviews,
4. coding and description of conceptual labels emerging through the worldview analyses as well as identification of emerging conceptual categories and saturation of the labels and categories,
5. identification of new data sets to be collected and analyzed,

and ended when a single conceptual category that had become saturated and a new theory emerged. Beyond the variation to Gadamerian hermeneutics previously mentioned (that there were no 'real' people engaged in the discussion), my collage also varied from most grounded theory in that the identification of additional data sets happened at one time, in response to the analysis of my story, rather than emerging one after the other over time.

The next section will briefly summarize the results of the analysis of my story and the additional data sets. After the analysis of my story, I will continue with analysis of the philosophies of mathematics, the math wars, the struggle of many Indigenous students with mathematics and the field of ethnomathematics, and risk education.

ANALYSIS RESULTS

My first data set, the story of mathematics and me, begins with my earliest memories of 'doing' and being aware of mathematics and ends with my aforementioned engagement with Little Bear's text at the start of my PhD program. Because the story captures numerous aspects and events with respect to my relationship with mathematics and the teaching and learning of mathematics and because that relationship is challenged and often revised throughout the story, I chose to analyze my story according to chronological sections: before starting school; in grades 1-8, my two undergraduate degrees; teaching high school mathematics, being a pilot teacher, implementation leader, and a master's student; working at the Ministry of Education; and my PhD studies. Throughout these periods, I endeavoured to recall as many memories about mathematics and the teaching and learning of mathematics that I had. Using coding of the dialogues between the epiphanies found within my story and the two worldviews, I identified conceptual labels within each of the sections of my story. In the first section (before starting school), I identified the concepts of categorization and isolation, abstraction, singularity, relationship, context, and power and authority. In the next section, these same concepts also

emerged as did the concepts of hierarchy and specialization. All of these concepts continued to be present throughout the remainder of my story with no additional concepts emerging. Moreover, within each successive section of my story, more instances of the merging of various combinations of these concepts was occurring. As such, the concepts of hierarchy, specialization, singularity, categorization and isolation, relationship, power and authority, abstraction, and context became consistent in their defining with much saturation and merging within and amongst the categories.

As a result of further application of grounded theory, four areas of tension and interest emerged:

1. *how people view and think about what mathematics is (specifically, the philosophies of mathematics),*
2. *how it is believed that mathematics should be taught and learned (specifically, the math wars),*
3. *how mathematics relates to culture and individuals (specifically Indigenous students' struggles with mathematics and implications of ethnomathematics), and*
4. *how curriculum represents mathematics content with an eye to emerging mathematics curricula content (in this case, risk education).* (Russell, 2017, p. 122)

In response to these four areas, I next collected data to better inform my understanding and thinking in relation to them, and then began the analysis cycle described above again working with each data set separately and then pulling the analyses together in the end. As I continued to analyze these new data sets, the same concepts (categorization and isolation, abstraction, singularity, relationship, context, power and authority, hierarchy, and specialization) continued to be present, and no additional concepts emerged. Moreover, the concepts themselves became heavily saturated with no new understandings related to them emerging, and the concepts ultimately merged together to form a single conceptual category of the attributes of mathematics and the teaching and learning of mathematics, which then led to my proposal of a new theory: the Transreform Approach to the teaching and learning of mathematics.

TRANSREFORM APPROACH TO THE TEACHING AND LEARNING OF MATHEMATICS

The conceptual category that emerged was consistently demonstrating an important relationship between the kinds of knowledge and ways of knowing that are valued (or not valued) from the perspectives of the two worldviews: where the Traditional Western worldview denied most of the knowledge and ways of knowing valued by an Indigenous worldview, an Indigenous worldview had space within it for the knowledge and ways of knowing valued by the Traditional Western worldview. Thus, this conceptual category and its associated understandings led me to propose that there exists an approach beyond the reform and traditional approaches that are the existing battlegrounds for the math wars. This approach, the Transreform Approach to the teaching and learning of mathematics, is the result of grounding the teaching and learning of mathematics (and how we think about mathematics in general) within an Indigenous worldview. In so doing, both the reform and traditional approaches and thinking about mathematics can co-exist, with the people, place, and time determining what kinds of knowledge and ways of knowing mathematics are of greatest value within that particular context for the person or people involved. Thus, what mathematical knowledge and ways of knowing are to be valued is responsive to the context in which they are being required. This approach however does not only serve to 'house' both traditional and reform approaches and understandings of mathematics, it also provides space for mathematics and the teaching and learning of mathematics in between the two approaches as well as beyond either one.

This theory then has many potential impacts related to the specific areas that my dissertation focused in on:

1. Because the Transreform Approach values diverse ways of knowing and kinds of knowledge, Indigenous and non-Indigenous students would be provided the opportunity to 'find themselves' within mathematics. The Transreform Approach would encourage students to bring to bear their stories, emotions, intuition, physical understandings, spiritual understandings, and even cultural or traditional understandings to the teaching and learning of mathematics, along with the intellectual, abstract and rational thinking that has traditionally been valued.

2. The Transreform Approach changes the boundaries of mathematics and the teaching and learning of mathematics without negating anything that existed within the previous boundaries; rather, it extends and creates new boundaries that allow previously unresolvable mathematical contexts and needs to be addressed in meaningful ways.

3. The Transreform Approach consequently can expand and strengthen mathematical understanding and ability, which is of advantage for everyone—traditionalist or reformer.

4. The Transreform Approach could eliminate the existence and even the perceived need for the math wars.

5. The Transreform Approach would work against the cultural biases and even non-universality of mathematics as it exists in schools.

In order for the Transreform Approach to come to fruition however, it quickly becomes apparent that none of the existing (documented) philosophies of mathematics can sufficiently support such an approach. Thus, I next propose a new philosophy of mathematics: transreform radical humanism.

TRANSREFORM RADICAL HUMANISM

In defining the mathematics philosophy of transreform radical humanism, I necessarily needed the philosophy to be similarly grounded within an Indigenous worldview. I also wanted, because of the categorization and understandings of existing philosophies of mathematics, to be sure that this philosophy was indeed seen as fallibilist or humanist; however, I also wanted to be sure that this philosophy was distinctly removed from the aspects of humanism which in the past had allowed for individuals and cultures to be segregated and devalued—othered—by the dominant culture. In the past, humanism relegated many people, including Indigenous peoples, to being less-than human, and consequently their knowledge and ways of knowing were deemed inferior at best. For this reason, I have chosen to speak in terms of 'radical humanism', in that I wish to radicalize the understanding and use of the term humanism in order to deny the othering of any person, group of people, or culture in ways that would deny significance for alternative ways of knowing and kinds of knowledge. This philosophy of mathematics contends that mathematics is a human endeavour in which there are a possibly unlimited number of ways of knowing and kinds of knowledge possible. Moreover, the value of any particular way of knowing and kind of mathematical knowledge is bound to the context in which it is sought and used. Mathematics is related to notions of quantity, space, movement, patterns, relationships, certainty, and uncertainty but how those ideas are represented, preserved, and applied is dependent upon the place and time of the knowers and seekers of the knowledge. There is no one best way to use mathematics, show mathematics, or preserve mathematics.

ON INCOMMENSURABILITY

Stemming from this new theory and philosophy of mathematics, I include one last claim in my dissertation which counters that of Van Eiijck and Roth (2007) who stated that an Indigenous worldview and the Traditional Western worldview (my terms) are "are incommensurable with each other" (p. 935). If one looks through the lens of the Traditional Western worldview, the mathematics (and science) valued within an Indigenous worldview would seem insignificant and irrelevant (hence, incommensurable); however, if one looks through the lens of an Indigenous worldview, there is space for the mathematics (and science) that is valued within the Traditional Western worldview as well as all others. Thus, the Transreform approach, and its associated philosophy, open mathematics teaching and learning to commensurability between the mathematical knowledge and ways of knowing that are valued within both worldviews by choosing to position oneself within an Indigenous worldview's set of values for mathematical ways of knowledge and kinds of knowledge.

ADMISSIONS AND ACKNOWLEDGMENTS

I would be remiss to not end any discussion of my dissertation work without a set of admissions and acknowledgements. First, I admit that my work is necessarily and undeniably biased by my interpretations and understandings of mathematics, the teaching and learning of mathematics, and of the work of others with respect to the two worldviews. This cannot be avoided, but neither should it be kept hidden. So to, the reader who in some way engages with my work should also recognize that the same will be true of their own thinking, researching, and documentation.

Second, out of gratitude and in the spirit of reconciliation, I acknowledge and thank all of the elders and knowledge keepers who either directly or through their work with others have influenced my thinking and feeling within my research. Without their generosity and willingness to take a risk, my life would not have undergone (nor continue to undergo) the transformations for which I am truly thankful.

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OPENING SPACE: COMPLEXITY THINKING, CLASSROOM DISCOURSE, AND MATHEMATICS LEARNING IN THE ELEMENTARY CLASSROOM

LA TECHNOLOGIE DU FORUM OUVERT: COMPLEXITÉ, LE DISCOURS DE CLASSE, ET L'APPRENTISSAGE DE MATHÉMATIQUES EN ÉCOLE ÉLÉMENTAIRE

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INTRODUCTION

A TEACHER'S DILEMMA

The genesis of this research began through observations made in my own classrooms across Canada and from the occasions I had to be part of the living systems that are elementary classrooms. My experiences as a teacher and as a teacher educator could best be described as a complex network of occurrences that hold more meaning for me as an interconnected web than if I considered each happening individually. A complex perspective provides valuable insights into the nature of living systems and how a collective may become greater than the sum of its parts through its interactions and web of connections. My personal teaching philosophy arose through observation of students' interactions and their growth and learning made possible through collaboration and engagement. As a mathematics teacher, however, my observations are less about students engaging in meaningful conversation about their mathematical thinking and learning and more about their procedural practice of skills-oriented problems. The following key questions guide my teaching in the elementary mathematics classroom where I aim to create conditions for students learning through collaborative engagement and conversation. In short, how do I

- Initiate and foster mathematical discourse in my classroom?
- Engage my students in mathematical thinking and problem solving rather than problem practicing?
- Promote and develop a complex mathematical learning community in my classroom?

Recalling a specific classroom event that I experienced as a substitute teacher during a self-imposed, mid-career sabbatical, I experienced a pedagogical shift in my teaching that would help me answer these questions. Faced with a seemingly unruly group of students one day, I recognized the futility in trying to 'direct' the energy in the classroom. What stands out for me now is the understanding that I could no more control the situation than I could the weather or the traffic. Instead, I invited the student body to reach some level of self-organized order by

choice. Rather than trying to manipulate the students to change who and what they were as individuals and as a group, I joined with them and proceeded to have my own interactions. I listened from the inside and created the space for the student body to adapt and self-organize. I recall their efforts to accommodate me as a new participant in what I would come to call a system: more a learning collective than a collection of learners, to paraphrase Davis and Simmt (2003). This experience prompted me to pursue more literature around complex systems and, most importantly, how I might create the conditions for complexity in my own classroom. My research emerges from one social meeting methodology, Open Space Technology (OST), that was designed to respond to these conditions. Through OST, I moved my thinking and instructional questions forward by finding connections among the threads of complexity thinking, classroom discourse and student engagement.

THEORETICAL FRAMEWORK

COMPLEXITY THINKING

Complexity thinking provides a lens to describe complex adaptive systems, i.e., anthills, bird flocks, weather patterns, et cetera (Johnson, 2001). Complexity thinking clarifies not only the self-organizing, adaptive capabilities of systems, like a classroom, but also illuminates the emergent qualities of the 'whole' that is at once self-similar to, yet somehow 'more than', the simple combination of its agents. Emergence, as a phenomenon of complexity, speaks of how possibilities, new ideas and ways of being arise through engagement and, as connectivity strengthens, converges into a new order. Davis and Simmt (2003) describe minimum requirements (deep simplicity) to foster complex learning systems in mathematics classrooms:

- Internal diversity
- Redundancy
- Self-organization
- Decentralized control
- Neighbor interactions

As complexivists tend to think in terms of *occasioning* emergence; that is, providing the conditions necessary to sustain the complex system, these requirements provide such conditions and support emergence in the classroom.

OPEN SPACE TECHNOLOGY: COMPLEXITY IN ACTION

OST is a social technology for large-group facilitation that promotes meaningful conversation, allows for new ways of thinking, and generates creativity and change. OST principles create a 'space' for learning where participants self-organize according to interests and needs around a common purpose (Owen, 1997).

The OST principles are

- *Whoever comes is the right person*—It is not necessary to have every person in the group join a conversation, just those people who care the most about that particular topic. If only one person comes, it might be a focused occasion for thinking and writing on the issue;
- *Whatever happens is the only thing that could have*—Participants let go of expectations and work with whatever unfolds;
- *Whenever it starts is the right time*—Creativity does not happen on a schedule. Everyone enjoys the flexibility of relaxed time constraints;

- *When it is over, it is over*—If a group finds a solution or reaches a conclusion in 20 minutes, people move on to the next group. If it takes two hours, people keep talking as others join in and/or leave independently (Owen, 1997).

One law, the Law of Mobility, reminds participants that if they are neither learning nor contributing to a group, they must leave and join a session in progress where they feel more useful and inspired or not (Owen, 1997). The Law of Mobility sits at the heart of OST: participants have full freedom of choice and accountability for their own contributions and can take responsibility for their own learning.

The OST design has three phases. First, from an opening circle, individuals voluntarily create conversations by posting topics of concern or interest. Second, self-organization allows all participants to choose where they would most like to contribute, and the Law of Mobility allows them to switch to another conversation at any time. Participants who choose not to join a conversation immediately become, in Owen's terms, Butterflies and Bumblebees. 'Social' butterflies may visit many small groups, choosing to remain on the periphery of the conversations and to make occasional contributions. Bumblebees, as their namesake in nature, are participants who move around, cross-pollinating ideas as they go. Finally, a closing circle concludes the session. Participants reconvene in the original opening circle formation to reflect on their involvement in the conversations.

