CANADIAN MATHEMATICS EDUCATION STUDY GROUP

GROUPE CANADIEN D’ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

PROCEEDINGS / ACTES
2018 ANNUAL MEETING / RENCONTRE ANNUELLE 2018

CMESG GCEDM

Quest University
Squamish, British Columbia
June 1 – June 5, 2018

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Proceedings of the 2018 Annual Meeting of the
Canadian Mathematics Education Study Group /
Groupe Canadien d'Étude en Didactique des Mathématiques
are published by CMESG/GCEDM.
They were published online in May 2019.

ISBN: 978-0-88920-020-3

PROCEEDINGS OF THE 2018 ANNUAL MEETING OF THE CANADIAN MATHEMATICS EDUCATION STUDY GROUP / ACTES DE LA RENCONTRE ANNUELLE 2018 DU GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

42nd Annual Meeting
Quest University
Squamish, British Columbia
June 1 – June 5, 2018

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INTRODUCTION

Peter Liljedahl – President, CMESG/GCEDM
Simon Fraser University

In June 2018 the Canadian Mathematics Education Study Group/Groupe Canadien d'étude en didactique des mathématiques (CMESG/GCEDM) held its 42nd meeting in the idyllic setting of Squamish. With the mountains as a backdrop the local organizers, Richard Hoshino and Asia Matthews, and their entire conference team anticipated and attended to our every need. A special thanks goes out to the energetic and enthusiastic students who helped this meeting run so smoothly: Kailyn Pritchard, Anika Watson, Jacob Richardson, Ahmed Faarah, Graham King, and Kika MacFarlane.

This meeting marked the first time CMESG/GCEDM had been in British Columbia since 2010 and the first time it had been held at Quest University. Among the more than 150 attendees were 35 teachers, 17 of whom had been funded by the BC Association of Mathematics Teachers (BCAMT) and the Pacific Institute for Mathematical Sciences (PIMS) to attend the meeting. Together we were stimulated by a scientific program featuring plenary sessions by Merrilyn Goos and Donald Violette. While Donald moved us to be passionate about our teaching, Merrilyn inspired us to think about what can be achieved when mathematicians and mathematics education researchers work together in teacher education. The scientific program also challenged us in choosing among five working groups, six topic sessions, eight new PhD presentations, 16 gallery walk presentations, and six AdHoc sessions. At every moment of the program I was keenly aware that there were sessions happening that were just as good as the one I was in.

In this regard, the proceedings of the 42nd meeting of CMESG/GCEDM is an opportunity to catch up on all the things that we missed when we were forced to choose one amazing session over another. On behalf of the CMESG/GCEDM membership I would like to thank the contributors to this proceeding as well as the proceeding editor, Jennifer Holm, for their dedication in creating a written record of our meeting. As you read these proceedings, I hope that you will be able to recapture some of the magic that occurs when 157 people, dedicated to the betterment of mathematics education, come together to share, discuss, and create the future of mathematics education in Canada.

Horaire

Vendredi 1 juin	Samedi 2 juin	Dimanche 3 juin	Lundi 4 juin	Mardi 5 juin
Réception des amis de FLM vendredi, 1 juin 15h30 – 16h20	8h45 – 10h15 **Groupes de travail**	8h45 – 10h15 **Groupes de travail**	8h45 – 10h15 **Groupes de travail**	8h45 – 9h30 **N$^{\text{lles}}$ thèses**
14h30 – 18h45 **Inscription**	10h15 – 10h45 Pause café	10h15 – 10h45 Pause café	10h15 – 10h45 Pause café	9h45 – 10h30 **Séances ad hoc**
17h00 – 18h30 **Souper**	10h45 – 12h15 **Groupes de travail**	10h45 – 12:15 **Groupes de travail**	10h45 – 12h15 **Groupes de travail**	10h30 – 11h00 Pause café
18h30 – 19h30 **Séance d'ouverture**	12h30 – 13h45 Dîner	12h30 – 13h30 Dîner	12h30 – 13h45 Dîner	11h15 – 12h30 **Séance de clôture**
19h30 – 20h30 **Plénière I**	13h45 – 14h15 **Petits groupes**	13h00 – 14h00 **Café, dessert et galerie mathématique**	13h45 – 14h15 **Petits groupes**	
20h30 – 22h00 **Réception**	14h25 – 15h25 **Discussion de la plénière I**	14h00 – 15h00 **Plénière II**	14h25 – 15h25 **Discussion de la plénière II**	
	15h30 – 16h15 **Séances thématique**	15h15 – 16h00 **Excursions**	15h30 – 16h00 **Temps pour préparer las présentation de « clôture »**	
	16h15 – 16h45 Pause café	19h00 – ? **Souper Social/spectacle de talents/micro ouvert…**	16h00 – 16h45 **Séances thématique**	
	16h45 – 17h25 **N$^{\text{lles}}$ thèses**		16h55 – 17h25 **Séances ad hoc**	
	17h30 – 18h45 **Assemblée générale annuelle**		17h30 – 22h00 **Souper Social/Danse**	
	18h45 – ? **Souper libre**			

Schedule

Friday[i] June 1	**Saturday June 2**	**Sunday June 3**	**Monday June 4**	**Tuesday June 5**
	8h45 – 10h15 **Working Groups**	8h45 – 10h15 **Working Groups**	8h45 – 10h15 **Working Groups**	8h45 – 9h30 **New PhDs**
				9h45 – 10h30 **Ad hoc**
	10h15 – 10h45 Break	10h15 – 10h45 Break	10h15 – 10h45 Break	10h30 – 11h00 Break
Friends of FLM Reception Friday, June 1 15h30 – 16h20	10h45 – 12h15 **Working Groups**	10h45 – 12:15 **Working Groups**	10h45 – 12h15 **Working Groups**	11h15 – 12h30 **Closing Session**
	12h30 – 13h45 Lunch	12h30 – 13h30 Lunch	12h30 – 13h45 Lunch	
		13h00 – 14h00 **Dessert, Coffee & Mathematics Gallery**		
	13h45 – 14h15 **Small Groups**		13h45 – 14h15 **Small Groups**	
	14h25 – 15h25 **Plenary I Discussion**	14h00 – 15h00 **Plenary II**	14h25 – 15h25 **Plenary II Discussion**	
14h30 – 18h45 **Registration**	15h30 – 16h15 **Topic Session**	15h15 – 16h00 **Excursions**	15h30 – 16h00 **Time to organize 'Closing' presentation**	
	16h15 – 16h45 Break		16h00 – 16h45 **Topic Session**	
17h00 – 18h30 **Dinner**	16h45 – 17h25 **New PhDs**		16h55 – 17h25 **Ad Hoc**	
	17h30 – 18h45 **Annual General Meeting**		17h30 – 22h00 **Dinner Social/Dance**	
18h30 – 19h30 **Opening Session**	18h45 – ? **Dinner on your own**	19h00 – ? **Dinner Social/talent show/open mic…**		
19h30 – 20h30 **Plenary I**				
20h30 – 22h00 **Reception**				

Plenary Lectures

Conférences plénières

ET SI ON ENSEIGNAIT LA PASSION?

Donald Violette
Université de Moncton

RÉSUMÉ : DEUX ADAGES M'ONT TOUJOURS GUIDÉ DEPUIS MES PREMIERS BALBUTIEMENTS DANS LE MONDE DES MATHÉMATIQUES :

- « Rien ne s'accomplit dans ce monde, sans passion » et
- « Être plus que paraître ».

Ces deux adages m'ont amené à me questionner sur mon enseignement dès ma première année d'enseignement universitaire. J'ai vite compris qu'un professeur ne devait pas seulement transmettre des connaissances, mais également de la passion. Car la passion est contagieuse et elle rend les cours plus intéressants, plus vivants et plus stimulants.

J'avais des rêves : créer des activités intellectuelles de haut niveau pour les élèves doués des écoles francophones du Nouveau-Brunswick et publier un roman jeunesse dans lequel les mathématiques seraient au cœur de l'intrigue. Ces rêves sont devenus réalité en fondant, entre autres, trois concours et trois camps de mathématiques et une fondation mathématique, la seule au Canada, qui englobe tous mes projets pour les jeunes. De plus, mon livre *Les mathémagiciens* a été lancé en mars 2017.

MA PASSION

Depuis mes premiers pas dans le monde des mathématiques, deux adages m'ont toujours inspiré et guidé. Ce sont « Être plus que paraître » et « Rien ne s'accomplit dans ce monde sans passion ».

La passion permet de créer, d'imaginer, d'avoir de l'inspiration, d'accomplir de grandes choses et d'aller jusqu'au bout de nos rêves. En outre, la passion est contagieuse, communicative et par conséquent on ne peut faire autrement que la transmettre aux autres. N'est-ce pas qu'un professeur passionné dans une salle de classe peut faire toute une différence et faire des miracles.