In mathematics, where thinking, or cognition, and communication necessarily go hand-in-hand (Sfard, 2007), it is vital that teachers and students find ways to participate meaningfully when they communicate understanding. OST offers a simple framework that could foster such communication thereby changing the quality of mathematics discourse and students' take-up of it in significant ways.

RESEARCH QUESTION

The primary question guiding my research is

> *What changes to students' participation in learning communities and take up of the discourse of mathematics are brought about by the introduction of Open Space Technology in the mathematics classroom?*

OST provided those conditions necessary for participants to engage in meaningful exchanges that, over time, helped build connectivity among diverse participants and a space for complex emergence and convergence.

METHODOLOGY

ACTION RESEARCH

Data were generated through five recursive action research cycles in which four OST sessions were held (See Figure 1). The research took place at an elementary school in Canada. Participants were 24 grade 6 students and one classroom teacher. Data sources included transcripts from audio and video recordings of the conventional mathematics classroom and of the OST sessions; student and teacher semi-structured interviews; field notes and observations; and student journals. An integrated approach to classroom discourse analysis (Cazden, 2001) used Sfard's (2007) framework for mathematics discourse analysis and Gee's (2011) identities building tool to hear more fully the students' exchanges. I defined meaningful exchanges as interactive behaviors among semi-autonomous agents that were self-generated within the shared container of the classroom which included a common purpose, an open physical environment and mutual constraints. These behaviors could, in my analysis, lead first to

emergence in the form of shared mathematical information, ideas as well as insights and consequently, to convergence toward a new collective understanding, a new order, or change in how students understand numeracy. Three types of talk emerged from the data as student conversations showed sequences of such meaningful exchanges: (1) sharing information; (2) building knowledge; and (3) exploring possibilities.
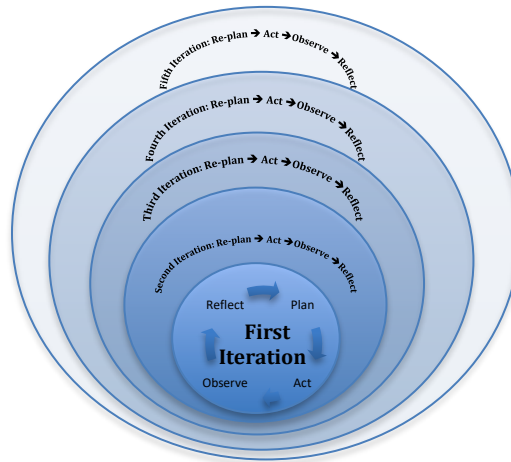
Figure 1. Recursive action research cycles.

Complexity thinking: Characteristics and necessary minimum conditions	Discourse focus: Structure and moves	In mathematics classroom discourse, to what extent do we see, hear and/or experience:
Internal diversity and redundancy	Participant structure	Whole class, small group, dyad, individual
Decentralized control (Self-organization)	Discourse phase	Univocal—transmitting, receiving Dialogic—generating, listening, questioning
	Social talk	Initiation, response, evaluation, teacher mimicry, illusory participation
Organized Randomness (Liberating Constraints)	Sequence development	Adding on, complementing, challenging
	Discourse function	Asking, explaining, giving information, checking in, offering opinion, clarifying, observing, accepting, disagreeing, rejecting, blocking, re-voicing, summarizing
Neighbor Interactions	Sequence orienting	Commenting, organizing, problem solving, reviewing
	Forms of mathematical knowledge	Procedures, concepts, strategies
	Mathematical discourse use	Mathematical words, narrative, routine, visual representations
Emergence	Types of talk	Sharing (storytelling, commenting) Building (organizing, adding on) Exploring (testing ideas, playing with)
	Blocking	Interrupting, inhibiting, controlling space, telling more than listening to others

Table 1. Complexity thinking analytic lens for classroom discourse: Third iteration.

ANALYTIC LENS

My intention in developing the coding instrument for the data and building the *complexity thinking analytic lens for classroom discourse* was to illuminate, from the perspective of complexity thinking, the structure of discourse and account for the variety of discourse moves arising in the classroom (See Table 1). The coding, developed iteratively at each analysis phase, makes visible the students' participation in the OST sessions (e.g., participants structure, social talk, blocking, etc.) and accounts for their take-up of the discourse (e.g., univocal, dialogic, discourse use, and types of talk including sharing, building, and exploring, etc.) all the while maintaining a complexivist view of the complex learning system and the minimum conditions that ensure its viability.

FINDINGS

SHARING

As students began to self-organize into circle conversations, the most apparent instances of meaningful exchanges appeared as sharing opportunities with students offering insights and personal reflections and stories on a variety of topics as indicated in the diagram below. The contributions of each student as they shared in turn their ideas about patterns and their usefulness, served to create a common sense of awareness around their topic. As the discourse continued to develop among them, the significant examples of sharing among students set the stage for their mathematical thinking.
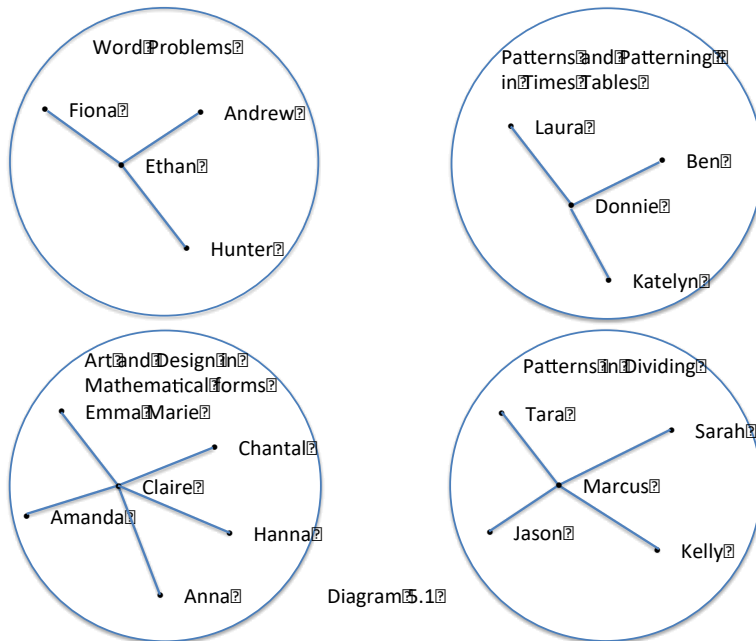
Figure 2. Conversation circles.

In some instances, exchanges became significant as students made connections to personal life experiences and used mathematical words to offer examples about the topic that were meaningful to them. Their examples showed understanding of mathematical concepts as students used patterning narratives to describe their use. In turn, students related them to familiar routines in shared understanding of operations, such as the multiplication tables.

Engagement with the topic was clearly articulated with each contribution that the students shared. Students' input propelled the conversation forward as students accepted each comment and often inspired others to make additional connections and share from their own perspective. Students often used the device of storytelling and the term 'story' to motivate the conversation and participate in the classroom discourse through sharing.

BUILDING

The OST sessions provided the space for students to choose freely their areas of interest and the manner of their contributions with the purpose of building knowledge together by conversing within a collaborative setting. From the Latin *to work together*, collaboration changed the nature of the work in this mathematics classroom and relocated the subject as the work of mathematicians who make and find patterns; rather than calculate answers to textbook problems and record solutions on the whiteboard. The experience of the circle conversations provided opportunities for students to think actively, to 'own' their work, and to take responsibility for completing it efficiently and accurately. Evidence of students questioning, elaborating on what others said and offering their own new perspective on issues became apparent through subsequent iterations of the complexity thinking analytic lens.
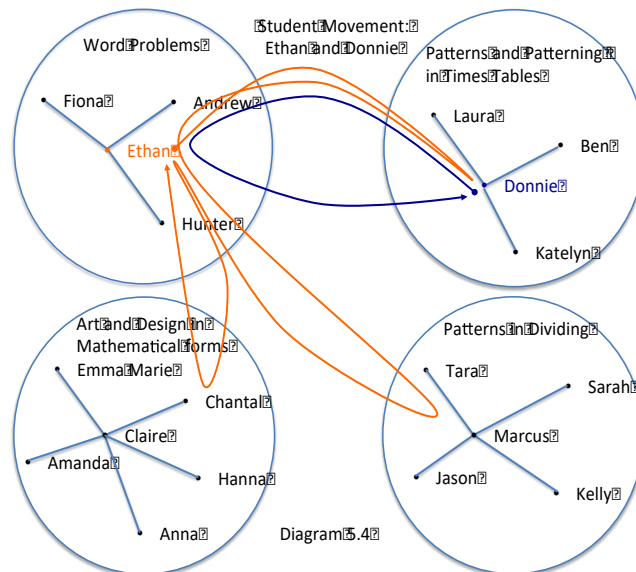
Figure 3. Individual student movement.

Taken together, students' verbal explanations and visual representations of problems in their journals with the increased student movement through the open space of the classroom indicated efforts in the circle groups to clarify their mathematical thinking with each other via verbal and visual information and to build upon their mathematical knowledge through increased connectivity within the classroom network. As indicated in the diagrams of student movement, overall connectivity among the students increased through the iterations of the research cycles and strengthened meaningful exchanges as they were more often shared through the growing classroom network.

EXPLORING

Asking questions like, "What would happen if you multiplied an even number of odd numbers (e.g., $3 \times 3 \times 3 \times 3$) or an even number of even numbers (e.g., $2 \times 2 \times 2 \times 2$)?" or "What would

happen if you multiplied an odd number of odd numbers (e.g., $3 \times 3 \times 3$) or an odd number of even numbers (e.g., $2 \times 2 \times 2$)?" opened the door to a wider conversation that integrated patterning concepts naturally with multiplication concepts. Students began to consider these patterns from the level of multiplicative thinking rather than relying solely on earlier conceptions of multiplication as repeated addition or, perhaps, memorization of facts. A systematic investigation of such issues in number theory allowed students to consider what makes sense in any given situation, to deepen their understanding of number relationships and develop a broader sense of numeracy and confidence in problem solving. This example showed the extent to which the conversation among participants explored possibilities to obtain a new understanding.
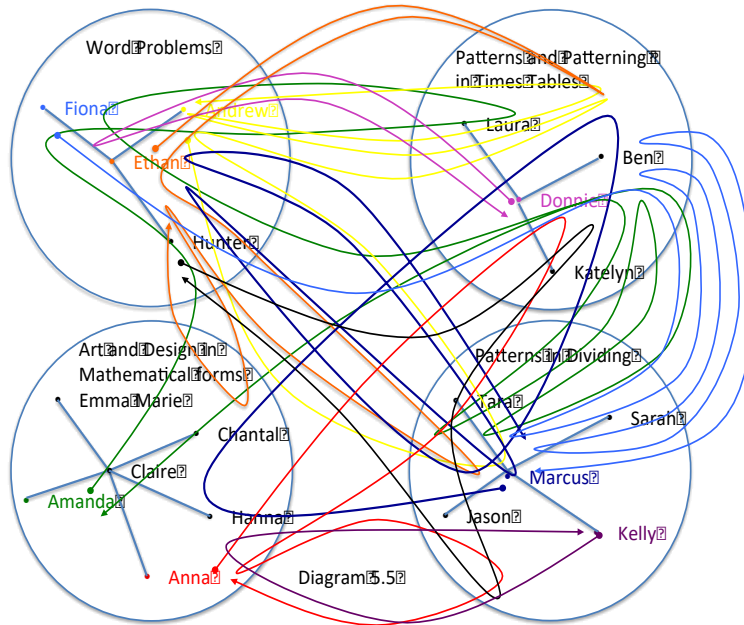
Figure 4. Whole class movement.

This most complex level of classroom discourse emerged gradually through several iterations of the research cycle as students expressed new ways of thinking and openings for exploring and investigating new possibilities. Efforts to deviate from the otherwise monotonous task of completing basic facts or computation refocused students' attention on deeper understandings of number relationships and created a broader sense of numeracy and confidence in problem solving.

FINDINGS I: STUDENT ENGAGEMENT

Initial findings centered around student engagement. An increase in student participation and sustained engagement through sharing became apparent in the early iterations of analysis as I considered data from the traditional classroom activities in contrast to data generated through the first iterations of OST. Students relocated consistently in the classroom space to engage in conversation as Bumblebees, successfully cross-pollinating information as they went. The contributions of disengaged learners in the classroom, rather than lost or ignored, were counted among more engaged students as Butterflies.

Three types of talk in mathematics were noticed: first, how the students relocated in the classroom space to engage in conversation; second, how the students relocated the subject into

a discursive space where they could build upon their ideas; and third, how the classroom discourse relocated toward a more dialogic space open to possibilities.

FINDINGS II: MATHEMATICS DISCOURSE

Subsequent findings focused on students' take-up of the discourse of mathematics. Students relocated the subject into a discursive space, different from their traditional classroom, through building. In this way, doing mathematics became a new activity with students showing self-directed responsibilities to engage in collaborative problem solving and accountability to communicate understanding with peers. Students empowered with freedom and choice to pursue meaningful exchanges where they could, increased student connectivity in the classroom and enriched the classroom network. Classroom discourse shifted from predominantly univocal toward a more dialogic space open to exploring possibilities. Close analysis of classroom discourse revealed students' efforts to take up the features of discourse in meaningful ways through mathematical words, routines, narratives, and visual mediators.

FINDINGS III: PARTICIPATION

Additionally, findings were noted in the commitment to and sustainability of student participation in small-group learning communities around the open space of the classroom. Student interviews and journal records indicated that while OST was useful to students because many people could participate and they could get ideas from others; it was difficult as well because of the frustration felt when students could not be at every conversation. Evidence of the emergence of mathematics as a collaborative process of problem solving became apparent as students sustained meaningful conversations and, in some instances, proposed original strategies for problem solving. Significant obstacles to participation and more meaningful exchanges emerged through blocking as some students demonstrated antagonistic, heedless, and excessive methods for silencing and defying other students' efforts to share or build knowledge together.

EDUCATIONAL IMPLICATIONS

MATHEMATICS EDUCATION AND STUDENTS

OST offers teachers an effective methodology that they can incorporate in their approach to mathematics education. Firstly, it empowers students to become active participants in their own learning while the reduced role of authority, found traditionally in the teacher, becomes a shared responsibility among the students. Secondly, OST places the onus on students to self-organize around topics they feel are important to discuss, thereby warranting fuller and prolonged engagement. Thirdly, OST ensures that all students are given full choice over their actions and full accountability for their learning so the issue of student compliance becomes irrelevant as the system accommodates all degrees of participation. Therefore, there is no need to opt out or remove oneself entirely from the system. In general, the minimum conditions of complexity thinking as embodied in the principles of OST serve to *occasion* learning rather than cause it and may promote a more sustainable learning system.

CLASSROOM DISCOURSE AND ANALYSIS

OST provides opportunities for mixed-ability groupings and collaborative activity, thereby enriching classroom discourse. Critical, in this regard, is the teacher/facilitator's role in supporting learners through the modeling of productive discourse and the holding open of space by acknowledging the self-organization of the complex learning system where possibilities might emerge for meaningful participation and fruitful conversations.

OST also opens the space for students to make choices, give and take feedback productively, and communicate their mathematical thinking in effective ways, thereby encouraging a growth mindset. The facilitation of OST in this classroom provides space and opportunities for all students to remain active participants in their mathematics learning. As such, this meeting methodology holds potential for teachers looking for ways to encourage choice, freedom, and accountability in their students. As the data indicated to me in many exchanges, students need to learn how to give and take feedback more productively to advance their thinking rather than inhibit themselves or others from building new understandings. Continuing to use OST regularly, along with skill development in creating and maintaining conversations through effective communication strategies and encouraging growth mindsets may be beneficial to many students.

ACADEMIC RESEARCH

Complexity thinking's minimum conditions (self-organization, decentralized control, diversity, redundancy, neighbor interactions) ask researchers to view and design classroom research differently. My own observations and subsequent analysis, in this way, echo Owen's (1997) proviso for OST participants: "*Be prepared to be surprised*" (p. 101). In complexity terms, the introduction of one small change may bring about unexpected transformation on a much larger scale. This, in many ways, was evident for me in the research and it is my hope that further research involving OST and mathematics might assist other teachers in experiencing their classroom differently as well.

OST is complexity in action and opens a space for future possibilities to emerge. In concert with action research, teachers may find ways to illuminate student engagement and increase classroom connectivity with students as their knowledge network develops in an open space of learning. They may also embrace action research as a methodology to illuminate how students interact and communicate in an effort to acknowledge fully their complicity in the development of classroom discourse. Contributions to a growing body of classroom research in this way will inform and enrich their practice as well as the practice of others.

CONCLUDING THOUGHTS

When this research classroom met the minimum conditions of complexity thinking that generated self-organization, students increased their participation in mathematical activity and gained insights into their mathematics knowledge through sharing, building, and exploring. This was created through the intervention of OST as a social meeting methodology enabling more meaningful exchanges and collaboration among students. Promoting such opportunities for complexity in action and opening the classroom space so that these minimum conditions are met supports mathematical discourse and increases knowledge building and understanding for students.

Many questions continued to arise for me as I considered the implications of the findings. For example, how best to support students in creating meaningful exchanges to build capacity for increased sharing, building, and exploring, and how best to assist students in the art of hosting a conversation? Could the sharing, building, and exploring increase by investing in the development of communication skills for participants specifically? As well, based on some students' reflections after the OST sessions, I recognized their expression of enjoyment in a classroom where having fun is not always appreciated. Future questions then occurred to me including Is playfulness an avenue into deeper 'exploring' of mathematical concepts (e.g., patterns, multiplicative thinking)? And, is 'having some fun' (student journal) usurping the traditional importance of memory recall of facts (e.g., times tables drill) hence supporting

convergence and, specifically, numeracy? Investigating these and other questions yet to emerge now become the foci of subsequent iterations of my complexity thinking action research cycle.

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INVESTIGATING MATHEMATICS TEACHERS' KNOWLEDGE FOR TEACHING AND THEIR LEARNING TRAJECTORIES

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This study investigated three secondary mathematics teachers' knowledge for teaching and their learning trajectories from their own schooling through their establishment as experienced teachers in their education system. Three case studies were conducted through semi-structured research instruments and interviews. Other data such as prior and current official curricula and materials related to teachers' professional development were also collected. The findings indicate that there are five categories of professional knowledge for teachers: subject matter knowledge, curriculum knowledge, knowledge of students, mathematics pedagogy, and knowledge of professional development. Each has its subcategories. The categories and subcategories have their own characteristics and are connected with each other at some levels. The findings also indicate that the process of teacher professional development is complex. Teachers learn from various formal and informal sources. Teachers' knowledge for teaching is not static but is dynamic. Their knowledge is shaped by changes in school curriculum and the teachers' choice of approaches to and learning directions for their professional development.

INTRODUCTION

Over the last four decades, research has provided evidence that teachers' professional knowledge is closely related to students' achievement and quality of instruction (Baumert et al., 2010; Hill et al., 2008). Many researchers believe that content knowledge and pedagogical content knowledge are vital components of teachers' knowledge that affect students' achievement (Baumert et al., 2010; Kleickmann et al., 2013).

Since Shulman (1986, 1987) developed a seminal model of teachers' knowledge, many researchers have refined this model and investigated both teachers' knowledge (e. g., Ball, Thames, & Phelps, 2008) and the development of their professional knowledge (Depaepe, Verschaffel, & Kelchtermans, 2013). In recent decades, researchers have focused on teachers' subject matter knowledge and pedagogical content knowledge. They have designed various research instruments to measure teacher's knowledge (e.g., Baumert et al., 2010; Krauss et al., 2008). At present, there is no universally accepted way to measure teachers' knowledge for teaching.

Measuring mathematics teachers' knowledge for teaching and understanding their process of knowledge growth is valuable for mathematics education. It would be helpful to determine what teachers need to know in order to teach elementary or secondary school mathematics. This

information could improve teachers' programs and their professional development. Hence, this study investigated three secondary mathematics teachers' knowledge in subject matter, official curriculum, students' misconceptions, and pedagogical content knowledge and professional learning trajectories.

LITERATURE REVIEW

INFLUENTIAL MODELS OF TEACHERS' KNOWLEDGE

Since Shulman (1986, 1987) developed a model of teachers' knowledge, educational researchers have refined and expanded the model to investigate mathematics teachers' knowledge. In this section, I will explain Shulman's model and a few other influential models that have been developed in recent decades.

Shulman (1986) suggested that teacher knowledge consists of content knowledge, pedagogical content knowledge, and curricular knowledge. He defined content knowledge as

> the account and organization of knowledge per se in the mind of the teachers….To think properly about content knowledge requires going beyond knowledge of the facts or concepts of a domain. It requires understanding the structures of the subject matter. (p. 9)

Subject matter knowledge includes substantive and syntactic structures:

> The substantive structures are the variety of ways in which the basic concepts and principles of the discipline are organized to incorporate its facts. The syntactic structure of a discipline is the set of ways in which truth or falsehood, validity or invalidity, are established. (p. 9)

Pedagogical content knowledge is the "blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented and adapted to the diverse interests and abilities of learners for instruction" (Shulman, 1987, p. 8). Teachers' curriculum knowledge encompasses understanding the scope of programs designed for the teaching of particular subjects and topics at a certain level, and the variety of available instructional materials in the programs. Curriculum knowledge also includes knowledge of alternative curriculum materials for a given subject or topic in a grade. In addition, it includes lateral and vertical curriculum knowledge. Lateral curriculum knowledge "underlies the teacher's ability to relate the content of a given course or lesson to topics or issues being discussed simultaneously in other classes" (Shulman, 1986, p. 10). Vertical curriculum knowledge "is familiarity with the topics and issues that have been and will be taught in the same subject area during the preceding and later years in school" (p. 10).

Ball et al. (2008) refined Shulman's (1986, 1987) model and its categories of mathematics teachers' knowledge based on their empirical research. Their model includes two major categories: subject matter knowledge (SMK) and pedagogical content knowledge (PCK). Each consists of subcategories. SMK includes common content knowledge (CCK), horizon content knowledge (HCK), and specialized content knowledge (SCK). They defined CCK as "the mathematical knowledge and skill used in settings other than teaching" (p. 399). The SCK refers to "the mathematics knowledge and skills unique to teaching" (p. 400). HCK "is an awareness of how mathematical topics are related over the span of mathematics included in the curriculum…. It also includes the vision useful in seeing connections to much later mathematics ideas" (p. 403). PCK includes knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum (KCC). KCS is knowledge that combines knowing about students and knowing about mathematics; KCT combines knowing

about teaching and about mathematics. They placed Shulman's (1986) third category, curriculum knowledge, as KCC.

The model of Fennema and Franke (1992) included four components of teachers' knowledge: knowledge of mathematics, knowledge of learners' cognition in mathematics, pedagogical knowledge, and beliefs. The *content of mathematics* is teachers' knowledge of concepts, procedures, and problem-solving processes within related content domains. It includes knowledge of concepts underlying procedures, relationships between concepts, and knowledge organization. *Pedagogical knowledge* refers to teachers' knowledge of teaching procedures, including planning, classroom routines, and general pedagogical strategies. *Knowledge of learners' cognition* is defined as knowledge of how students think and learn, and how this happens within specific mathematics content. It also consists of knowing how students acquire mathematics knowledge, and understanding the processes students use and difficulties they may have (Fennema & Franke, 1992).

Other researchers working in various countries around the world have also created models of mathematics teachers' knowledge. German mathematics education scholars (Baumert et al., 2010; Krauss et al., 2008) investigated pedagogical content knowledge (PCK) and mathematics content knowledge (CK) at the secondary school level. In their research, pedagogical content knowledge (PCK) is constructed of three components: mathematics tasks, student misconceptions and difficulties, and mathematics-specific instructional strategies; i.e., tasks, students, and instruction. The *tasks* dimension assesses teachers' ability to identify multiple solution paths; the *students* dimension assesses teachers' ability to recognize students' misconceptions, difficulties, and solution strategies; and "the instruction dimension assesse[s] teachers' knowledge of different representations and explanations of standard mathematics problems" (Baumert et al., 2010, p. 149). Mathematics content knowledge (CK) in these studies refers to in-depth background knowledge of the secondary school mathematics curriculum in Germany.

Rowland and his colleagues (2005) created an empirically-based conceptual model called the "Knowledge Quartet" for observation and analysis of classroom teaching. It focuses on classroom application of teachers' mathematics-related knowledge. It includes four dimensions: foundation, transformation, connection, and contingency. Each category is composed of a few subcategories. *Foundation* includes overt subject knowledge, use of textbooks, and reliance on procedures. *Transformation* comprises use of instructional materials, choice of examples, and presentation. *Connection* includes connections between procedures and concepts, decisions about sequencing, and recognition of conceptual appropriateness. *Contingency* comprises responding to students' ideas and teacher insight.

RESEARCH ON PROFESSIONAL DEVELOPMENT

Teachers' professional development takes place through both formal and informal learning. Formal learning refers to a structured and organized environment, while informal learning refers to activities in which a teacher participates in the workplace (Grosemans, Boon, Verclairen, Dochy, & Kyndt, 2015). Many educational scholars have proposed general ideas for teacher professional development. They have suggested: collaboration; collective participation; a focus on crucial problems of curriculum and instruction; continually gaining knowledge, professional habits and norms; providing opportunities for teaching and feedback; a focus on students' learning; follow-up activities; long-term support; coaching teachers; peer observation; inquiry-oriented learning approaches; and school-based learning communities (Desimone, 2009; Hill, Beisiegel, & Jacob, 2013; Little, 2006).

However, because teacher education lacks a common theoretical, well-defined, and shared conception of the mathematical knowledge required for teaching, researchers have not reached an agreement on what SMK and PCK can be tailored for teaching (Hoover, Mosvold, Ball, & Lai, 2016). Researchers understand that effective research instruments are vital to measure teachers' knowledge and further improve teachers' programs and teaching. They have suggested the further investigation of teachers' knowledge for teaching, including designing effective instruments to examine mathematics teachers' content knowledge and pedagogical content knowledge (Blömeke & Delaney, 2012; Hoover et al., 2016).

METHODOLOGY

PARTICIPANTS

The participants were selected from among Ontario mathematics teachers who worked in an urban metropolis. They were from three different schools and all three participants were male. In this study, they go by the pseudonyms of Ron, Alan, and Hardy. They were mathematics department heads. They studied under a five-year high school system when they were students themselves while the current Ontario high school system spans four years. Ron attained a bachelor's degree in the mathematics teaching stream at the University of Waterloo, and a bachelor of education degree at the University of Western Ontario in 1993. He attained a masters' degree in mathematics for teachers at the University of Waterloo in 2013. Alan's majors were in applied mathematics and history at the University of Toronto. He attained a bachelor of education degree from the Ontario Institute for Studies in Education at the University of Toronto. He has been a teacher since 1993. Hardy's major was bioscience at the University of Toronto. He earned a bachelor of education degree from Queen's University. He had taught for 24 years when he was interviewed in 2014. At the time of the study, Ron taught in an independent school, and Alan and Hardy taught in public schools.

DATA COLLECTION

Data were collected through multiple resources: interviews, documents written by the participants, classroom observation, emails with participants, documents published on a participant's website, and some documents related to participants' knowledge, such as official curriculum (K-12) and textbooks recommended on official websites and used by participants. I also collected and examined first- and second-year undergraduate linear algebra and calculus courses and course outlines designed by undergraduate mathematics instructors. Moreover, I collected published articles about the changes in the Ontario secondary school education system and curricula and their impact on teaching and learning.