J'aime les mathématiques depuis ma tendre enfance et j'ai eu le bonheur et la chance d'avoir une excellente enseignante de mathématiques durant mes études secondaires. Cette enseignante était jeune, tout feu tout flamme, muni d'une passion incommensurable, d'une grande générosité et d'un dévouement sans borne. Elle m'a inculqué de grandes valeurs morales, dont le respect et le goût du travail bien fait, me permettant de développer ma conscience professionnelle. Comme je m'ennuyais dans les cours de mathématiques, elle s'en est rapidement rendu compte et m'a conseillé de m'avancer par moi-même dans la matière et de faire des chapitres supplémentaires. Déjà à cet âge, j'adorais aider mes camarades de classe, leur expliquer et enseigner différents concepts. J'apportais ainsi une grande aide à mon enseignante de mathématiques tout en allant chercher une certaine confiance que je n'avais pas forcément à cet âge surtout à cause de ma timidité. En outre, cela m'a permis de m'épanouir. Cette enseignante de mathématiques fut mon premier modèle dans le monde des mathématiques et eut une très grande influence positive sur moi. Grâce à elle, je savais déjà à 15 ans que je deviendrais un mathématicien, plus précisément un professeur d'université.

Je vais vous raconter une petite histoire qui montre sans l'ombre d'un doute l'amour et la passion que j'avais pour les mathématiques à l'école secondaire. Un matin je me suis levé avec une vilaine grippe : fièvre, courbatures, toux profonde et la tête qui voulait éclater. Comme je n'arrivais pas à dormir, je me suis levé et me suis mis à faire des mathématiques à la table de la cuisine. Je revois encore ma mère assise dans sa berceuse qui me regardait et qui ne pouvait pas croire que je faisais des mathématiques étant aussi malade. Elle m'a dit que ça n'avait pas de bon sens. Je lui ai expliqué que ça me faisait du bien, que ça me détendait, car j'adore les mathématiques, et me faisait oublier un peu la douleur. Il ne faut pas seulement aimer les mathématiques pour faire une telle chose; il faut être diablement passionné.

Voici une petite anecdote qui démontre, en plus d'être un élève doué, l'étonnante autonomie ainsi que le niveau élevé de maturité et le sens aigu des responsabilités que je possédais à l'âge de 16 ans. La direction de mon école m'a demandé de remplacer mon enseignante de mathématiques qui devait subir une intervention chirurgicale. Pour faire ce travail, le ministère de l'Éducation du Nouveau-Brunswick m'a accordé un permis local d'enseignement alors que j'étais toujours un élève et j'ai été suppléant durant une huitaine de jours dans tous les cours qu'elle enseignait. Comme je ne pouvais pas suivre mes autres cours, je devais faire mes tests après l'école et j'obtenais toujours une note parfaite. Cette expérience unique et enrichissante fut l'étincelle qui allait allumer un grand feu dans ma vocation et ma passion de professeur. À ma connaissance, je suis le seul élève qui a eu la chance de vivre cette expérience unique, enrichissante, motivante et rarissime. Elle m'a apporté une grande confiance en moi et m'a conforté dans mon désir de devenir non seulement un mathématicien, mais un professeur au niveau universitaire.

Il paraît que j'ai une façon unique d'enseigner; en tout cas, les étudiants me le disent régulièrement. Je ne transmets pas seulement de la matière et des connaissances, mais également de la passion. Quelle différence pour les étudiants de voir leur professeur amoureux et passionné du sujet qu'il enseigne et de le transmettre avec toute sa fougue, sa passion et son amour! Lorsque j'introduis une nouvelle théorie, je le fais toujours de la façon la plus naturelle qui soit, faisant des parallèles avec des notions qu'ils connaissent déjà, ce qui leur donne des points de repère. Je suis un professeur qui va jusqu'au fond des choses et donc pas question de présenter les concepts de façon superficielle. En classe, je dis régulièrement aux étudiants que les mathématiques sont belles et captivantes et pour la majorité d'entre eux, c'est la première fois qu'ils entendent quelqu'un leur dire que les mathématiques sont jolies. Ils ont le réflexe de rire, mais dès qu'ils y réfléchissent un peu, ils prennent conscience de cette réalité. Ainsi, de façon subtile, je leur fais aimer les mathématiques.

Un exemple qui montre la reconnaissance de mes étudiants envers moi : d'une façon relativement régulière, les étudiants, peu importe leur niveau, m'applaudissent à la dernière rencontre (séance) du cours. Je ne connais pas d'autres professeurs qui reçoivent une telle marque d'affection. Il n'y a pas de plus belle récompense, c'est tellement émouvant et ça démontre l'amour que mes étudiants me portent.

La capacité d'un professeur de s'enthousiasmer, de rayonner et de s'emballer devant une classe est un facteur extrêmement important et déterminant dans la réussite des étudiants. Se développe alors une complicité entre le professeur et ses étudiants. Ceux-ci n'ont pas le choix d'embarquer; ils passeront donc plus de temps à étudier et à travailler dans votre cours. Y a-t-il quelque chose de plus gratifiant pour un professeur que de voir ses étudiants travailler plus fort et mieux dans ses cours? On sent que les étudiants sont avec nous. Lorsque j'ai à présenter une notion difficile et subtile, j'en rêve souvent les nuits précédentes, car je me pose toujours la question comment je vais aborder ce concept, comment je vais le transmettre. Mais quand je vois de grands yeux s'allumer et des sourires sur les visages de mes étudiants, je sais que j'ai relevé défi. Et là c'est le bonheur total; c'est comme du bonbon. Il n'y a pas de plus belle récompense.

Mes étudiants forment une grande famille. J'ai un très grand respect pour eux et je les traite tous de la même façon, avec le même respect, la même rigueur, la même exigence et la même justice. De façon générale, ils sont sympathiques et respectueux et me voient comme un modèle. Ce qui me permet de tenir le coup après plus de 41 années d'enseignement universitaire, c'est ma passion incommensurable, mon amour pour ma discipline, pour ma profession et mes étudiants.

J'AVAIS DES RÊVES

Ma passion et mon amour des mathématiques et de l'enseignement m'ont permis d'aller au bout de mes rêves. Le plus grand, à l'exception de mon enseignement, est la fondation de camps de mathématiques pour les élèves francophones du Nouveau-Brunswick. Ce rêve date du tout début de ma carrière. Au Canada, seuls les francophones du Nouveau-Brunswick n'avaient pas de camps mathématiques dans leur langue. Il fallait donc corriger cette énorme injustice. Je m'y suis attelé et en 2012, le premier de mes camps de mathématiques voyait donc le jour. Maintenant, nous en avons trois, un pour les élèves de la 12^{e} année, un autre pour les élèves de la 5^{e} année et enfin un camp pour les élèves de la 8^{e} année. Nous sommes partis de loin et de rien, mais maintenant, nous sommes en avance sur les autres provinces canadiennes, car à ma connaissance, seuls les francophones du Nouveau-Brunswick ont des camps de mathématiques au niveau primaire. Ces trois camps représentent des activités intellectuelles de haut niveau et sont précédés par trois concours de mathématiques qui me permettent d'aller chercher les plus doués, à savoir la crème de la crème, ceux qui en mangent des mathématiques.

La création d'une fondation mathématique, la seule au Canada, était un autre de mes rêves. Cette fondation est un organisme à but non lucratif qui englobe tous mes projets de promotion des mathématiques, tels que mes camps de mathématiques, mes concours et autres. L'idée d'avoir une telle fondation est partie du fait que c'était plus facile d'aller chercher des dons et des subventions pour la tenue des camps et des concours.

J'ai cessé de faire de la recherche en mathématiques pures à partir du moment où mon énergie et mes efforts allaient surtout pour promouvoir et démystifier les mathématiques. Ce n'est pas facile de faire de la recherche en mathématiques pures; la publication d'un article substantiel peut prendre quelques années de travail. En outre, très peu de personnes lisent les articles de mathématiques pures, car ça intéresse qu'une petite portion de mathématiciens. Comme, dans mon cas, l'enseignement universitaire a toujours primé sur la recherche, j'ai donc décidé de concentrer mon temps et mes efforts à créer des activités intellectuelles de haut niveau pour les jeunes et à fabriquer de petits mathématiciens, ce qui me permettaient en même temps de réaliser mes rêves.

Je me suis tourné également vers la publication d'articles de vulgarisation mathématique portant sur la magie et la beauté des mathématiques; j'en ai publié quelques-uns dans le *Bulletin de l'Association mathématique du Québec*. Dans cette direction, l'un de mes plus grands projets a été la sortie de mon premier roman jeunesse en 2017, ce qui m'amène à vous parler maintenant de mes contributions.

MES CONTRIBUTIONS

La publication de mon livre *Les mathémagiciens* est un beau et grand projet s'adressant aux jeunes de 8 à 13 ans, mais qui peut plaire à tous. Avec cette publication, j'ai dû sortir des sentiers battus, car je ne suis pas un écrivain, mais un mathématicien, un professeur et un conférencier. Écrire un livre n'a jamais été un rêve, mais la vie nous apporte d'agréables surprises et on rencontre des gens qui nous inspirent et qui sans le savoir nous influencent. C'est ce qui s'est passé avec ce projet; deux de mes petits protégés possédant un grand potentiel en mathématiques m'ont incité à écrire ce roman jeunesse. L'un des buts de ce livre est de faire croître l'intérêt des petits francophones envers les mathématiques. De plus, cette publication représente un autre outil de démystification, de promotion et de rayonnement des mathématiques. Parallèlement à ce projet, j'ai développé une fiche pédagogique pour accompagner les enseignants à réaliser, à partir de ce livre, des explorations ou activités mathématiques dans la salle de classe. Cette fiche se retrouve sur le site Internet de la maison d'édition *Bouton d'or d'Acadie*.