DATA CODING AND ANALYSIS

To derive evidence of mathematics teachers' knowledge from the data, pattern-matching strategies (Creswell, 1998; Yin, 2014) were used for logic analysis. I designed categories and subcategories of participants' professional development trajectories and four types of mathematics teachers' knowledge based on the literature. When examining each case, a comparison method was used to analyze the data. Through the comparison of the three cases, the categories and subcategories of teachers' knowledge were modified or changed.

I used inductive reasoning to seek some unanticipated and core categories and subcategories for teachers' knowledge. "Inductive analysis refers to approaches that primarily use detailed readings of raw data to derive concepts, themes, or a model through interpretations made from the raw data by an evaluator or researcher" (Thomas, 2006, p. 238). For each case, I also sought salient themes and foci on professional development and the participant's knowledge for

teaching. Through cross-case analysis, I sought participants' common concerns and foci in their knowledge and learning trajectories.

FINDINGS

TEACHERS' LEARNING TRAJECTORIES:

The three participants first ensured the proficiency of their mathematics content knowledge for teaching mathematics. Ron thought he was prepared with enough mathematics knowledge because his bachelor's degree was in a mathematics department's teaching stream. He was taught every subject related to secondary school mathematics. He believed that he did not need to upgrade his mathematics knowledge when he became a teacher. After some years of teaching, he felt bored with teaching the same content daily, and then earned a master's degree in mathematics for teaching. As a result, he felt he had refreshed his mathematics knowledge for teaching. When Alan and Hardy became secondary school teachers, they upgraded their mathematics content knowledge. They made sure that they did not make mistakes when teaching. Alan felt that he had not understood a few concepts in depth, such as logarithms and operations. He upgraded these weaknesses for teaching. Because Hardy's major was not in mathematics, he made up for some mathematics content he felt he needed to upgrade. Hardy and Alan also stated that, as novice teachers, they encountered challenges in that some students did not want to learn, or they did not know how to interpret mathematics content in classes. They took some time to make their teaching flexible and effective.

They took various paths in their professional development. They learned from formal and informal activities. They learned from workshops and conferences organized by the Ministry of Education or their board of education. They learned from their own experience of teaching and learning, their secondary school and undergraduate teachers, their colleagues, their students, textbooks and curriculum materials, internet information, and self-directed learning. For example, when Hardy encountered a challenge that one of his students did not want to learn in classes, he asked his colleagues how to deal with this problem. When Ron's students asked him how to solve difficult questions, he tried various ways to solve these problems. When he invigilated mathematics contests, he saw students' answers and found good ways to solve problems.

TEACHERS' MATHEMATICS KNOWLEDGE

The participants were asked to solve nine mathematics questions, which were related to the mathematics concepts of absolute value, trigonometric identity, quadratic equations and function transformation, simplifying rational expressions, and proof; these included one algebra question and one geometric question. They told me that the concepts of absolute value and radical equations had been removed from the official curriculum. They did very few proof questions. Alan said that he had last seen most of the questions they were asked to solve in this study 15 years ago. He had not seen them since he left university. When the three teachers solved an absolute value question: *How many integers satisfy $|2x − 5| < 9$?*, all of them drew graphs to get solutions first, and then they used the absolute value concept to solve the questions. When I asked them why they drew graphs first to solve this problem, they answered that students were used to using calculators and drawing graphs to solve problems.

One algebraic proof question was used in this study:

1. Proof: if $a, b \in \mathbb{R}$, then $a^2 + b^2 \geq 2ab$, if and only if when $a = b$, $a^2 + b^2 = 2ab$
2. Suppose $x + y = 7$ ($x, y \in \mathbb{R}$), what is the maximum value of xy?

Ron proved question 1) and solved 2) with quadratic function or derivative methods. He did not use the conclusion of 1) to solve 2). After I reminded him that he could do this, he said: "*They [students] actually understand the tool. They know the square maximum area. So usually, when you do that, when a = b, ab gets maximum value*". Hardy and Alan solved question 2) with a derivative of a quadratic function. They did not prove question 1), and did not realize the relationship between questions 1) and 2). They told me that the curriculum had moved away from proof questions and inequality questions.

PERSPECTIVES ON THE ONTARIO CURRICULUM

The three participants thought that the official curriculum emphasized precalculus and the use of technology. The secondary school curriculum had moved away from or de-emphasized some important mathematics concepts. The participants also knew the changes in the elementary mathematics curriculum. Ron stated that "*most of the high school math curriculum is precalculus. There is a little bit of geometry and a little bit of statistics, but the vast bulk of it is for preparing students for calculus*". He reported that proofs were taken away from all the mathematics courses except for a little content. "*The only good thing left was trigonometric identities. But it is downgraded. It is a small component for the grade 11 course and a little in the grade 12 course,*" Ron said. Conic sections, axis rotation, polar coordinates and equations, complex numbers and inductive reasoning were also taken out of the high school curriculum. Combination and permutation content only appears in the Data Management course, which is taken by students who are not in the stream to pursue advanced mathematics.

Alan believed that technology was good for showing and demonstrating mathematics content and ideas. However, as students were doing less abstract proving mentally with pencil and paper, they did not get a deep understanding and appreciation of mathematics. Hence, Alan thought, "*the weakness of the curriculum is for the academic level* [students]*, it is emphasizing technology too much for the academic level. It emphasizes on the technology at the expense of more abstract thinking*". He believed "*if we were still doing the proofs and the technology added to that, that would be good. But because they do everything by technology now, they are missing an abstract appreciation for math*".

Hardy worried about the official expectations for students in the content changes. He said, "*I think that our expectations have changed in terms of what we expect. There is also content that we used to think was important, that I think now, at least at the provincial level, they do not think as important anymore*".

When the participants chose curriculum materials, they used standard textbooks to make sure all students in their schools had the same learning progression. However, they might add some mathematics content that they had discussed with colleagues. They also provided proofs and reasoning for important content. They selected supplemental documents for students' learning. They also selected internet materials or websites from which students could learn.

IDENTIFYING STUDENTS' MISCONCEPTIONS

Five typical students' wrong answers were provided to the three participants and they were asked to identify the students' misconceptions. They identified reasons for the wrong answers from the perspectives of students' understanding of the concepts, their prior learning in elementary mathematics, their behavior of learning, and teachers' teaching methods. The participants pointed out which concepts students might not understand and how to address them. Hardy and Alan also stated that students might not do assignments and some brought misconceptions from elementary mathematics, which might lead to mistakes.

Ron recognized the student mistakes in the case of simplifying $(xa + xb) / (x + xd) = (x^2ab) / (x^2d)$. Ron thought that, in this case, when students collected like terms, they had multiplied them. He believed that the students knew the difference between addition and multiplication. However, they might not know that addition was repeated multiplication, and essentially, these two operations are the same thing. They might think that, if there was an x and another x, then the result was x squared. He thought, in this situation, the students actually did not know that the meaning of x squared was x times x.

When Alan was asked why students made mistakes such as $\frac{x}{2x+3} + \frac{3x+1}{3x+5} = \frac{x+3x+1}{2x+3+3x+5} = \frac{4x+1}{5x+8}$, he said that students' skills in fractions were often very low, and they did not understand fraction operations because they had not done enough drills in this area. He said that, because students did not understand fraction operations, then they made mistakes when calculating rational expressions; he felt this was the source of the mistake above. Hardy thought because students memorize a lot of rules and their understanding is rule-based, they might mix up these rules unless they know them well and how to apply them. Hardy pointed out when students learned fraction operations, they did not understand the concept of a unit fraction. Hence, when they added two rational expressions, they made similar mistakes. He reminded students that "*remember that you can only add and subtract things which are 'alike'. These are not alike; you cannot add them*".

PEDAGOGICAL CONTENT KNOWLEDGE

In this study, teachers' pedagogical content knowledge was tested, how teachers blended mathematics content knowledge, knowledge of students, knowledge of curriculum and pedagogy in teaching. Participants were asked about five topics related to teaching. The questions asked in this part of the study were about how the participants introduced variables, concepts of trigonometry, and polynomial division, how they used technology, and how they used analogies or metaphors in their teaching.

When the participants taught, the curriculum materials they chose integrated students' prior knowledge, new knowledge, the students' familiar environment or life, analogy, metaphor, comparison and technology. In their teaching, they used mathematics teaching software, projects, and real-life materials to demonstrate mathematics content. They also considered the alignment of algorithms and fundamental ideas between elementary, secondary, and undergraduate mathematics. For example, when the three participants taught polynomial division, such as $(6x^2 + 5x + 6) / (2x + 3)$, they all refreshed the algorithm of long division. Each teacher also had his personal teaching characteristics. Ron said:

> When I teach polynomial division, I usually go back to long division. They learned long division, but they just learned it as algorithm, they did not learn why it works. So they just know steps. So I go back and do some long division and talk about why it works……I talk about counting systems in base 10, because we count in base 10, but students do not realize base 10 is just one base. There are bases—you can count in other bases if you want to. And I say, in my opinion, you count in base 10 because we have 10 fingers. If we had 8 fingers only, we would probably develop base 8 in our counting system. So we do division questions with base 8, we see the process is the same, and then we do division questions in base x, and you see the process is the same, and you see that is polynomial division.

When Alan taught this topic, he asked students if they could do long division of integers. He found that his students did not remember the method of long division because they used calculators in elementary school. However, the textbook authors assumed students already knew long division when they were exposed to polynomial division. He said, "*I do one with integers* [division] *on one side on the board, and I do the same thing with polynomial division on the other side. So they* [the students] *can see the same procedures*".

Before Hardy taught polynomial division such as $(6x^2 + 5x + 6) / (2x + 3)$, he reviewed how to turn improper fractions into mixed fractions. Then he reminded students that the methods and procedures of polynomial division were similar to improper fraction operations. He said that he encouraged students to do polynomial division in the same way that they used the algorithm for long division because this method was similar to what they were used to in terms of division. Of course, he also taught students the synthetic division method.

CONCLUSIONS AND IMPLICATIONS

In this section, I draw conclusions about the five aspects of teachers' knowledge. First, teachers' subject matter knowledge will concentrate on the requirements of the official curriculum. There is a positive association between the depth and breadth of teachers' mathematics knowledge and the requirements of their school curriculum. If teachers are not prepared with enough mathematics knowledge as is required in the school curriculum before becoming teachers, they will make up for this knowledge when they are teachers. If they learned mathematics content that does not appear in the curriculum, this knowledge will fade gradually.

Second, teachers can recognize the strengths and weaknesses of an official curriculum. Because of the strong mathematics knowledge and learning experience of the participants in this study, they provided supplementary materials to their students to make up for the weaknesses in the curriculum and ensure students' learning continuity, especially for STEM major students.

My third conclusion is about teachers' identification of students' misconceptions. Teachers can recognize which concepts students understand or are vague They also recognize the possible roles of misbehavior and motivation in learning.

Fourth, teachers' pedagogical content knowledge integrates mathematics content, curriculum, students' prior knowledge and cognition, pedagogy, analogy, comparison, metaphor and technology together. Teachers teach flexibly. Each teacher may have different approaches to teaching, but the destination is the same: making students grasp mathematics content and fundamental ideas. Teachers' enacted curriculum materials are based on the curriculum and textbooks, but are adjusted based on students' situations.

Finally, the process of teachers' professional learning is complex. Teachers develop their professional knowledge through formal and informal activities: organized activities and workshops, self-selected activities, the influence of their own teachers, students' challenges, colleagues, textbooks, and internet materials. Through comparing the content they teach, the challenges they receive from students, and their reflections on their own teaching, teachers make judgments about their knowledge deficiencies and seek approaches to upgrade their knowledge. Each teacher makes sure to teach correct mathematics content and uses proper ways to make content comprehensible to students. Their self-monitoring and self-motivation drive teachers to seek opportunities to upgrade their professional knowledge. Teachers' knowledge is also shaped by changes in the school curriculum and their choice of approaches to and learning directions for their professional development.

This study implies that educational organizations need to create opportunities for teachers, especially novice teachers, to upgrade their knowledge as needed. Preservice programs may evaluate student teachers' mathematics content knowledge and add mathematics content that fosters teachers deeply understanding content, and add fundamental ideas that align elementary, secondary, and undergraduate mathematics. In-service teacher programs need to be tailored to teachers' needs in teaching such as how to teach core mathematics concepts, how to identify students' misconceptions, and how to use technology to enhance their teaching. Teacher

programs may refresh teachers' subject matter knowledge after they have taught for some years. This study also implies that teacher programs may add content related to self-monitoring of one's knowledge for teaching and ways to upgrade professional knowledge. Moreover, pedagogical content knowledge is the art of integrating or blending mathematics content, students' cognition, curriculum, and pedagogy together. Different teachers with different students may have different ways to teach the same content. Hence, teacher programs may provide exemplary lessons for teachers' observation, discussion, and imitation. This also implies that novice teachers should have confidence in themselves and seek opportunities to develop their professional knowledge.

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Ad Hoc Sessions

Ad hoc Session

WHAT/HOW CAN WE LEARN FROM THE DEAF MATHEMATICS CLASSROOM?

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This *ad hoc*-session had its roots in the idea of looking from different perspectives and with entirely new eyes on a piece of data that has been gathered in the course of a larger project on deaf students' learning and conceptualization of mathematics—and the *ad hoc* interest in participating in such a session expressed in conversations during the poster session. The video data (Krause, 2018) presented an episode from a grade 5 deaf geometry classroom to provide an insight into how mathematics is taught and learned in such a specific setting, serving as input for the discussion, especially for those who came to the session without any previous experience with deaf education.

SOME SPONTANEOUS IMPRESSIONS

A first spontaneous insight concerned the observation that when first encountering a mathematical idea through discourse, 'everything' has to be an action to make sense of the signs as means to communicate such that conceptual and referential understanding goes hand in hand. Moreover, mathematical experience might become more likely to be gathered by being the mathematical object, being put in the centre of the mathematical activity, in this case rotating a figure around a point for accessing the idea of point symmetry. While this action-based approach might be beneficial to provide concrete context to understand basic concepts that are accessible through perception, deriving at more complex and 'abstract' concepts that cannot be approached through action might become an issue. This sheds an interesting perspective on deaf students' struggles with abstraction as reported by practitioners (e.g., Brinkley 2011).

The discussion quickly moved away from the concrete case of deaf learners but towards finding similarities and differences in their way of learning to the case of others, e.g., the way dyslexic learners process information. One participant brought up that dyslexic learners process information holistically and struggle with a linear representation of information, just as deaf signers do. Insights on how mathematical knowledge is represented holistically through gestures or, more specifically, in sign language, and how this benefits the deaf learners might also become beneficial to support dyslexic learners. This potential interaction between two fields in special education that have not necessarily been seen as linked before provides an excellent example of how looking at a specific group of learners might provide a benefit beyond. But also much more in general, we agreed that we can learn a lot from "asking" different groups of learners, maybe also heading towards universal task and teaching design. However, an important methodological question with which we left the discussion is How we can become better able to come to understand the approaches of the special students? Certainly, this demands for combining a mixture of different methods of gathering and analysing data, both taking into account the specific background of the learners. Non-verbal expression might be taken into account through drawings to provide an additional resource, e.g., for the case of deaf or dyslexic learners, allowing them to fix their thoughts more holistically than possible through language.

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THE MATHEMATICAL ASSOCIATION OF TANZANIA (MAT) EXPERIENCE IN CMESG CONFERENCE

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It was a great pleasure for Mathematical Association of Tanzania (MAT/CHAHITA) to be represented by a chairperson of the association among the more than hundred participants of the 41st conference of Canadian Mathematics Education Study Group (CMESG), which took place at McGill University, Montréal from the 2nd to 6th June, 2017. The chair attended this conference under the full sponsorship of *Capacity Development for Mathematics Teaching in Rural and Remote Communities in Tanzania* simply known as in Tanzania as *Hisabati ni Maisha*. This project is of three universities which are University of Dodoma (Tanzania), University of Alberta and Brock University (Canada). MAT/CHAHITA would like to thank the project for sponsoring him to establish and strengthen the cooperation already in place.

The MAT was established in January 1966 by a group of mathematicians at the Mathematics Department, University of Dar es salaam. The Department of Mathematics is the patron and also a Headquarter of the association. The association has been active since 1966.

OBJECTIVES AND ACTIVITIES

The main objective is to facilitate improvements in the teaching and learning of mathematics and its applications in primary, secondary and tertiary levels. The association is bridging these education levels. The MAT activities are publication of books, MAT Bulletins, organizing Mathematics contests for primary and secondary schools, conducting zonal and national mathematics seminars, Pi day celebrations, and other outreach activities.

CHALLENGES

The main challenges are an inadequate number of mathematics teachers particularly at secondary level. In 2016, there was *a scarcity of more than 7,000 mathematics teachers in the country* (Basic Education Statistics in Tanzania (BEST), 2016). Other challenges are few mathematics teachers, most of the teachers have low mathematical skills, lack of ICT teaching resources, and lack of funds for implementing MAT activities. These challenges are the main reason for the association to find means to alleviate or reduce poor performance of mathematics in national examinations results. For instance, the ordinary secondary level average pass rate in the national examinations from 2008 to 2017 is *17.46%.*

FUTURE PLANS

Future plans include to reach more schools/teachers through cooperation with District councils, stimulate interests through seminars and Pi day celebrations, write new text books, booklets to facilitate teaching and learning and to attract funds from donors who are ready to help or work with the association on MAT activities. We also envisage initiating a teaching volunteer scheme through projects in order to reduce the scarcity of mathematics teachers. Furthermore, we need to educate the society to use ICT on teaching and learning mathematics. In these activities, we need partners that we can jointly write the projects.

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Mathematics Gallery

Gallérie Mathématique

TOWARD A HYBRID MODEL TO DESCRIBE CREATIVE ACTS IN MATHEMATICS LEARNING ENVIRONMENTS

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In my poster, I presented parts of the findings of a larger research study that explored collective creativity in elementary mathematics classroom settings. Two claims underlying the rationale of my research study are "in this changing world, those who understand and can do mathematics will have significantly enhanced opportunities and options for shaping their futures" (NCTM, 2000, p. 5), and doing and understanding mathematics are creative processes (Martin, Towers, & Pirie, 2006) that should be fostered at both the individual and the collective levels. However, what does creativity look like in mathematics classroom settings? Scholars in pedagogy, mathematics education, and teacher education have generated a rich literature base promoting learning for fostering, and characterizing mathematical creativity. However, few of the current definitions for creativity are suited to the distributed and collective enterprise of the classroom.

In my research study, I adopted a design-based research methodology (DBR) to explore mathematical creativity with(in) a collaborative problem-solving environment. I worked closely with participant teachers toward co-developing classroom tasks, and studying the design, implementation, and re-design of these tasks. I also collaborated with scholars in the field of mathematics education in designing worthwhile tasks and learning environments, developing theories of learning, and re-designing the tasks and the learning environments.

Data analysis and interpretation revealed that creativity in mathematics classroom settings is a multifaceted phenomenon that is located in, and stems from, students' (co)actions, and interactions. I suggested four metaphors (namely, overcoming obstacles, expanding possibilities, divergent thinking, and assembling things in new ways) to describe students' creative acts while they are working on an engaging problematic situation. In addition, I have drawn on both the origins of the word creativity and the assumption that creativity is a social phenomenon to approximate a description of creativity as it emerges in classroom settings. In my description, I argue that *creative acts are the (co)actions, and interactions, of a group of curious learners, while they are working collaboratively on an engaging problematic situation. Such acts, which may include (1) overcoming obstacles, (2) expanding possibilities, (3) divergent thinking, and (4) assembling things in new ways, trigger the new and the crucial to emerge and evolve.* Thus, creativity, for me, is not the final end-product that results from students' actions and doings, rather creativity is located in, and stems from, those actions and doings themselves that result in what might be considered as new and significant to, at least, the local classroom community.

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PROBLEM DRIFT: IMAGING EMERGING CURRICULAR SIGNIFICANCE

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This poster depicted an example of problem drift, the method of analysis used in my master's work that explored the implications for adopting an enactivist lens of cognition while teaching in a culture of curriculum (Banting, 2017). Through the enactivist tradition (Maturana & Varela, 1987), as well as complexity theory as applied to education (Davis & Sumara, 2006), the classroom structure of the small group was conceptualized as a complex knowing system capable of maintaining coherence with(in) an environment by continually posing problems to coordinate the group's knowing (Varela, Thompson, & Rosch, 1991).

Attending to problem drift, then, is the process of observing the re-posing of the problem relevant to the group's attention as they know—that is to say, act—together (Banting & Simmt, 2017). For the teacher, the identification of problem drift creates the opportunity to observe the emergent character of the group's cognition in relation to particular curriculum mandates. In this particular study, each observed shift in the problem coordinating knower and environment was coded to contain correlative strength with two anticipated curricular outcomes. Charting the problems in direct concert with curricular outcomes provided an image of the contexts (the relevant problems) in which curricular processes became known.

Problem drift has pragmatic value as a method of analysis for teachers and researchers observing the knowing action of classroom groups. It stresses the processes through which the curriculum becomes known, thus providing information as to what perturbations may trigger further interaction with anticipated outcomes. Observing knowing is no longer accomplished by seeking answers to the question, "How did they solve the problem?": this insinuates a static knower working on an external problem. Rather, problem drift orients the observer's attention to the question, "What problem are they solving?" This shifts focus away from describing knowledge as held by human beings and toward observing the knowing actions of humans being.

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INCORPORATING WRITING INTO THE UNDERGRADUATE MATHEMATICS CURRICULUM

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Writing is essential in almost all careers, subjects, and disciplines. Many mathematics instructors have been using writing extensively in their courses and researchers testify to its benefits. These benefits include facilitating deeper understanding, developing communication skills, and providing instructors with deeper insights into students' learning and reasoning.

Writing in the math classroom can take a variety of forms including reflective (students reflect on their personal thoughts, feelings, and experiences that relate to the content being learned), expository (students explain complex ideas to non-experts), and excogitative (students explain their mathematical thinking carefully and thoroughly, often coupled with a mathematical proof or computation). In my poster, I documented the writing projects I have tried in my university math classrooms and students' reactions to them.

In the summer of 2015, I taught a second-year differential equations class at McMaster University. Students were asked to use short narratives in a variety of ways on five written assignments, which emphasized writing quality over mathematical correctness.

By analyzing students' written assignments and the results of an administered survey, Miroslav Lovric and I (2017) found that students held a positive attitude towards the written assignments and appeared to view writing as an effective learning strategy (unified course concepts, pushed students to work harder, question their understanding, seek help). However, many students viewed writing and mathematics as distinct entities and did not tend to view the skills gained through mathematical writing as transferable.

In the spring of 2017 I taught a Cryptology & Number Theory class at the University of Minnesota. Students kept a course journal and submitted a weekly response to a writing prompt (e.g., Did you have any 'aha' moments this semester?). Since many students are rather weak in the metacognitive processes of reflecting on their own approaches to learning, accurately accessing what they do and do not know, and modifying their behaviour as a result (Benassi, Overson, & Hakala, 2014), these prompts were designed to encourage self-reflection. Students were also asked to submit solutions to four Professional Problems, which were graded based on both mathematical correctness and quality of writing. In these problems, students were asked to write for a specified audience (e.g., client, community member).

At the end of the course, the majority of students (24/39) claimed that journal-writing enhanced their experiences in the course. Almost all students (10/11) who did not see value in journal writing expressed that they did, however, find the Professional Problem writing valuable.

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INFORMING LOCAL MATERIALS USED FOR CONCEPT-RICH INSTRUCTION PRACTICE IN MATHEMATICS CLASS WITH PRE-SERVICE TEACHERS IN TANZANIAN CONTEXT

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This paper addresses the local materials used during implementing concept-rich instruction (Ben-Hur, 2006) in a study of pi in a Tanzanian context. I designed and facilitated concept-rich instruction of pi with pre-service teachers for five day sessions. We used teaching and learning resources including local materials such as sunflower, coconut, pawpaw stem, stems of maize, stems of Cassava, and stems of sunflower. We used these local materials in the three components of the concept-rich instruction. The three elements include *practice* (for instance, during demonstration on how to find the value of pi using local materials); *decontextualization* (for example, during measuring the diameter and circumference of the circular local material in a group); and *recontextualization* (for instance, the applications of pi). These local materials helped a facilitator to engage university pre-service teachers in learning the concept of pi in the class of concept-rich instruction. This notion of using local materials to engage pre-service teachers in learning the concept of pi helped pre-service teachers to see the mathematics embedded in their life. In particular, pre-service teachers connected their mathematics teaching and learning the concept of pi with circular local materials available in their daily practice that are helpful to develop students understanding of pi as a concept in their classroom.

The use of local materials in the class of concept-rich instruction of pi taught pre-service teachers to start using local materials for teaching and learning of the concept of pi in their classroom. For instance, in the last day sessions, pre-service teachers developed the lesson plan and performed a micro-teaching experience on how to teach pi using local materials as teaching and learning resources including using stems of pawpaw. They used these local materials in the class of concept-rich instruction of pi while performing their micro-teaching in finding the value of pi. They noticed that it was easy to build a better understanding of pi using local circular materials in the class.

I engaged the pre-service teachers in a concept rich instruction of pi using local materials in learning the concept of pi. This engagement helped pre-service teachers to see other possible circular local materials as resources that they noticed to be promising for teaching and learning the concept of pi including one hundred coins (Tanzanian shillings), sugarcane, water lemon, and orange.

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LES JEUX TECHNOLOGIQUES DANS LES COURS DE MATHÉMATIQUES AU SECONDAIRE

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Au secondaire, il semble y avoir un manque de motivation de la part des élèves dans les cours de mathématiques (Wigfield, 1994, cité par Chouinard, 2001), et ceci « correspond à un désengagement des apprentissages liés à ce domaine et bien entendu, des carrières qui y sont associées » (Leblond, 2012, p. 15). Certains auteurs affirment que les jeux représentent une bonne façon d'augmenter le niveau d'intérêt des jeunes envers les mathématiques (Sauvé, Renaud, & Gauvin, 2007). Les technologies de l'information et de la communication (TIC) sont aussi pertinentes pour motiver les jeunes, puisque leur utilisation permet un meilleur niveau d'engagement de la part des élèves et une meilleure compréhension du matériel à l'étude (Muir-Herzig, 2004).

Cette recherche vise à vérifier si les jeunes sont plus motivés à apprendre dans les cours de mathématiques au secondaire si les jeux technologiques sont intégrés aux activités de la salle de classe. Ce projet, encore en cours, est une étude de cas. Quatre jeux technologiques, incluant un jeu de Jeopardy et de Battleship Mathématique qui portent sur le programme d'études de mathématiques de 11e année, furent intégrés aux cours réguliers et les élèves furent observés à l'aide d'une liste de vérification, dans le but de vérifier si les indicateurs de motivation étaient présents chez ces derniers.

Afin de nous permettre de mieux comprendre l'impact des jeux sur le niveau de motivation des élèves, cinq d'entre eux ont participé à des entrevues semi-dirigées à la fin de la période de quatre mois. De plus, l'enseignant fut questionné à la suite de l'intégration de chaque jeu. Les résultats recueillis nous permettent d'observer, notamment, le fait que les élèves sont engagés, demeurent à la tâche plus longtemps, ont hâte de venir au cours, ont du plaisir, démontrent des habiletés mathématiques et veulent explorer de nouveaux concepts pendant l'intégration du jeu.