Je joue également un rôle important dans les écoles francophones du Nouveau-Brunswick ainsi qu'au Ministère de l'Éducation du Nouveau-Brunswick. J'ai été membre du Comité des programmes ainsi qu'expert-conseil externe dans l'évaluation des examens du ministère durant près de 25 ans. Jusqu'à présent, j'ai prononcé environ 500 causeries et conférences dans les écoles francophones du Nouveau-Brunswick auxquelles ont assisté plus de 20 000 élèves. Mes conférences portent sut la beauté et la magie des mathématiques tout en faisant un clin d'œil à la nature. Le ruban de Möbius et les carrés magiques sont les gros coups de cœur des élèves. À la fin de mes présentations, les jeunes me demandent souvent des autographes, me voyant comme le mathématicien-vedette du Nouveau-Brunswick. Lorsque je termine une conférence, je leur pose toujours la question « que devrait-on faire si les mathématiques n'existaient pas? » Il y a toujours au moins un élève qui répond « il faudrait les inventer », ce qui est évidemment la bonne réponse.

Comme autres contributions, j'ai fondé trois concours de mathématiques ainsi que trois camps de mathématiques pour les niveaux de 5^{e} année, 8^{e} année et 12^{e} année. Chaque année, 50 jeunes de ces trois niveaux participent à mes camps de mathématiques. Ces camps sont d'une durée de huit jours et chacun de mes concours est le prélude à chacun de ces camps. Les jeunes adorent ces activités intellectuelles de haut niveau; d'ailleurs, ils en redemandent et si ces camps n'existaient pas, il faudrait les créer. Les camps mathématiques Donald-Violette ont pour but de rassembler des jeunes francophones doués qui font preuve d'enthousiasme pour les mathématiques. Ils leur permettent de partager leur passion avec d'autres jeunes et de faire des mathématiques avec des mathématiciens chevronnés et professionnels dans un climat convivial et propice à l'apprentissage et dans un environnement de concurrence saine.

Afin de continuer le travail de démystification des mathématiques, je vais mettre sur pied cette année une galerie d'objets d'art mathématiques grâce à ma fondation mathématique. Elle mettra en vedette de beaux objets mathématiques fascinants comme le ruban de Möbius, le tétraèdre de Sierpinski, la réalisation géométrique de la bouteille de Klein et des formes fractales, ainsi que des affiches de mathématiciens célèbres.

La création en 2017 du *Cercle des jeunes mathématiciens du sud-est du Nouveau-Brunswick* est une autre de mes réalisations. Les doués en mathématique des écoles secondaires viennent nous rencontrer afin de résoudre des énigmes mathématiques et le but de cette activité est de leur apprendre à réfléchir, non à calculer. Tous les mois, nous nous rencontrons un samedi matin et résolvons des énigmes pour une période d'environ deux heures. Lors de la première rencontre du Cercle, j'ai fait une belle découverte. En effet, les jeunes étaient disposés à aller au tableau pour montrer et expliquer leur solution. Quelle ne fut ma surprise, lorsqu'un autre jeune dans la salle me demanda d'aller montrer sa solution, différente de l'autre. Je ne peux que constater l'engouement des jeunes pour ce genre d'activités intellectuelles de haut niveau.

Le samedi 30 juin 2018, entre deux camps de mathématiques, je faisais le dévoilement d'une structure mathématique géante dans l'entrée de la cour de l'école Mgr-Martin de Saint-Quentin. Cette structure est un ruban de Möbius confectionné en aluminium mesurant près de quatre mètres de hauteur et près de deux mètres de diamètre. Grâce à ce monument, Saint-Quentin devient donc la ville des mathématiques en Atlantique.

À chaque année depuis 2013, nous célébrons les mathématiques dans le cadre de la journée internationale du nombre pi. Durant cette journée, nous organisons des activités telles que des jeux mathématiques, des combats, des rallyes, nous prononçons des conférences et j'incite mes étudiants à faire de courtes présentations.

J'ai créé une bourse d'excellence en mathématiques remise annuellement à un diplômé de la polyvalente A.-J.-Savoie de Saint-Quentin qui obtient la plus haute note dans le cours de mathématiques avancées. En outre, de concert avec la compagnie *Assomption Vie* et la *Fondation mathématique Donald-Violette,* nous remettons annuellement trois bourses d'excellence en mathématiques dans chacun des districts scolaires francophones du Nouveau-Brunswick. Ces bourses mettent en évidence l'importance et l'utilité des mathématiques dans notre société en constante évolution. De plus, elles encouragent l'excellence et permettent de démystifier et de promouvoir les mathématiques tout en enrayant certains préjugés à l'égard des mathématiques. Combien de fois entend-on des adultes avancer que les mathématiques sont compliquées et ne servent à rien?

Le projet *Mathématiques en mouvement* a vu le jour en 2013 et consiste en des soirées de vulgarisation pour les adultes. On ne peut aimer ce qu'on ne connaît pas. Il faut donc donner la chance aux adultes d'apprivoiser les mathématiques en leur montrant la magie et la beauté qui les caractérisent si bien. En présentant des soirées de vulgarisation mathématique, les adultes en repartent étonnés et sont surpris de constater qu'ils prennent goût aux mathématiques, qu'ils s'amusent et ont beaucoup de plaisir. Les adultes se laissent prendre aux jeux. On entend des rires et des « Mon Dieu ». Souvent certains adultes se regroupent et participent activement au défi que je leur ai lancé. De telles activités aident à enrayer des préjugés et des attitudes négatives à l'égard des mathématiques.

MA VISION

Toutes ces activités intellectuelles de haut niveau que j'ai créées aident énormément au recrutement et à la rétention d'étudiants universitaires et l'impact est loin d'être négligeable. Il faut à tout prix augmenter le nombre d'inscriptions dans les programmes de mathématiques dans les universités et c'est en multipliant ce genre d'activités intellectuelles que nous allons y arriver. Il est impératif d'encourager les différents médias à parler plus souvent des mathématiques. Par exemple, les médias encensent les progrès remarquables réalisés dans la conquête de l'espace, mais ne mentionne jamais jusqu'à quel point les mathématiques ont été utiles, voire essentielles, à leur réalisation. Dans de telles situations, j'aimerais voir un encadré qui expliquerait le rôle indispensable joué par les mathématiques. Dans l'Acadie du Nouveau-Brunswick, il n'y avait pas de culture mathématique il y a quelques années. Avec tous les projets que j'ai mis sur pied, j'ai réussi à créer une culture mathématique. Mon prochain défi est de maintenir cette culture.

CONCLUSION

Il reste énormément de travail à faire dans la démystification et la promotion des mathématiques pour que celles-ci retrouvent leur lettre de noblesse. Chaque activité intellectuelle que l'on fait avec les jeunes et les adultes est importante et aide à prévenir les préjugés et les attitudes négatives envers les mathématiques. Continuons à multiplier ce genre d'activités et nous vaincrons!

MAKING CONNECTIONS ACROSS DISCIPLINARY BOUNDARIES IN PRE-SERVICE MATHEMATICS TEACHER EDUCATION

Merrilyn Goos
University of Limerick, Ireland

This paper provides an augmented recount of the plenary address I gave at the 2018 CMESG conference, about an Australian project that brought together mathematics educators and mathematicians to design new interdisciplinary approaches to initial teacher education. It draws on conversations I had with conference participants before and after my talk—conversations that caused me to reflect on the circumstances that led to the project as well as implications for the future. Thus the paper is organised in three main sections: Prologue, Project, and Epilogue. The Prologue looks back on key experiences that, in hindsight, prepared me to envision and lead the project. The section describing the Project explains its theoretical rationale, design, and outcomes. The Epilogue considers the potential impact of this work.

PROLOGUE

Mathematics and mathematics education are fields that differ in the types of knowledge they produce as well as ways of pursuing knowledge. Michael Fried (2014a, 2014b) has written with great sensitivity and insight about the relationship between mathematics and mathematics education, and how as disciplines they have been moving further apart for many years. Fried noted that mathematics education as an academic discipline has increasingly turned towards the social sciences for its sense of identity, while at the same time mathematicians are deeply concerned about the apparent disappearance of mathematical content from mathematics education research. Thus we have witnessed a gradual divergence—and perhaps estrangement—between two fields that are historically closely connected. Fried (2014b) argues that one of the outcomes of this divergence is reduced cooperation between mathematicians and mathematics education researchers, leading to a lack of coherence in mathematics education research "regardless whether it is the mathematicians or the mathematics education researchers who take charge" (p. 10).