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DE JEUNES CHERCHEURS À LA RENCONTRE DE LA PHILOSOPHIE DES MATHÉMATIQUES

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Pour qui un cours de philosophie des mathématiques? Nous, trois étudiantes au doctorat en didactique des mathématiques, accompagné d'un professeur, nous sommes aventurés à développer nos idées quant à la nature des mathématiques et de l'activité mathématique. Les questions abordées nous ont forcées à sortir de notre zone de confort : philosophie des mathématiques (Tymoczko, 1998a), démonstrations (Tymoczko, 1998a), mathématiques pures et appliquées (Pincock, 2009), certitude, fondements et faillibilité (Tymoczko, 1998a), technologie (Tymoczko, 1998b), mathématiques de l'école (Watson, 2008), etc.

Nos réactions face à ce cours sont variées. Nous avons découvert l'importance d'entrer dans les éléments philosophiques d'une discipline comme les mathématiques, qui peut sembler très rigide, et d'y découvrir l'envers du décor. Nous avons remis en question l'enseignement des mathématiques comme une série de vérités et de règles. Nous avons questionné la tendance à prendre « le mathématicien chercheur » comme point de référence pour définir ce que signifie « faire des mathématiques ». Nous en savons maintenant plus sur la diversité des conceptions qui existent et l'impact qu'elles peuvent avoir sur la façon dont nous interagissons les uns avec les autres. Pour nous, ce cours fut une étape importante dans notre parcours d'études supérieures, car elle nous a conduites à nous forger individuellement une vision des mathématiques en tant que futures chercheures; et elle nous a convaincues de l'intérêt pas seulement de développer une telle vision, mais aussi de la rendre plus explicite dans nos travaux.

La possibilité d'expliciter nos conceptions et d'en débattre par des discussions nous fait croire que la philosophie des mathématiques devrait être abordée par tous les étudiants en didactique des mathématiques. Un cours de philosophie des mathématiques est une opportunité de réfléchir sur la discipline en donnant l'occasion de s'exprimer sur des idées fondamentales. C'est aussi en tant que (futurs) formateurs d'enseignants et de mathématiciens que nous avons trouvé essentiels de réfléchir à la place des mathématiques à l'école et à la manière de les y aborder.

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EXAMINING PARENT PERSPECTIVES OF MULTIPLE STRATEGIES

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But the fact is there isn't great research behind it. And the other fact is that […] students who are coming out of learning these strategies—the math scores have weakened. Because I'm all for doing a different strategy if you have proof that that strategy is even better than what we've been doing all along. However, at the end of the day the scores indicate that, no, it hasn't been the best approach to teaching math.

So like if they're learning multiplication, they'll show them all five different ways that you can come to the answer, which could be frustrating because like I had a conversation with [my daughter] *where she knew the answer just by looking at it. But her teacher made her do this whole worksheet to find the answer and she's like, "I already know it. Why would I do this whole thing?" But I get it that different strategies work for everybody and how can a child find the strategy that works best for them without trying them all?*

The two previous quotes show differing perspectives related to parents' perceptions regarding the use of multiple strategies when teaching mathematics. The first quote shows an obvious tension with a parent and the second shows a concern but a belief that the idea is inherently positive. These differing beliefs of parents provide a potential obstacle when implementing multiple strategies in the classroom. When curricular goals support the use of multiple strategies (e.g. Alberta Education, 2016), considering the concerns of parents can be important when implementing these ideas in classrooms. The literature itself is not universally supportive of multiple strategies which compounds this tension for teachers implementing these skills. Some shows potential benefits, such as improve adaptivity and flexibility (e.g., Hatano, 2003); greater procedural flexibility and increased understanding (e.g., Verschaffel et al., 2009); cultivates appropriate attitudes and conceptions of mathematics (e.g., Verschaffel et al., 2009). Some researchers suggested that they are not appropriate for all students: may exacerbate the difficulties of low achieving students (e.g., Auer, Hickendorff, & Putten, 2016); flexibility needs a foundation of skills to build upon (e.g., Geary, 2003); and solving problems in more than one way is difficult to operationalize in classrooms (Silver et al., 2005).

Using a qualitative thematic analysis (Braun & Clarke, 2006), we identified key assertions by parents with regard to their perceptions of multiple strategies from the ten focus groups and fifteen interviews that were conducted as data collection in this study. The overarching themes identified in the parents' comments mirrored the research: adaptivity and flexibility of strategies, conceptual and procedural knowledge, disposition and identity, implementation of multiple strategies, and not effective for all students.

Through this research we have noted that parents have varied perspectives related to the use of multiple strategies with their children. Many were able to provide concrete examples of their experiences of the use of multiple strategies with their children, and these examples could be positive or negative depending on their child's situation or context. The counter-examples provided by the parents are important not to ignore in considering changes with the curriculum and classroom implementation.

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LA DIALECTIQUE PERFECTION/IMPERFECTION : UNE NOUVELLE MANIÈRE DE PENSER L'ACTIVITE MATHEMATIQUE

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Pour plusieurs, atteindre le vrai et pouvoir le démontrer est le propre des mathématiques. Pourtant, on trouve des exemples de différentes natures où l'on peut observer un côté plutôt imparfait des mathématiques. Je vous en présente quelques-uns.

Certains philosophes ont parlé de la nature des mathématiques et de façon très différente; on connait la vision de Platon, selon laquelle les idées mathématiques sont l'idéal de la perfection. D'autres sont en contradiction avec ce dernier. Hersh (1998), par exemple, présente une perspective sur l'*existence* des objets mathématiques qui contraste avec cette « perfection ». Pour lui les objets mathématiques ne sont ni mentaux ni tangibles : ce sont des objets socioculturels et historiques. Les mathématiques n'auraient pas de réalité en dehors de la culture où elles existent. Cette existence, bien « imparfaite », ne serait « que » culturelle. Ensuite, un évènement marquant de l'histoire et de la philosophie des mathématiques est « la crise des fondements », dont le sommet est le théorème d'incomplétude de Gödel. Ce théorème démontre que dans un système *cohérent*, il y a toujours des énoncés mathématiques qui sont indémontrables, tandis que dans un système où tout serait *démontrable*, il y aurait nécessairement des incohérences. On doit donc faire un choix au niveau de l'im-perfection entre cohérence et démontrabilité. De plus, l'analyse de textes mathématiques montre des manières pour les mathématiques d'être à la fois parfaites et imparfaites. On voit que ce qui est *conventionnel* dans les mathématiques varie beaucoup. Le papyrus de Rhin par exemple montre que les nombres, les symboles et les algorithmes manipulés sont à la fois « parfaits » au sens où ils sont adoptés par tous et qu'ils font bien le travail, mais aussi « imparfaits », car pas du tout universels ou intemporels. Aussi, les textes mathématiques ne sont jamais vraiment « complets » : même aujourd'hui, l'auteur choisit les éléments qu'il présente (Livingston, 2015) et l'acceptation de nouveaux résultats est en partie une question de crédibilité. Par ailleurs, une analyse historique permet de voir comment des aller-retour entre la perfection et l'imperfection furent des moteurs importants de développement pour les mathématiques. Lakatos (1976) discute des exemples intéressants, comme des conjectures incorrectes et des résultats partiels ayant fait partie d'activités mathématiques fructueuses.

Or, ce travail de *perfectionnement* n'est possible que si l'on se situe quelque part entre les deux pôles perfection-imperfection. Comment cette nouvelle manière de penser l'activité mathématique peut-elle être mise au service de l'activité mathématique des élèves ? Peut-on penser explorer certains de ces « malaises » mathématiques avec des élèves ? Nous croyons que oui, et dans ce projet de recherche nous travaillons présentement à la mise au point de situations pour la classe permettant de mettre en valeur ces aspects.

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UNDERSTANDING SCHOOL LEADERS' DISCOURSE IN REGARD TO MATHEMATICS ACHIEVEMENT

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The notion of discourse in school mathematics has been of considerable interest to researchers. Scholars largely associate mathematical discourse to classroom practices (Moschkovich, 2007) and tend to overlook the political dimensions of discourse that are largely the prerogatives of school leaders. There is a significant body of literature pointing to the fact that school leaders are critical in supporting effective schools (Fullan, 2011). Further, school leadership is widely considered to impact student achievement and success (Leithwood, Patten, & Jantzi, 2010). Despite that evidence, not much attention seems to be given to school leaders' discourse on math achievement even though it is well known that school leadership discourse impacts school culture (Webster, 2012).

This poster presents findings from a pilot project involving 10 school administrators and systems leaders representing two different jurisdictions: five from the Northern Haiti and five from the French-language schools in Ontario. The participants from Haiti were from private schools (religious and secular), and the ones from Ontario were from both the catholic and public systems. The research examines school leaders' discourse in the context of math achievement for all students. What are some of the commonalities of school leaders' discourse when it comes to math achievement? What do these leaders perceive to be success factors, roadblocks and challenges to students' math achievement? To what extent is their discourse a reflection of implicit inequities in school math? Do they allude to deficit assumptions in their understanding of math achievement for all students?

Using primarily semi-structured interviews, this research used a qualitative framework to explore some of these questions. Early analysis of the transcripts yielded to several themes including the importance of teachers' impact, the students' and teachers' attitudes effect, the fixed mindsets regarding math achievement, and the challenge to make math meaningful to students. These four themes are examined in light of literature arguing that "effective school leadership is needed to support the transformation of teaching practice and school culture" (Vale et al., 2010, p. 47).

This research offers insights on school leaders' discourse related to students' math achievement and addresses a gap in the literature (Herbel-Eisenmann, Choppin, Wagner, & Pimm, 2011). This project initiates a framework for further studies on how school leaders' discourse interplay with math achievement. As scholars examine equity issues in school math and as policymakers discuss ways to increase math achievement for all students, it is critical to also consider how school leaders can be supported in developing discourses compatible to more equity in school math.

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I SEE WHAT YOU'RE SAYING: VISUALIZATION AND THE GROWTH OF MATHEMATICAL IMAGES

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University of Calgary

In this poster, I explored the interconnection between images and visualization, asking the question: What role might visualization play in the growth of mathematical images? This research is grounded in the theoretical framework of the Pirie-Kieren Dynamical Theory for the Growth of Mathematical Understanding (1994). The theory is an eight-levelled model of nested ovals, which characterizes growth as a dynamic, levelled but non-linear, transcendently recursive process (Pirie & Kieren, 1994). My research within this framework is focused on the preliminary process of transitioning from *Image Making* to *Image Having* (Pirie & Kieren, 1994). This reference to *having* is not referring to a static state of completeness, for there is always movement, but rather one of usefulness in the absence of the perceived act. Husserl (1970) has made a set of distinctions which has offered direction to my understanding of this transition. He refers to three ways that an object can be presented—signitively, imaginatively, and perceptually. For instance, I can talk about a withering oak which I have never seen but have heard is standing in a nearby field (signitive); I can view a detailed drawing of the oak tree (imaginative); or I can perceive the oak tree myself by standing in front of it, hearing the rustle of its leaves, feeling the roughness of its bark, and seeing its arching branches overhead (perceptive). Husserl discusses these experiences as having different levels of perceptual offerings ranked from the least direct to the most direct. I describe these Husserlian ideas for I am interested in relating them to meaning rather than object. Within this reorientation, I suggest the signitive as being the linguistic form of mathematics which is unchanged from Husserl's original intention, the oral discussion of mathematics and the written symbolic form, which is the most indirect and emptiest way of presenting the meaning within the mathematical idea. The imaginative as the utilization of our created images—the act of visualizing, just as a picture offers some level of meaning, so also does visualizing (Driskell, Copper, & Moran, 1994). The final category, perceptual, and the richest offering of meaning, I suggest can be found in the enactment of a mathematical idea (Maturana & Varela, 1987). It is within the enactment of a mathematical idea that meaning is offered in the most direct, original, and optimal way. The use of this leveled categorization of offerings is for the exploration of what I believe to be an often-ignored aspect of mathematics education—the growth of imagination and visualization.

The data collection phase of this study will take place at a school for students with learning differences (LD). I believe this group will create a rich environment of observation, as there is some discussion about this group being more visual than 'typical' learners (West, 1991).

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STUDENTS' RELATIONSHIPS WITH MATHEMATICS REVEALED THROUGH DRAWINGS

Jennifer Plosz, Jo Towers, Kori Czuy, Ayman Aljarrah, Andrew Vorhies, & Miwa Takeuchi
University of Calgary

In our poster, we presented examples of students' images, as explored through the drawings they created in response to prompts about how they feel when doing mathematics. This research is grounded in the theoretical framework of enactivism, a theory of embodied cognition which explores the intertwining relationship of emotion and cognition (Colombetti, 2013). This theoretical context encouraged us to delve beneath the surface of students' drawings to interpret the ways in which their experiences of doing mathematics in schools may have contributed to their relationships with mathematics. The use of drawings offers an unusual approach in mathematics education, and there are very few studies that have explored students' drawings as a means to understand emotional relationships with mathematics (for one pertinent example see Perkkilä & Aarnos, 2009). Our initial analysis of interview transcripts revealed four broad categories of students' relationships with mathematics: positive, negative, mixed, and changed/changing (Towers, Takeuchi, Hall, & Martin, 2017). We then studied the drawings contributed by the participants whose narratives had been categorized into these four domains. The drawings added further implicit messaging through the choice of subject matter, use of colour, et cetera. One such implicit message relates to the connection between students' emotions and bodies (exemplified by a focus on heads, tears, steam coming out of the head, etc., in many of the drawings), which had not been obvious to us through transcript analysis. Another interesting element of the drawings was a noticeable absence of detail in many of the drawings in the positive category. Within this category, there were very few specifics given about what students enjoyed about mathematics and all that was typically offered for the drawing was a generic smiley face or the written word "*happy*". On the contrary, within the negative category there seemed to be an explosion of detailed emotion, such as "*I'm mad or I'm sad, I'm kind of both*", "*I get really frustrated*", "*It's just like cutting my head in half*", or "[Math] *wants to kill me*" and drawings were more unique to the individual and often very detailed. In the mixed category, some students chose dramatic imagery (e.g., a heart split in two) to portray their conflicted relationship with math, while in the changed/changing category students often chose to represent their emotions through pairing a sad face with a smiley face.

In this study, the activity of drawing provided a contextualized expression of the participants' relationships with mathematics. Drawing seemed to stimulate, ripen, and occasionally contradict students' verbal responses to interview questions. It appeared to act as a tool for the participants to "come to know […] through making" (Cain, 2010, p. 19), and as a means for us as researchers to gain access to students' unspoken thoughts about learning mathematics.

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TEACHER PROFESSIONAL LEARNING IN TANZANIA: EXPERIENCES OF MATHEMATICS TEACHER LEADERS

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This poster presents preliminary research findings of the study that explored the experiences of mathematics teacher leaders (MTLs) in leading the professional learning of primary school mathematics teachers in rural and remote communities in Tanzania. The study stems from my observation that teacher-led professional learning is ultimately effective in promoting professional growth of teachers, yet when it comes to locating it in schools, the education system often falls short. Given such a situation and the need to inform the transition to teacher-led professional learning in Tanzanian schools, this study sought to develop an understanding of how MTLs experience their work of leading the professional learning of mathematics teachers in order to suggest ways to improve teacher professional learning in Tanzania. To achieve the aim, the study drew on symbolic interactionism, perspective consciousness, and Dewey's notion of experience to develop in-depth insights related to MTLs' leadership of teacher professional learning. It employed qualitative case study methodology (Yin, 2014) to gather experiences of eight MTLs from their real-life contexts with a range of sources, including an open-ended questionnaire, in-depth interviews, metaphors, and vignettes. The poster presents, amongst others, the perspectives on teacher-led professional learning; the influence of local leadership on teacher leadership; teachers' perceptions of their leadership; and hopes, outcomes, and challenges of teacher leadership in rural schools in Tanzania. This study will expand the understanding of teacher-led professional learning of mathematics teachers. It should richly inform relevant teacher education and mathematics education research across Tanzania as well as similar educational developments in other parts of the world.

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"PERIMETER IS EASIER TO WORK WITH THAN AREA": STUDENTS' LANGUAGE USE IN GROUP WORK

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Growing linguistic and ethnic diversity in the classroom has prompted us to better understand the characteristics of language in mathematics classrooms. Paying close attention to the language use is thus one of the important aspects of pedagogical design for mathematics classes. Linguistic complexity during students' group work has been documented, especially in relation to its affordance for mathematics learning (Barwell, 2009; Takeuchi, 2016). The goal of this poster is to understand how students' use of language influences opportunities to learn mathematics, by focusing on the context of group work. Language in this study is conceptualized from Vygotsky's (1934/1986) theory of *sense* and *meaning*. Vygotsky distinguished the sense of a word from its meaning. The sense of a word develops and will be enriched; whereas, its meaning remains stable, as definitions presented in the dictionary. The data for this study were collected in two urban schools in Canada. Both schools were linguistically and ethnically diverse: More than 45 home languages of students were represented and more than 50% of the students were categorized as English language learners. The data used for this paper include video recordings of group work sessions and individual video-mediated interviews. Mathematics problems were taken from the provincial standardized assessments. The research question addressed in this poster is: How do students make sense of words in mathematics problems? I employed a social semiotics approach to classroom discourse analysis (Lemke, 1990) that focuses on the web of semantic relationships. Many students got stuck with the presented mathematics problem because they did not make a meaningful connection among mathematically-related concepts (such as area and perimeter). For example, students defined area and perimeter with merely with numerical operation (e.g., "adding is perimeter and multiply is area") or as interchangeable concepts (e.g., "I'll just say perimeter is easier to work with than area. We're working with outside not inside 'cause inside will be harder so we're going to work on the outside."). This finding raises a concern about "verbalism" (Vygotsky, 1934/1986, p.148): Students knew the meaning of mathematical words but did not develop its sense. As previous research has identified, the relationships among key concepts are often implicit in mathematics classrooms (Herbel-Eisenmann & Otten, 2011). This poster suggests the significance of paying close attention to students' sense-making in language in mathematics classrooms, especially for linguistically diverse learners.

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TOWARD AN UNDERSTANDING OF MATHEMATICS TEACHERS' PARTICIPATION IN PROFESSIONAL LEARNING NETWORKS

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Nowadays, more and more mathematics teachers have extended their professional learning by participating in Professional Learning Networks (PLNs). However, few studies have been conducted on the exploration of understanding their doing in PLNs (Dash, de Kramer, O'Dwyer, Masters, & Russell, 2012). Thus, this study is intended to address the gap by investigating mathematics teachers' learning processes through PLNs to seek to understand their learning actions as well as the phenomenon and nature of teachers' engaging in PLNs.

The research will be conducted under the umbrella of the theory of complexity systems and the methodology of interpretive inquiry. Complexity systems serve as the theoretical framework of the research based on the considerations that PLNs should be understood as a kind of complexity system according to Mitchell's (2009) common properties and definition of complexity systems. An interpretive inquiry in the constructivist paradigm is used to approach the understanding of participants' learning actions or expressions in a more intense, careful, and self-conscious way when their meanings or reasons are not clear (Smith, 1992).

Four types of triangulated data will be collected from these two PLNs, including 1) archived documents such as logs, posts, comments, or responses; 2) online survey; 3) online open-ended interviews with related participants; and 4) my own reflections. The data analysis will be undertaken simultaneously with the collection of data by interpretive inquiry.

Several data analysis techniques will be adopted to analyze the archived documents, online survey and open-ended interviews so as to provide "enough illustrative material(s)" (Ellis, 1998, p. 32). The models such as 1) Knowledge of Mathematics for Teaching (KMT), Mathematics-for-Teaching(M4T), Entangled Dynamics, or Decentralized Networks will be applied to investigate interested topics, knowledge emergence, and idea interaction from archived documents. 2) Necessary Conditions for Complexity Systems will be employed to understand the rationale behind the emergence and the interaction. 3) Some statistical concepts will be adopted to examine the potential reasons of teachers' participation in PLNs through the online survey. 4) Thematic analysis will be taken to make sense of the teachers' experiences of participating in PLNs and the attached meaning to their professional learning, possibly revealing the depth of understanding mathematics teachers' participation in PLNs.

The results could help us understand what they possibly need in their professional learning, offer a valuable reference for improving the design of and evaluation on both online and even conventional professional development for teachers, and contribute to the rapidly increasing literature on teachers' professional learning, particularly in an online learning community.

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Appendices

Annexes

Appendix A / Annexe A

WORKING GROUPS AT EACH ANNUAL MEETING / GROUPES DE TRAVAIL DES RENCONTRES ANNUELLES

1977   *Queen's University, Kingston, Ontario*

· Teacher education programmes
· Undergraduate mathematics programmes and prospective teachers
· Research and mathematics education
· Learning and teaching mathematics

1978   *Queen's University, Kingston, Ontario*

· Mathematics courses for prospective elementary teachers
· Mathematization
· Research in mathematics education

1979   *Queen's University, Kingston, Ontario*

· Ratio and proportion: a study of a mathematical concept
· Minicalculators in the mathematics classroom
· Is there a mathematical method?
· Topics suitable for mathematics courses for elementary teachers

1980   *Université Laval, Québec, Québec*

· The teaching of calculus and analysis
· Applications of mathematics for high school students
· Geometry in the elementary and junior high school curriculum
· The diagnosis and remediation of common mathematical errors

1981   *University of Alberta, Edmonton, Alberta*

· Research and the classroom
· Computer education for teachers
· Issues in the teaching of calculus
· Revitalising mathematics in teacher education courses

1982   *Queen's University, Kingston, Ontario*

· The influence of computer science on undergraduate mathematics education
· Applications of research in mathematics education to teacher training programmes
· Problem solving in the curriculum

1983   *University of British Columbia, Vancouver, British Columbia*

· Developing statistical thinking
· Training in diagnosis and remediation of teachers
· Mathematics and language
· The influence of computer science on the mathematics curriculum

1984   *University of Waterloo, Waterloo, Ontario*

· Logo and the mathematics curriculum
· The impact of research and technology on school algebra
· Epistemology and mathematics
· Visual thinking in mathematics

1985   *Université Laval, Québec, Québec*

· Lessons from research about students' errors
· Logo activities for the high school
· Impact of symbolic manipulation software on the teaching of calculus

1986   *Memorial University of Newfoundland, St. John's, Newfoundland*

· The role of feelings in mathematics
· The problem of rigour in mathematics teaching
· Microcomputers in teacher education
· The role of microcomputers in developing statistical thinking

1987   *Queen's University, Kingston, Ontario*

· Methods courses for secondary teacher education
· The problem of formal reasoning in undergraduate programmes
· Small group work in the mathematics classroom

1988   *University of Manitoba, Winnipeg, Manitoba*

· Teacher education: what could it be?
· Natural learning and mathematics
· Using software for geometrical investigations
· A study of the remedial teaching of mathematics

1989   *Brock University, St. Catharines, Ontario*

· Using computers to investigate work with teachers
· Computers in the undergraduate mathematics curriculum
· Natural language and mathematical language
· Research strategies for pupils' conceptions in mathematics

1990   *Simon Fraser University, Vancouver, British Columbia*

· Reading and writing in the mathematics classroom
· The NCTM "Standards" and Canadian reality
· Explanatory models of children's mathematics
· Chaos and fractal geometry for high school students

1991   *University of New Brunswick, Fredericton, New Brunswick*

· Fractal geometry in the curriculum
· Socio-cultural aspects of mathematics
· Technology and understanding mathematics
· Constructivism: implications for teacher education in mathematics

1992   *ICME–7, Université Laval, Québec, Québec*

1993   *York University, Toronto, Ontario*

· Research in undergraduate teaching and learning of mathematics
· New ideas in assessment
· Computers in the classroom: mathematical and social implications
· Gender and mathematics
· Training pre-service teachers for creating mathematical communities in the classroom

1994   *University of Regina, Regina, Saskatchewan*

· Theories of mathematics education
· Pre-service mathematics teachers as purposeful learners: issues of enculturation
· Popularizing mathematics

1995   *University of Western Ontario, London, Ontario*

· Autonomy and authority in the design and conduct of learning activity
· Expanding the conversation: trying to talk about what our theories don't talk about
· Factors affecting the transition from high school to university mathematics
· Geometric proofs and knowledge without axioms

1996   *Mount Saint Vincent University, Halifax, Nova Scotia*

· Teacher education: challenges, opportunities and innovations
· Formation à l'enseignement des mathématiques au secondaire: nouvelles perspectives et défis
· What is dynamic algebra?
· The role of proof in post-secondary education

1997   *Lakehead University, Thunder Bay, Ontario*

· Awareness and expression of generality in teaching mathematics
· Communicating mathematics
· The crisis in school mathematics content

1998   *University of British Columbia, Vancouver, British Columbia*

- Assessing mathematical thinking
- From theory to observational data (and back again)
- Bringing Ethnomathematics into the classroom in a meaningful way
- Mathematical software for the undergraduate curriculum

1999   *Brock University, St. Catharines, Ontario*

- Information technology and mathematics education: What's out there and how can we use it?
- Applied mathematics in the secondary school curriculum
- Elementary mathematics
- Teaching practices and teacher education

2000   *Université du Québec à Montréal, Montréal, Québec*

- Des cours de mathématiques pour les futurs enseignants et enseignantes du primaire/Mathematics courses for prospective elementary teachers
- Crafting an algebraic mind: Intersections from history and the contemporary mathematics classroom
- Mathematics education et didactique des mathématiques : y a-t-il une raison pour vivre des vies séparées?/Mathematics education et didactique des mathématiques: Is there a reason for living separate lives?
- Teachers, technologies, and productive pedagogy

2001   *University of Alberta, Edmonton, Alberta*

- Considering how linear algebra is taught and learned
- Children's proving
- Inservice mathematics teacher education
- Where is the mathematics?