Fried's writing provides an entry point into the circumstances leading to my interest in interdisciplinary collaboration between mathematics education researchers and mathematicians, and ultimately to the *Inspiring Mathematics and Science in Teacher Education* (IMSITE) project that was the subject of my CMESG plenary. Although I am a mathematics educator who has spent most of my academic life located in a university School of Education, for nearly five years (2008-2012), I had a different job as Director of my university's Teaching and Educational Development Institute. In that role I was responsible for leading programs to enhance the quality of teaching and learning across the whole university and for developing cooperative relationships with schools and disciplines to support and disseminate good teaching practices. This position created unique opportunities for me to work with academics in disciplines other than education—including mathematics—and to learn how to integrate our complementary knowledge and perspectives to enhance teaching and learning. At around the same time (2010-2014) I became President of the Mathematics Education Research Group of Australasia (MERGA), an organisation similar to CMESG. As President, I represented MERGA on various committees and boards that sought input from mathematics educators and mathematicians. These included Science and Technology Australia, the national peak body representing scientists, industry, governments and community organisations; the National Committee for the Mathematical Sciences convened by the Australian Academy of Science, the adhering organisation to the International Mathematical Union; the Australian Council of Heads of Mathematical Sciences, representing the Heads of university mathematics departments; and various working groups contributing to development of a new national mathematics curriculum for schools. All of these experiences provided opportunities for dialogue between mathematics educators and mathematicians and developed interdisciplinary networks of academics from both fields who were interested in the teaching and learning of mathematics.

Although I worked hard on building mutual understanding between mathematicians and mathematics educators and respect for each other's disciplinary expertise, it is unlikely that any major initiatives would have resulted without another significant confluence of events. In 2011 the Australian government appointed a new Chief Scientist, Professor Ian Chubb, who began writing or commissioning a suite of policy and strategic reports on science, technology, engineering and mathematics—the so-called STEM disciplines (Office of the Chief Scientist, 2012, 2013, 2014). It was in response to the recommendations from the first of these reports that the Australian Government, Department of Education and Training (2016) funded the *Enhancing the Training of Mathematics and Science Teachers* (ETMST) program. The goal of the program was to drive "a major improvement in the quality of mathematics and science teachers by supporting new pre-service programs in which faculties, schools or departments of science, mathematics and education collaborate on course design and delivery, combining content and pedagogy so that mathematics and science are taught as dynamic, forward-looking and collaborative human endeavours" (para. 1). The ETMST program offered $12 million to fund three-year national projects (2014-2016) addressing this ambitious goal. Here was an opportunity to test and further develop my emerging ideas about interdisciplinary collaboration between mathematics educators and mathematicians. I assembled a project team, jointly led by myself and a mathematician colleague; we wrote a project proposal; and we were successful in obtaining funding. The story of the project continues to unfold in the next section.

THE PROJECT

Although my main interest is in mathematics education, it was strategically advantageous to propose a project that also included science education—and so the IMSITE project was born. It was one of five national projects funded under the ETMST program. The IMSITE project was a partnership between six Australian universities and involved 23 chief investigators who were mathematicians, scientists, mathematics educators and science educators. The universities and participants were chosen based on two criteria: (1) there were pre-existing collaborations—or at least a willingness to collaborate—between education academics and discipline (mathematics or science) academics; and (2) the universities represented a variety of institutional groupings, geographical locations, initial teacher education program structures, and student demographic characteristics.

AIMS AND RATIONALE

The overall ETMST program aimed to promote strategic change in the Australia higher education sector by developing and disseminating new interdisciplinary approaches to mathematics and science initial teacher education. The significance of this aim needs to be understood in the context of teacher education program structures in Australian universities (summarised in Table 1). Content knowledge and content-specific pedagogical knowledge are usually taught in separate courses by academics in separate disciplines—for example, mathematicians and mathematics educators—who rarely collaborate on course design and may not even know each other. These courses are also typically taught at different times during a teacher education degree program, so there is a temporal as well as spatial separation of content from pedagogy in the university-based preparation of future teachers.

Level/duration	Entry requirement	Preparation for teaching	Content
4-year undergraduate Bachelor of Education	Completed secondary school	Primary or secondary school	Mathematics and science content Mathematics and science pedagogy
4- or 5-year undergraduate dual degree (BSc/BEd)	Completed secondary school	Mainly secondary school	**First:** Mathematics and science content (taught by discipline academics) **Then:** Mathematics and science pedagogy (taught by education academics)
1- or 2-year postgraduate Diploma or Masters degree	Completed initial undergraduate degree with mathematics or science content	Mainly secondary school	Mathematics and science pedagogy

Table 1. Teacher education program structures in Australian universities.

The IMSITE project aimed to address identified national priorities in teacher education by fostering genuine, lasting collaboration between the mathematicians, scientists, and mathematics and science educators who prepare future teachers and by identifying and institutionalising new ways of integrating the content expertise of mathematicians and scientists and the pedagogical expertise of mathematics and science educators. The conceptual framework for the project drew on two sources, involving policy and theory. From a policy perspective, the project was grounded in the Australian Professional Standards for Teachers (AITSL, 2011), which identify what teachers are expected to know and be able to do at four career stages. In the domain of Professional Knowledge, teachers should “know students and how they learn” and “know the content and how to teach it” (p. 3). Our argument was that knowing the content and knowing how to teach it needed to be brought into closer alignment rather than being separated in most teacher education degree programs (see Table 1). From a theoretical perspective, we referenced Wenger’s (1998) social theory of learning, and in particular the notion of communities of practice and boundary practices, to understand how the perspectives of mathematicians and mathematics educators can be coordinated and connected. At the time the project began there were few known instances of productive collaboration in the design and deliver of initial mathematics teacher education programs in Australia.

The project built on existing initiatives in the participating universities with these universities collaborating to develop, test and evaluate the following approaches:

- recruitment and retention strategies that promote teaching careers to undergraduate mathematics and science students;
- innovative curriculum arrangements that combine authentic content and progressive pedagogy to construct powerful professional knowledge for teaching;
- approaches by which universities can build long term relationships with teacher education graduates, enabling them to continually renew their professional and pedagogical knowledge of mathematics and science.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

Collaboration between discipline academics and education academics connects the perspectives of mathematicians, scientists, and teacher educators. Wenger's (1998) social theory of learning, and in particular the notion of communities of practice, provided the theoretical framework for the project. Communities of practice are everywhere—in people's workplaces, families, and leisure pursuits, as well as in educational institutions. Communities of practice are characterised by mutual engagement of participants, negotiation of a joint enterprise that coordinates participants' complementary expertise, and development of a shared repertoire of language and concepts.

Mathematicians and mathematics educators are members of distinct, but related, communities of professional practice, and one of the aims of the IMSITE project was to find ways of connecting these communities (and corresponding communities of scientists and science educators) across the boundaries that define them. Wenger (1998) writes of boundary encounters as potential ways of connecting communities. Boundary encounters are events that give people a sense of how meaning is negotiated within another practice. They often involve only one-way connections between practices, such as one-on-one conversations between members of two communities. However, a two-way connection can be established when delegations comprising several participants from each community are involved in an encounter. Wenger suggests that if "a boundary encounter—especially of the delegation variety—becomes established and provides an ongoing forum for mutual engagement, then a practice is likely to start emerging" (p. 114). Such boundary practices then become a longer term way of connecting communities in order to coordinate perspectives and resolve problems.

There is an emerging body of research on learning mechanisms involved in interdisciplinary work on shared problems. This type of work is becoming increasingly important because of growing specialisation within domains of expertise that requires people to collaborate across boundaries between disciplines and institutions. Akkerman and Bakker's (2011) review of this research literature emphasised that boundaries are markers of "sociocultural difference leading to discontinuity in action or interaction" (p. 133). Boundaries are thus dynamic constructs that can shape new practices through revealing and legitimating difference, translating between different world views, and confronting shared problems. As a consequence, boundaries carry potential for learning.

Akkerman and Bakker (2011) identified four potential mechanisms for learning at the boundaries between domains. The first is *identification*, which occurs when the distinctiveness of established practices is challenged or threatened because people find themselves participating in multiple overlapping communities. Identification processes reconstruct the boundaries between practices by delineating more clearly how the practices differ: discontinuities are not necessarily overcome. A second learning mechanism involves *coordination* of practices or perspectives via dialogue in order to accomplish the work of translation between two worlds. The aim is to overcome the boundary by facilitating a smooth movement between communities or sites. *Reflection* is nominated as a third learning mechanism that is often evident in studies involving an intervention of some kind. Boundary crossing—

moving between different sites—can promote reflection on differences between practices, thus enriching one's ways of looking at the world. The fourth learning mechanism is described as transformation, which, like reflection, is found in studies investigating effects of an intervention. Akkerman and Bakker state that transformation is a learning mechanism that can lead to a profound change in practice, "potentially even the creation of a new, in-between practice, sometimes called a boundary practice" (p. 146). They go on to label processes of transformation as including

- confrontation—encountering a discontinuity that forces reconsideration of current practices;
- recognising a shared problem space—in response to the confrontation;
- hybridisation—combining practices from different contexts;
- crystallisation—developing new routines that become embedded in practices;
- maintaining the uniqueness of intersecting practices—so that fusion of practices does not fully dissolve the boundary;
- continuous joint work at the boundary—necessary for negotiation of meaning in the context of institutional structures that work against collaboration and boundary crossing.

Akkerman and Bakker note that, although transformation is rare and difficult to achieve, it carries promise of sustainable impact. They also propose that identification and reflection, both of which involve recognising and explicating different perspectives, are necessary pre-conditions for transformation to occur.

While the IMSITE project was designed to address practical priorities in initial teacher education, it also contributed to my long-term research program that conceptualises learning from a sociocultural perspective (see Goos, 2014). This research program had previously investigated the learning of school students and teachers (Goos, 2004; Goos & Bennison, 2008), and it was now being extended to explore opportunities to learn through the exchange of expertise across disciplinary boundaries in mathematics education. I addressed the following research questions through the IMSITE project:

1. What boundary practices emerged between the two communities?
2. What conditions enabled or hindered sustained interdisciplinary collaboration?
3. How did learning occur at the boundaries between the two communities?

RESEARCH DESIGN AND METHODS

Two rounds of interviews were conducted with the project chief investigators. Interviews were semi-structured to allow for consistency in the topics of inquiry and flexibility in the depth and sequencing of questions. Question prompts included

- To what extent is there interdisciplinary collaboration between mathematicians and mathematics educators in your university?
- Can you describe any barriers to, and enablers of, such collaboration?
- What types of exchanges and activities that bring together mathematicians and mathematics educators do you consider to be most successful?

Interviews lasted from 20 to 40 minutes; they were audio-recorded and later transcribed. Analysis of the interviews was guided by the research questions listed earlier. To answer question (1), regarding boundary practices that emerged between the communities of mathematicians and mathematics educators, evidence was initially sought from participating university annual reports describing new models of teacher education that integrated content and pedagogy. Transcripts of interviews were then searched for evidence of exchanges and activities that brought mathematicians and mathematics educators together to work on these initiatives and that subsequently institutionalised new practices. To answer question (2), regarding enabling/hindering conditions, a content analysis of transcripts identified relevant excerpts and developed a minimal set of categories that allowed similarities and differences in the responses to be highlighted. To answer question (3), regarding emergence of learning mechanisms at the boundary between disciplinary communities, the transcripts were scrutinised for evidence of the mechanisms theorised by Akkerman and Bakker (2011).

EMERGENT BOUNDARY PRACTICES

Working between discipline communities and education communities requires the development of new practices that draw on the expertise of individuals from both communities. Evidence of these practices was seen in two ways: co-developed and co-taught courses, and approaches to building communities of pre-service mathematics teachers.

Co-developed and co-taught courses integrating content and pedagogy

Courses that integrate content and pedagogy for secondary pre-service teachers were either further developed or designed and implemented in four of the participating universities. Two examples are summarised below.

Mathematical Content Knowledge for Lower Secondary Mathematics Teachers is offered in the Bachelor of Education (Secondary) program at University A. It was co-designed and is co-taught by a mathematician and mathematics educator who had a prior history of collaboration. The mathematician described the motivation for developing the course:

> *Well it's a subject specifically aimed for [...] pre-service maths teachers. The university has never had a subject like that and I'm not aware of many around the planet even. But it was a need that [mathematics educator] had expressed to me early on. [Her] opinion had been formed by her own students that they were getting a bunch of subjects in the maths department, that they felt as though didn't really prepare them for the maths they were going to teach in the classroom.*

Student learning outcomes include developing deep content knowledge of lower secondary mathematics represented in the *Australian Curriculum* (Australian Curriculum, Assessment and Reporting Authority, n.d.), understanding the links between these mathematics content areas, investigating and communicating mathematical ideas, and understanding the historical and socio-cultural development of mathematical ideas.

Reflective Communication in Mathematics was developed and is delivered collaboratively by a mathematician and a mathematics educator from University B to give non-education students an opportunity to explore teaching (e.g., students are enrolled in Bachelor of Engineering, Bachelor of Mathematics, Bachelor of Advanced Mathematics, Bachelor of Arts programs). The course was also made available to pre-service Bachelor of Mathematics Education students. In addition to coursework, students undertake private mathematics tutoring and participate in a range of mathematics outreach activities that bring secondary school students and their teachers to the university, for example, in 'Work like a mathematician' excursions. Intended learning outcomes include demonstrating the ability to analyse one's own understanding of mathematical concepts, demonstrating pedagogical content knowledge to explain mathematical concepts, and demonstrating technical and communication skills to explain mathematical ideas in creative ways. The intent of the course is to provide a 'risk free' experience in teaching to students who are not enrolled in a teaching degree, in order to encourage them to consider a future career in this field. However, the mathematician and mathematics educator who delivered the course also recognised unanticipated benefits for pre-service teacher education students who were taking the course as an elective. The mathematician commented:

> *We both realised that [these students] had not made the connections between their maths subjects, their pedagogy subjects, and the maths that they were going to be teaching at school. This was the first subject that they'd had where we were talking about both at the same time, taking it further than anything had been taken – like take the syllabus from high school, push it into where it goes to university where they come back and talk about how might you teach it so that you get those outcomes.*

Communities of pre-service mathematics teachers

At the time the IMSITE project was conducted, pre-service teacher education programs for secondary mathematics teachers typically involved either an undergraduate Education degree, a dual degree such as Bachelor of Science/Bachelor of Education (BSc/BEd), or an initial discipline-specific Bachelors degree followed by a one-year postgraduate Diploma in Education or a two-year Master of Teaching (see Table 1). In all models, content and pedagogy are taught in separate courses. In dual degree programs it is typical for mathematics content courses to be taught first, in the BSc component of the program, and pedagogy courses some years later, in the BEd component. This means that pre-service mathematics teachers take their mathematics content courses together with a much larger group of BSc students who are not planning to become teachers, and they may not even be aware that there are other aspiring teachers in their content classes. The lack of a cohort experience in the early years of a pre-service teacher education program makes it difficult to build a sense of community amongst prospective mathematics teachers and could lead to unwanted attrition. This was a shared problem identified by the mathematician and mathematics educator at University C when they realised that they taught the same pre-service secondary teacher education students:

> *Then I think you and I just started chatting one day … and we thought, you know what? You teach the students maths and I teach them education. We should at least be sharing what we know about the students; starting to compare contrast, talk about issues, retention. We started talking about the fact that we would lose some of them.* [mathematics educator]

University C offers a five-year dual degree Bachelor of Science/Bachelor of Education program, where overlap between mathematics and education courses does not occur until the third year of the program. For this reason, the mathematician and mathematics educator participating in the IMSITE project collaborated to create early cohort experiences for pre-service mathematics teachers. For example, in a compulsory first-year mathematics course, rather than randomly mixing mathematics education students in tutorial groups with non-education students, they are allocated together to special tutorials taught by former secondary school mathematics teachers. Regular lunches and social events are also held to bring together later year pre-service students with first year mathematics students who have not yet begun their education studies, for networking and sharing of experiences. An alumni conference was initiated where mathematics pre-service teachers nearing the end of their program participate with recent graduates, mathematicians, and mathematics educators in a professional development day. The purpose of all these cohort-building activities is to create a strong sense of mathematics teacher identity and community from the earliest stages of the degree program and extending beyond graduation.

CONDITIONS ENABLING OR HINDERING COLLABORATION

All participants who were interviewed referred to personal qualities, including open mindedness, trust, mutual respect, shared beliefs and values, as being crucial to enabling interdisciplinary collaboration. Such qualities allow for productive disagreements and challenges:

> *I like the fact that you* [mathematician] *are challenging what I say, my views of the world. I really value that. Obviously, there's trust there because, I guess, if there wasn't trust I wouldn't be happy.* [University B, mathematics educator]

One interviewee (a mathematics educator) identified the importance of having confidence in one's own disciplinary knowledge of mathematics while at the same time being willing to admit ignorance:

> *I'm sure that sometimes education people might feel a bit inferior to … mathematicians when they talk to them. Possibly vice versa as well, when they're talking about pedagogy and they [mathematicians] think "I don't know anything about that, that's strange language". So I guess there's that fear of looking like a fool in front of the other, which you've kind of got to get over at some point somehow.* [University E, mathematics educator]

A second condition, explicitly mentioned by interviewees from three universities, was identification of a *common or shared problem*. In one case the problem became shared when the mathematician and mathematics educator realised that they could help each other solve problems that were initially unrelated:

> *A lot of the stories that X* [mathematics educator] *told me about what she was facing in terms of challenges with her maths students or the people training to be maths teachers caught my attention; stories of students who weren't capable enough when they were out in the classroom as pre-service teachers. So at that point I knew that I had to put in some effort in terms of meeting X's needs. At the same time X was able to put in effort in meeting my needs because we were having challenges in our first year maths classes around tutorial engagement and that sort of thing. X was able to offer some as a sort of mentoring type of role in an action research project where she was the facilitator.* [University A, mathematician]

In other cases, a shared problem was identified when participants recognised that they taught the same pre-service secondary students—"*You teach the students maths and I teach them education, we should at least be sharing what we know about the students*" (University B, mathematics educator).