2002   *Queen's University, Kingston, Ontario*

- Mathematics and the arts
- Philosophy for children on mathematics
- The arithmetic/algebra interface: Implications for primary and secondary mathematics / Articulation arithmétique/algèbre: Implications pour l'enseignement des mathématiques au primaire et au secondaire
- Mathematics, the written and the drawn
- Des cours de mathématiques pour les futurs (et actuels) maîtres au secondaire / Types and characteristics desired of courses in mathematics programs for future (and in-service) teachers

2003   *Acadia University, Wolfville, Nova Scotia*

- L'histoire des mathématiques en tant que levier pédagogique au primaire et au secondaire / The history of mathematics as a pedagogic tool in Grades K–12
- Teacher research: An empowering practice?
- Images of undergraduate mathematics
- A mathematics curriculum manifesto

2004   *Université Laval, Québec, Québec*

· Learner generated examples as space for mathematical learning
· Transition to university mathematics
· Integrating applications and modeling in secondary and post secondary mathematics
· Elementary teacher education – Defining the crucial experiences
· A critical look at the language and practice of mathematics education technology

2005   *University of Ottawa, Ottawa, Ontario*

· Mathematics, education, society, and peace
· Learning mathematics in the early years (pre-K – 3)
· Discrete mathematics in secondary school curriculum
· Socio-cultural dimensions of mathematics learning

2006   *University of Calgary, Calgary, Alberta*

· Secondary mathematics teacher development
· Developing links between statistical and probabilistic thinking in school mathematics education
· Developing trust and respect when working with teachers of mathematics
· The body, the sense, and mathematics learning

2007   *University of New Brunswick, New Brunswick*

· Outreach in mathematics – Activities, engagement, & reflection
· Geometry, space, and technology: challenges for teachers and students
· The design and implementation of learning situations
· The multifaceted role of feedback in the teaching and learning of mathematics

2008   *Université de Sherbrooke, Sherbrooke, Québec*

· Mathematical reasoning of young children
· Mathematics-in-and-for-teaching (MifT): the case of algebra
· Mathematics and human alienation
· Communication and mathematical technology use throughout the post-secondary curriculum / Utilisation de technologies dans l'enseignement mathématique postsecondaire
· Cultures of generality and their associated pedagogies

2009   *York University, Toronto, Ontario*

· Mathematically gifted students / Les élèves doués et talentueux en mathématiques
· Mathematics and the life sciences
· Les méthodologies de recherches actuelles et émergentes en didactique des mathématiques / Contemporary and emergent research methodologies in mathematics education
· Reframing learning (mathematics) as collective action
· Étude des pratiques d'enseignement
· Mathematics as social (in)justice / Mathématiques citoyennes face à l'(in)justice sociale

2010   *Simon Fraser University, Burnaby, British Columbia*

· Teaching mathematics to special needs students: Who is at-risk?
· Attending to data analysis and visualizing data
· Recruitment, attrition, and retention in post-secondary mathematics
   Can we be thankful for mathematics? Mathematical thinking and aboriginal peoples
· Beauty in applied mathematics
· Noticing and engaging the mathematicians in our classrooms

2011   *Memorial University of Newfoundland, St. John's, Newfoundland*

· Mathematics teaching and climate change
· Meaningful procedural knowledge in mathematics learning
· Emergent methods for mathematics education research: Using data to develop theory / Méthodes émergentes pour les recherches en didactique des mathématiques: partir des données pour développer des théories
· Using simulation to develop students' mathematical competencies – Post secondary and teacher education
· Making art, doing mathematics / Créer de l'art; faire des maths
· Selecting tasks for future teachers in mathematics education

2012   *Université Laval, Québec City, Québec*

· Numeracy: Goals, affordances, and challenges
· Diversities in mathematics and their relation to equity
· Technology and mathematics teachers (K-16) / La technologie et l'enseignant mathématique (K-16)
· La preuve en mathématiques et en classe / Proof in mathematics and in schools
· The role of text/books in the mathematics classroom / Le rôle des manuels scolaires dans la classe de mathématiques
· Preparing teachers for the development of algebraic thinking at elementary and secondary levels / Préparer les enseignants au développement de la pensée algébrique au primaire et au secondaire

2013   *Brock University, St. Catharines, Ontario*

· MOOCs and online mathematics teaching and learning
· Exploring creativity: From the mathematics classroom to the mathematicians' mind / Explorer la créativité : de la classe de mathématiques à l'esprit des mathématiciens
· Mathematics of Planet Earth 2013: Education and communication / Mathématiques de la planète Terre 2013 : formation et communication (K-16)
· What does it mean to understand multiplicative ideas and processes? Designing strategies for teaching and learning
· Mathematics curriculum re-conceptualisation

2014   *University of Alberta, Edmonton, Alberta*

· Mathematical habits of mind / Modes de pensée mathématiques
· Formative assessment in mathematics: Developing understandings, sharing practice, and confronting dilemmas
· Texter mathematique / Texting mathematics
· Complex dynamical systems
· Role-playing and script-writing in mathematics education practice and research

2015   *Université de Moncton, Moncton, New Brunswick*

· Task design and problem posing
· Indigenous ways of knowing in mathematics
· Theoretical frameworks in mathematics education research / Les cadres théoriques dans la recherche en didactictique des mathématiques
· Early years teaching, learning and research: Tensions in adult-child interactions around mathematics
· Innovations in tertiary mathematics teaching, learning and research / Innovations au post-secondaire pour l'enseignement, l'apprentissage et la recherche

2016   Queen's University, Kingston, Ontario

· Computational thinking and mathematics curriculum
· Mathematics in teacher education: What, how… and why / Les mathématiques dans la formation des enseignants : quoi, comment… et pourquoi
· Problem solving: Definition, role, and pedagogy / Résolution de problèmes : définition, rôle, et pédagogie associée
· Mathematics education and social justice: Learning to meet the others in the classroom / Éducation mathématique et justice sociale : apprendre à rencontrer les autres dans las classe
· Role of spatial reasoning in mathematics
· The public discourse about mathematics and mathematics education / Le discours public sur les mathématiques et l'enseignement des mathématiques

2017   McGill University, Montréal, Québec

· Teaching First Year Mathematics Courses in Transition from Secondary to Tertiary
· L'anxiété mathématique chez les futurs enseignants du primaire : à la recherche de nouvelles réponses à des enjeux qui perdurent / Elementary Preservice Teachers and Mathematics Anxiety: Searching for New Responses to Enduring Issues
· Social Media and Mathematics Education
· Quantitative Reasoning in the Early Years / Le raisonnement quantitatif dans les premières années du parcours scolaire
· Social, Cultural, Historical and Philosophical Perspectives on Tools for Mathematics
· Compréhension approfondie des mathématiques scolaires / Deep Understanding of School Mathematics

Appendix B / Annexe B

PLENARY LECTURES AT EACH ANNUAL MEETING / CONFÉRENCES PLÉNIÈRES DES RENCONTRES ANNUELLES

1977	A.J. COLEMAN	The objectives of mathematics education
	C. GAULIN	Innovations in teacher education programmes
	T.E. KIEREN	The state of research in mathematics education
1978	G.R. RISING	The mathematician's contribution to curriculum development
	A.I. WEINZWEIG	The mathematician's contribution to pedagogy
1979	J. AGASSI	The Lakatosian revolution
	J.A. EASLEY	Formal and informal research methods and the cultural status of school mathematics
1980	C. GATTEGNO	Reflections on forty years of thinking about the teaching of mathematics
	D. HAWKINS	Understanding understanding mathematics
1981	K. IVERSON	Mathematics and computers
	J. KILPATRICK	The reasonable effectiveness of research in mathematics education
1982	P.J. DAVIS	Towards a philosophy of computation
	G. VERGNAUD	Cognitive and developmental psychology and research in mathematics education
1983	S.I. BROWN	The nature of problem generation and the mathematics curriculum
	P.J. HILTON	The nature of mathematics today and implications for mathematics teaching

1984	A.J. BISHOP	The social construction of meaning: A significant development for mathematics education?
	L. HENKIN	Linguistic aspects of mathematics and mathematics instruction
1985	H. BAUERSFELD	Contributions to a fundamental theory of mathematics learning and teaching
	H.O. POLLAK	On the relation between the applications of mathematics and the teaching of mathematics
1986	R. FINNEY	Professional applications of undergraduate mathematics
	A.H. SCHOENFELD	Confessions of an accidental theorist
1987	P. NESHER	Formulating instructional theory: the role of students' misconceptions
	H.S. WILF	The calculator with a college education
1988	C. KEITEL	Mathematics education and technology
	L.A. STEEN	All one system
1989	N. BALACHEFF	Teaching mathematical proof: The relevance and complexity of a social approach
	D. SCHATTSNEIDER	Geometry is alive and well
1990	U. D'AMBROSIO	Values in mathematics education
	A. SIERPINSKA	On understanding mathematics
1991	J .J. KAPUT	Mathematics and technology: Multiple visions of multiple futures
	C. LABORDE	Approches théoriques et méthodologiques des recherches françaises en didactique des mathématiques
1992	ICME-7	
1993	G.G. JOSEPH	What is a square root? A study of geometrical representation in different mathematical traditions
	J CONFREY	Forging a revised theory of intellectual development: Piaget, Vygotsky and beyond
1994	A. SFARD	Understanding = Doing + Seeing ?
	K. DEVLIN	Mathematics for the twenty-first century
1995	M. ARTIGUE	The role of epistemological analysis in a didactic approach to the phenomenon of mathematics learning and teaching
	K. MILLETT	Teaching and making certain it counts
1996	C. HOYLES	Beyond the classroom: The curriculum as a key factor in students' approaches to proof
	D. HENDERSON	Alive mathematical reasoning

1997	R. BORASSI	What does it really mean to teach mathematics through inquiry?
	P. TAYLOR	The high school math curriculum
	T. KIEREN	Triple embodiment: Studies of mathematical understanding-in-interaction in my work and in the work of CMESG/GCEDM
1998	J. MASON	Structure of attention in teaching mathematics
	K. HEINRICH	Communicating mathematics or mathematics storytelling
1999	J. BORWEIN	The impact of technology on the doing of mathematics
	W. WHITELEY	The decline and rise of geometry in 20th century North America
	W. LANGFORD	Industrial mathematics for the 21st century
	J. ADLER	Learning to understand mathematics teacher development and change: Researching resource availability and use in the context of formalised INSET in South Africa
	B. BARTON	An archaeology of mathematical concepts: Sifting languages for mathematical meanings
2000	G. LABELLE	Manipulating combinatorial structures
	M. B. BUSSI	The theoretical dimension of mathematics: A challenge for didacticians
2001	O. SKOVSMOSE	Mathematics in action: A challenge for social theorising
	C. ROUSSEAU	Mathematics, a living discipline within science and technology
2002	D. BALL & H. BASS	Toward a practice-based theory of mathematical knowledge for teaching
	J. BORWEIN	The experimental mathematician: The pleasure of discovery and the role of proof
2003	T. ARCHIBALD	Using history of mathematics in the classroom: Prospects and problems
	A. SIERPINSKA	Research in mathematics education through a keyhole
2004	C. MARGOLINAS	La situation du professeur et les connaissances en jeu au cours de l'activité mathématique en classe
	N. BOULEAU	La personnalité d'Evariste Galois: le contexte psychologique d'un goût prononcé pour les mathématique abstraites
2005	S. LERMAN	Learning as developing identity in the mathematics classroom
	J. TAYLOR	Soap bubbles and crystals
2006	B. JAWORSKI	Developmental research in mathematics teaching and learning: Developing learning communities based on inquiry and design
	E. DOOLITTLE	Mathematics as medicine

2007	R. NÚÑEZ	Understanding abstraction in mathematics education: Meaning, language, gesture, and the human brain
	T. C. STEVENS	Mathematics departments, new faculty, and the future of collegiate mathematics
2008	A. DJEBBAR	Art, culture et mathématiques en pays d'Islam (IXe-XVe s.)
	A. WATSON	Adolescent learning and secondary mathematics
2009	M. BORBA	Humans-with-media and the production of mathematical knowledge in online environments
	G. de VRIES	Mathematical biology: A case study in interdisciplinarity
2010	W. BYERS	Ambiguity and mathematical thinking
	M. CIVIL	Learning from and with parents: Resources for equity in mathematics education
	B. HODGSON	Collaboration et échanges internationaux en éduction mathématique dans le cadre de la CIEM : regards selon une perspective canadienne / ICMI as a space for international collaboration and exchange in mathematics education: Some views from a Canadian perspective
	S. DAWSON	My journey across, through, over, and around academia: "...a path laid while walking..."
2011	C. K. PALMER	Pattern composition: Beyond the basics
	P. TSAMIR & D. TIROSH	The Pair-Dialogue approach in mathematics teacher education
2012	P. GERDES	Old and new mathematical ideas from Africa: Challenges for reflection
	M. WALSHAW	Towards an understanding of ethical practical action in mathematics education: Insights from contemporary inquiries
	W. HIGGINSON	Cooda, wooda, didda, shooda: Time series reflections on CMESG/GCEDM
2013	R. LEIKIN	On the relationships between mathematical creativity, excellence and giftedness
	B. RALPH	Are we teaching Roman numerals in a digital age?
	E. MULLER	Through a CMESG looking glass
2014	D. HEWITT	The economic use of time and effort in the teaching and learning of mathematics
	N. NIGAM	Mathematics in industry, mathematics in the classroom: Analogy and metaphor
2015	É. RODITI	Diversité, variabilité et convergence des pratiques enseignantes / Diversity, variability, and commonalities among teaching practices
	D. HUGHES HALLET	Connections: Mathematical, interdisciplinary, electronic, and personal

2016	B. R. HODGSON	Apport des mathématiciens à la formation des enseignants du primaire : regards sur le « modèle Laval »
	C. KIERAN	Task design in mathematics education: Frameworks and exemplars
	E. MULLER	A third pillar of scientific inquiry of complex systems—Some implications for mathematics education in Canada
	P. TAYLOR	Structure—An allegory
2017	Y. SAINT-AUBIN	The most unglamorous job of all: Writing exercises
	A. SELDEN	40+ years of teaching and thinking about university mathematics students, proofs, and proving: An abbreviated academic memoir

Appendix C / Annexe C

PROCEEDINGS OF ANNUAL MEETINGS / ACTES DES RENCONTRES ANNUELLES

Past proceedings of CMESG/GCEDM annual meetings have been deposited in the ERIC documentation system with call numbers as follows:

Proceedings of the 1980 Annual Meeting . . . . ED 204120

Proceedings of the 1981 Annual Meeting . . . . ED 234988

Proceedings of the 1982 Annual Meeting . . . . ED 234989

Proceedings of the 1983 Annual Meeting . . . . ED 243653

Proceedings of the 1984 Annual Meeting . . . . ED 257640

Proceedings of the 1985 Annual Meeting . . . . ED 277573

Proceedings of the 1986 Annual Meeting . . . . ED 297966

Proceedings of the 1987 Annual Meeting . . . . ED 295842

Proceedings of the 1988 Annual Meeting . . . . ED 306259

Proceedings of the 1989 Annual Meeting . . . . ED 319606

Proceedings of the 1990 Annual Meeting . . . . ED 344746

Proceedings of the 1991 Annual Meeting . . . . ED 350161

Proceedings of the 1993 Annual Meeting . . . . ED 407243

Proceedings of the 1994 Annual Meeting . . . . ED 407242

Proceedings of the 1995 Annual Meeting . . . . ED 407241

NOTE

There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year.