A striking *hindrance* to interdisciplinary collaboration, mentioned by interviewees from four universities, was the *physical separation* of the buildings where mathematicians and mathematics educators worked. In one university these disciplines were located on separate campuses, and at the other universities the disciplines were typically on opposite sides of the same campus:

> *We are at polar ends of the campus. There's a big gully in between and there is a bridge. So we've got our metaphorical bridge. We alternate weekly meetings between the math and stat side and the education side. So we're walking over to the other side or the other side is coming to us.* [University C, mathematician]

A further structural hindrance identified by interviewees in four universities was embodied by *workload formulas* or *financial models* that did not recognise or reward interdisciplinary collaboration:

> *It's very difficult to get things like what we do [design and teach with a mathematics educator a course on mathematical knowledge for teachers] to be recognised in workload models. We do a lot of things under the radar but we don't actually get acknowledged on our workload. So in a sense we're doing extra stuff.* [University A, mathematician]

Despite respectful relationships having been established between the mathematician-mathematics educator pairs who participated in the project, interviewees in three universities referred to entrenched *cultural differences* between the disciplines in their institutions as hindrances to broader collaboration. More often than not, interviewees expressed frustration with the culture of their own discipline:

> *It annoyed me when I heard colleagues of mine complain about the other side, the people across the creek. When it came to the science pre-service teachers or the maths pre-service teachers, whatever problems they had, my colleagues blamed the other side.* [University A, mathematics educator]

> *I think my colleagues are free to let me do whatever I want to do, provide that it doesn't impact on their day to day workload and the way they approach what they look to do. So they're very supportive … "but we don't actually care what you're doing".* [University B, mathematician]

LEARNING MECHANISMS AT THE BOUNDARY

The following brief narrative presents a hybrid case constructed from all the interviews. The purpose is to illustrate what *transformation* can look like as a mechanism for learning at the boundary between disciplines (names are pseudonyms).

A mathematician (Carol) was working with a mathematics educator (Tess). Before the IMSITE project began, they got to know each other via an externally funded teaching and learning project. Carol was then allocated to the teaching of a first-year mathematics subject for pre-service teacher education students. She was surprised by students' apparent lack of mathematical knowledge after having completed 12 years of schooling:

> *I was lamenting, "Oh my goodness me, I can't believe they don't know any maths", like they know less that I had anticipated for someone who had come through the Australian schooling system.* [Carol, mathematician]

This experience represents a confrontation, a kind of discontinuity between the two worlds of school mathematics and university mathematics that prompted Carol to reconsider her current practice as a teacher of university mathematics. Recognising this confrontation led both to explore each other's worlds:

> *I learned a lot about how education works and Tess learned a lot about how we function. We broke down some of the scepticism that both sides can have.* [Carol, mathematician]

Carol discussed her observations with Tess, who was sympathetic and interested in exploring the differences between teaching mathematics and education in a university environment. Tess remembered "*noticing that my pre-service teachers, their content knowledge was not strong*", and she pointed out to Carol the areas that she wanted her to focus on in the first year mathematics course. Carol acknowledged that "*I was teaching her [Tess's] students at the time*", and both thus recognised a *shared problem space* in which both were contributing to the mathematical preparation of future teachers.

Given this problem space, Carol and Tess were working towards a *hybridisation* of practices from their respective disciplinary contexts. The hybrid result was a new mathematics content subject that is jointly planned and taught, as Tess explained:

> *We're in the class together, one of us leads and the other acts as a sort of sounding board. We planned the weeks so certain weeks are Carol's weeks and certain weeks are my weeks.* [Tess, mathematics educator]

There were encouraging signs that this new hybrid practice would become *crystallised*, or embedded into institutional structures. The teacher education program was under review, and the Heads of Mathematics and Education invited Carol and Tess to design two new mathematics-specific pedagogy subjects for the revised program. The subjects would be 'owned' by Education, with an income sharing arrangement to recognise the teaching contribution from Mathematics.

Despite the success in creating a new hybrid practice, Carol and Tess also *maintained the uniqueness of their established practices* as a mathematician and mathematics educator. Carol acknowledged their complementary expertise when teaching the mathematics subject together:

> *We go to class and there are times when she says to me "That's all yours because it's beyond what I understand" and that's fine. Likewise she'll come in and talk about the greats of education and I'm just going blank, no idea. As an educator it comes out very strongly that she's very well practised.* [Carol, mathematician]

The collaboration was sustained by *continuous joint work at the boundary* between the two practices. This included weekly project meetings, attending and teaching into each other's tutorials in mathematics and mathematics education subjects, joint supervision of Honours students, and jointly conducted professional development for practising teachers.

CONTRIBUTIONS TO KNOWLEDGE

Internationally it is rare to find research or teaching collaborations between mathematicians and mathematics educators. This study has shed some light on how such collaborations can work and what their outcomes might be. It contributes new insights in three ways. First, it provides an account of interdisciplinary boundary practices, proposed by Wenger (1998) as a means of stimulating long term connections between communities of practice. Second, it offers an evidence based classification of conditions that enable or hinder sustained collaboration across disciplinary boundaries. Third, it begins to develop an empirical grounding for Akkerman and Bakker's (2011) conceptualisation of learning mechanisms at the boundary between communities.

EPILOGUE

The IMSITE project had some unique features that, in my view, contributed to its success but are rarely mentioned in research reports. So let me highlight them now. It was the only one of the five ETMST projects to have co-leaders, one a mathematician and the other a mathematics educator, thus exemplifying the kind of interdisciplinary collaboration espoused in the ETMST program goals. It was also the only project to state an explicit aim of fostering interdisciplinary collaboration. We explained in our project proposal that our vision for pre-service teacher education involved more than simply creating new courses. Instead, we aimed for learning experiences that *cannot* be designed or implemented without the combined expertise of discipline and education academics. The third unique feature involved my approach to managing the project budget. Instead of keeping all or most of the funding in my own university, I made an annual allocation of funds to each of the participating universities to be used in the pursuit of project aims. This approach was in keeping with our project's emphasis on diversity: no single model of initial teacher education that privileged one structure for degree programs, one way of combining content and pedagogy, or one form of collaboration between discipline and education professionals was promoted. Instead, each university explored models to suit their own circumstances and adapted models borrowed from partner institutions.

It is becoming more common for governments to seek evidence of the impact of the projects they fund. For example, the Australian Research Council (2015) defines research impact as "the contribution that research makes to the economy, society, environment or culture, beyond the contribution to academic research" (para. 7). A research impact pathway typically documents potential outcomes such as implementation of programs and integration into policy, and these types of outcomes were certainly achieved by the IMSITE project. *Benefits* to society are more difficult to document but were observable in at least two ways. The first arises from the challenge experienced by academics who work at the boundaries between disciplines. As Akkerman and Bakker (2011) point out, these so-called 'brokers' between disciplines can feel like they belong to *both* one world *and* the other, or to *neither* one world *nor* the other. This was a challenge articulated by one of the mathematicians who participated in the IMSITE interviews:

> *I'm seeing myself more and more in between maths and education, caught a little bit in no man's land so I don't belong to either. I'm not unhappy with that because it's been quite an interesting and exciting mind-opening experience, but I do see that the expertise I'm gaining from being involved in the IMSITE project is not necessarily going to get my career furthered in terms of being a mathematician.*

Yet this mathematician, together with a colleague in one of the other IMSITE universities, successfully applied for academic promotion with a case for research excellence strongly grounded in their IMSITE experience. In other words, research in mathematics education 'crossed the boundary' in helping these mathematicians achieve career advancement.

I would like to think that a second important benefit to society of my work on the IMSITE project comes from sharing what we learned with diverse audiences, and especially audiences comprised of mathematicians and mathematics educators—as was the case at the CMESG conference. It was clear from my conversations with conference participants and the post-plenary discussion that the ideas I presented resonate strongly with the organisation's aims and practices. I look forward to seeing how these ideas are taken up and worked with by CMESG members in future conferences.

ACKNOWLEDGEMENT

This paper, and my plenary presentation, draws on an article that was published in the *Mathematics Education Research Journal* (Goos & Bennison, 2018).

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Working Groups

Groupes de travail

THE 21ST CENTURY SECONDARY SCHOOL MATHEMATICS CLASSROOM

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INTRODUCTION

A lot has been written about the '21st century mathematics curriculum', so much that most of us do not want hear about '21st century' any more. But the fact remains that some significant changes in mathematics education are due. Indeed they are perhaps 'overdue' as calls for change have been loud and clear for the past 100 years (Dewey 1938; Papert, 1980; Whitehead, 1929). Perhaps the problem is that there have been many such calls, often with conflicting priorities, and these have considerably muddied the waters (Kilpatrick, 1997).

Can we reasonably hope that this Working Group can make a difference when there is so much confusing history? Our answer is yes. This is a time of change at an unprecedented speed: change that requires not only new knowledge, but new ways of thinking and tackling problems. And it is increasingly realized that mathematics is the engine behind this change.

In our planning, one thing we were clear about is that in order to think about the mathematics curriculum, we needed to spend an hour each day doing mathematics. The problems we chose to work with were designed to point to these new ways of thinking and working with students in order to trigger discussions and reflections.

Days 1 and 3 were spent working on a challenging problem at (and pushing the boundaries of) the senior secondary school level. On day 1, the participants were given the problem in an algebraic form and used an algebraic analysis; on day 3, the participants were given a graphical form and used a graphical analysis. The emphasis here was on mathematical thinking in the analysis of a sophisticated structure. The day 2 problems were of a different kind: they were short elementary questions that yielded to a number of different approaches, and we were interested in generating a discussion around the relationships among the various approaches and the mathematical thinking that emerged from those. In fact, we wound up looking at only one of the problems. Unexpectedly, it had many avenues of resolution and generated a rich discussion. Working with the problem reminded us that much mathematical thinking can emerge from simple situations. Our discussion of these problems can be found at the end of this report.

SUMMARY OF THE DISCUSSION.

Over the three days, the discussion ranged over many topics. But there did seem to be some dominant themes we kept returning to. We have organized our comments under these various themes.

Curriculum, and problems

Hardly anything was said about content in the narrow sense, that is, should we put increased emphasis on functions or on geometry or on analysis of data or on discrete processes, etc. Rather we talked about the nature/style/atmosphere of the curriculum, covering many ideas.

One of these is that a focus on problem-solving and the choice and nature of the problems played a key role. Arguments were made for *open-ended* problems, with potential to be solved in different directions, and to be worked at *different levels* of knowledge and ability (what some call 'low floor, high ceiling') to lead student into a *structurally rich* exploration. Other arguments were about how important it was for problems to be *inviting* and foster play in solvers and teachers alike, facilitating *collaborative activity*. Thus, context is important, where problems can have *meaning* in the lives of the student and teachers, grasping along aspects of *affective* domain.

Together these make a tough set of requirements, and there are very few problems around that will exhibit all or even most of these. But they can act as key guiding principles.

It was clear to most that these objectives for problems could not be attained without a reduction in the amount of content to be covered. In fact, there is a better way to state this: there is a need to let go of the 'laundry list' of topics to be covered, and focus instead on the quality of the mathematical thinking. There was a felt sense that with a reasonable mixture of problems, the list of topics would usually look after itself… hence the 'letting go'! This mantra of less emphasis on content (i.e., not less content worked on, but less 'covering' of content) has been voiced at CMESG meetings since the very beginning in 1977. A notable milestone in this was the 2003 working group, "A mathematics curriculum manifesto", led by Brent Davis and Walter Whiteley (2004), which lamented both the amount of content and the tendency to over-prescribe its coverage. Given the present accelerated speed of change, those objectives seem even more important today.

Where are we to look to find problems that focus on mathematical thinking? An important and unexpected answer to this question is to look at the field of mathematics itself, drawing from the mathematics that mathematicians do. It is interesting that while this reasonable policy is certainly adopted in other disciplines (e.g., in English; Boaler, 2016), it is at times considered that the mathematics of mathematicians is beyond the capacity of high school students. However, as many insisted, if we attempt to take apart a piece of 'real' mathematics, it is surprising how often one or two high-school gems will appear. Of course, to adapt these gems to the high-school setting, there will be some design to be done. Ironically, the only danger in that process is to underestimate the caliber of the students and their readiness to address these:

students might not always learn what we want them to learn, but they surely can do lots of mathematics! A focus on problems embraces students' mathematical activity.

Expectations, or what we want of our students

What qualities, skills and characteristics do we want to develop in our students? Of course there is a fashionable list of adjectives that appears tireless in the literature—creative, innovative, curious, interactive (capacity to collaborate), and so on. Apart from these, we talked about mathematical thinking, the capacity to analyze a complex abstract structure, to take it apart and put it back together, perhaps in a somewhat different way. That is an important word, 'abstract', but along with that the student has to translate mathematics into concrete terms, to work in meaning: hence, the core of mathematics.

In a somewhat different direction, we need students who can master technical routines, who understand the importance of repetition in the development of automaticity. In another direction, we have communication skills, capacity to write clearly and effectively. In yet another direction, the capacity to convince, to know when an argument is coherent, logically sound, and to be able to construct such arguments, simply and clearly. Indeed it was observed that most kids do not write enough, or more precisely, that most of the writing they do is abbreviated, chopped up, hamburger-like text.

In another set of directions, arguments were raised to the importance of having students deal with uncertainty, be prepared to take risks (and here it is mostly about intellectual or academic risks) and be comfortable with, and even embrace, diversity. In sum, the student described, 'aimed at', is a student engaged in a meaning-making practice.

The curriculum document

What would the curriculum document look like for the 21st century? What would have to be hand(l)ed to the teacher? While less prescribed, the document would be filled with clear process objectives to give the teacher more freedom and thereby, more responsibility in which to work. All this would be supported with rich resources and more collegial time to ponder on those.

We discussed assessment in relation with it being more open and more flexible. Having said that, it is understood to be a challenge developing an assessment that aligns with an open curriculum of this nature. It was suggested that this is a time to do 'crazy things' with assessment. In line with this, someone asked, "Do we even need grades?" Of course this fits in with the tenor of the conversation as a whole—the development of the 21st century curriculum that we are imagining wanted to begin with a number of 'crazy ideas', or at least be inspired by some.

The role of the teacher

The nature of the problems called for, and the sort of curriculum talked about, inevitably put both teacher and student in unfamiliar situations, and in a sense that erases some of the hierarchy such that student and teacher can be more of a 'team' tackling a problem together. In this configuration, the teacher needs enough experience with the problem to know how much to give the student and how to replace an answer with a leading question but, as well as, being ready to interact with unforeseen ideas or a different management strategy than what could be expected. For teachers to be comfortable in this role, they have to accept that having students do mathematics implies that they will have themselves to do mathematics, to accept that not all is scripted, and that lots of mathematics also happens in the making (see also Barabé & Proulx, 2017, for more on these ideas concerning teaching).

Tertiary mathematics education.

To a large extent, we are shaped as teachers by our own learning experience, and certainly teachers' previous undergraduate experience will have a significant effect on how they view the subject of mathematics as well as on their view of how it should be taught. The views of most in the room were that current undergraduate mathematics courses do not, for the most part, provide a good model for the 21st century secondary mathematics classroom. Most universities have undergraduate offerings aimed at future teachers, but these courses are only a small part of the total load. It is a bit hard to put our finger on what exactly is not quite right about most university math courses, but as a start we might say that the emphasis seems also here to be on mathematical knowledge rather than mathematical thinking. Of course both were said to be needed, but the first without the second appeared of little use.

Having said that, the content of the tertiary mathematics curriculum certainly does have some of the desired features of our secondary curriculum. It works with problems that are structurally rich, often inviting, open-ended and playful (inspiring play for those who know how to play). That is, the raw material of the undergraduate curriculum is of the quality we are looking for. But the pedagogy often does not measure up to this standard. We definitely need to work on that, not only for the sake of our teachers, but for all undergraduates: this is also part of the 21st century curriculum.

THE PROBLEMS

We used a small-group approach: 4 or 5 sitting around a table sharing insights. On each of the three days, the problems seemed to generate a lively discussion.

PROBLEM FOR DAY 1: THE PARENT GAME—THE PAYOFF FORMULA

Consider the following 2-person game. Each player makes an investment which is a number x $(0 \leq x \geq 1)$, thought of as the player's contribution to the common good. The net payoff to a player playing x whose partner contributes y has the form $P(x, y) = B(x + y) - C(x)$ where B is the benefit to the player and depends on the sum of the two contributions, and C is the cost to the player and depends only on his own contribution. We suppose that these functions are given as

$$B(z) = x(4 - z)$$

$$C(x) = x^2$$

and these are graphed in Figure 1.

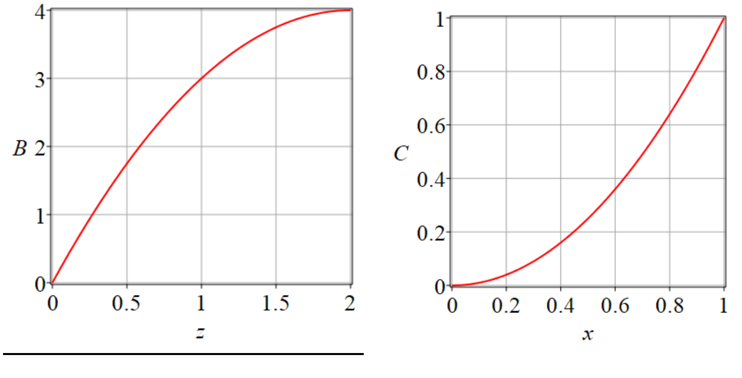


Figure 1.

Here we consider an asymmetric situation in which the two players are a female A who plays a and a male B who plays b, and furthermore we assume that A has to go first. That is, A chooses her contribution a before B chooses his, so that when B chooses b, *he knows the value of* a. The question we are interested in is how each player should play to maximize his or her own payoff (with no regard for the payoff to the partner).

Just to emphasize:

A's payoff:

$$P(a,b) = B(a+b) - C(a) = (a+b)(4-a-b) - a^2$$

B's payoff:

$$P(b,a) = B(a+b) - C(b) = (a+b)(4-a-b) - b^2$$

> Here's an interesting question—who has the advantage? Who will get the higher payoff, A who goes first, or B who goes last?

PROBLEM FOR DAY 3: THE PARENT GAME—THE PAYOFF GRAPH

In this version of the game, we work not with the *formula* of the payoff function, but with its *graph*, and we look for a *graphical* rather than an *algebraic* solution. Recall that the payoff function $P(x,y)$ is the payoff to a player investing x whose partner invests y. This is a function of two variables, so its graph is a surface in three dimensions and is in fact drawn in Figure 2.

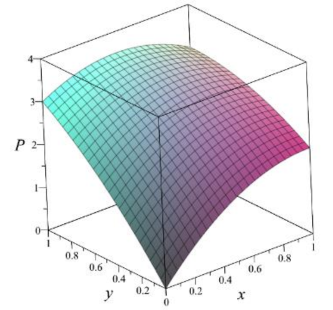


Figure 2.

Now it is certainly not easy to work with a 3-D graph drawn on a 2-D piece of paper. Fortunately there is an excellent 2-D representation for such surfaces: commonly found in topographical maps, known as a contour diagram. The contours here are the level curves of the function $P(x,y)$, curves along which P has a constant value. The curves in Figure 3 are spaced at intervals of 0.1.

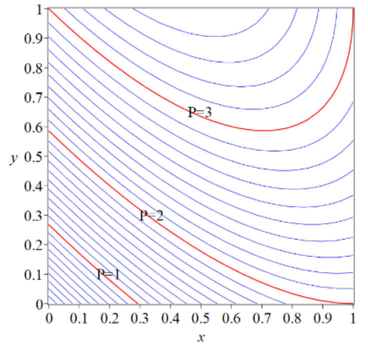


Figure 3.

Same situation as above—Two players A and B, and A plays first, so that when B chooses his investment b, *he knows the value of A's investment* a. How should each player play to maximize his or her own payoff?

PROBLEM DISCUSSION OF DAYS 1 AND 3

The problems interact with an advanced topic, often not encountered until university. This is a deliberate practice for the math 9-12 projects (Taylor, 2016). To ensure quality, the problems come from "the mathematics of mathematicians" (Taylor, 2018) and then they are morphed to work as 'low-floor, high-ceiling' activities at the secondary level (Boaler, 2016; Gadanidis, Borba, Hughes, & Lacerda, 2016). As a result there was considerable variation in the room in the extent to which the participants were familiar with the concepts and techniques. This variation of course was responsible for much of the discussion.

In terms of particular content, it was noted that these problems work with functions of two variables, and the functions that are studied at the secondary level are almost exclusively of a single variable. There are however a number of reasons that our students (and teachers!) would benefit greatly from the introduction of two-variable functions in the high school curriculum:

- they are interesting with scope for rich discussion and interaction;
- they enhance our understanding of slope and rate of change;
- they work in 3-dimensions and develop our spatial capacities;
- many students have trouble meeting them for the first time in university; and
- professional advancement—they offer rich challenges for many teachers.

In the discussion, the participants asked to what extent both the problem and their own thinking processes relate to our vision of the 21st century curriculum. Certainly the problem itself was somewhat sophisticated and required sustained thought and the asking of clarifying questions; for example, did the players care about the size of their partner's payoff? Also the handling of the asymmetry required clear and careful thought. A number of comments referred to the relationship between the algebraic and the graphical analysis. For most of the time, the analysis we used seemed quite different in the two cases. Having said that, there was excitement when correspondences between the two approaches were revealed. Most, but not all, found the graphical approach more challenging. There might be some right-brain, left-brain divide here.

In the two months since the working group, these problems, and closely related variants of them, have been workshopped in senior (Grades 11 and 12) high school classrooms in Ottawa and Toronto, and also in the 2018 Kingston SHAD camp. In all cases the students surprised us with their capacity and their success in analyzing the problems. However these were not research interventions, and no attempt was made to measure how successful the students were. This work remains to be done.

Solutions to these problems are found in Taylor (2016).

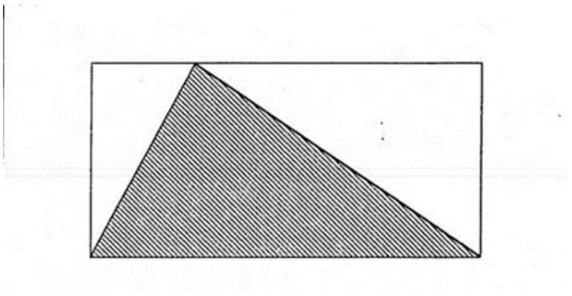
PROBLEM FOR DAY 2

The problems for day 2 were mental mathematics problems, meaning that each participant had to solve mentally, that is, without paper-and-pencil or any other material aid, a problem shown at the board. The activity was designed following J. Proulx's work on mental mathematics (e.g., 2015, 2017), following a structure of this sort (led by J. Proulx, taking the role of 'teacher'):

4. A task is shown on the board where oral instructions are given for the task to be solved.
5. Participants listen and then have about 20 seconds to think about their solutions.

6. At the signal, participants are asked to share their answer and explain their strategies to the entire group (and in some cases to come to the front to illustrate on the board).
7. If answers are given uniquely orally, the 'teacher' writes them on the board (and in many cases, explains them again).
8. The solutions given are treated as occasions for justification, and other participants are invited to question or intervene in some of the solutions if they are not fully convinced or do not understand what is offered.
9. The 'teacher' also invites other participants who may have solved differently (or who have thought of other ways of solving) to share their ideas and solutions.
10. The various solutions are compared if possible and discussed by the 'teacher' and participants with regard to their effectiveness, links, efficiency, advantages/inconveniences, possibilities for other problems, etc.
11. When all is completed for one task, another one is given and the cycle restarts.

Because of time constraints, and the fact that the interaction and sharing of strategies for the first problem lasted for about 30 minutes, only one problem was worked on. The task worked on is the following (taken from Clapponi, 1991-92):



Is the shaded part half of the rectangle?

Discussion then followed about the experience lived, but also for addressing the simple question: "What is the value of this sort of problem? And, is this sort of problem adequate for the mathematics curriculum of the 21st century?"

ACKNOWLEDGEMENTS

The members of this working group were an amazing resource for us both, and their enthusiasm and focus contributed hugely to the success of the meeting. A huge shout-out goes to Ann Arden for her careful and thorough note and image taking throughout the three-day working group. Without this resource, this report could hardly have been written. Finally, we are most grateful to the scientific committee for their invitation to lead this working group.

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CONFRONTING COLONIALISM

AFFRONTER LE COLONIALISME

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AN OPENING REFLECTION

Group: I'm afraid to start. **Old Woman:** It's emotional work. **Group:** I don't know where to start. There are no resources. **Old Woman:** There are resources all around you. **Group:** So what are they? I need the kit. **Old Woman:** There's a multitude of stories in this place. Listen.	**Groupe:** J'ai peur de commencer. **Vieille femme:** C'est un travail émotionnel. **Groupe:** Je ne sais pas par où commencer. Il n'y a pas de ressources. **Vieille femme:** Il y a des ressources tout autour de vous. **Groupe:** Alors, quelles sont-elles? J'ai besoin du materiel. **Vieille femme:** Il y a une multitude d'histoires dans cet endroit. Écoutez.

The dialogue above is modelled on similar dialogues written as meditations in the book Embers by the late Anishinabek (Ojibwe) author, Richard Wagamese, 2016. Our group found inspiration from Wagamese's words as we engaged in our discussions throughout the three days, so it seemed fitting to share our learning in a similar format. We are grateful to Cynthia Nicol and Gladys Sterenberg who helped us to construct this dialogue. In this report, we will weave in Wagamese's words as we recount our experiences as a working group focused on decolonizing mathematics education. We remember that "We live because everything else does" (Wagamese, 2016, p. 37).

BEGINNING IN A GOOD WAY

We began our session by inviting Edward Doolittle to help us open in a good way. Through prayer and thanksgiving, Edward invited us into a space of respectful sharing.

To enter into conversation, we invited everyone to introduce themselves with a particular focus on positioning ourselves in relation to the land, to Indigenous peoples, and to colonialism. Participants were asked to consider the following questions as they introduced themselves. Where are you from? What territory do you live in? How are you connected to that territory? Why have you come to this working group? What do you hope to do with what you learn in this group? These questions invited reflection upon complex ideas and resulted in complex positionings. While the majority of the group are settlers in this country we now call Canada, we each have had very different experiences in coming to this territory and living in this territory. Some of us have long histories on this land, and some are relative newcomers.

One recurring reflection during this time was that "*I live on their land*". It was a difficult place for many of us, recognizing that we lived on stolen land, but struggling with how to "*give it back*" without losing our place to live. This, in turn, lead some of us to struggle with the realization that "*I don't know who I am*", and where 'I' fit in and contribute to the continued colonization of Turtle Island. Some of us knew a great deal about our territory and the history of the First Peoples of that land, but most of us did not. Many of us were at a loss of where to begin learning. Many of us spoke of our own intimate relationship with the land alongside our ignorance regarding what the Indigenous names for the land we are so connected to are or mean.

Some of us had close connections to residential schools, having grown up near them, known people who had attended them, had relatives who taught at those schools, and had relatives who were missionaries on Turtle Island. Those relatives were described in the communities they worked in as respectful, yet as colonizers, it was recognized that their 'good' work was grounded in colonial beliefs and ideals.

Likewise, we came to this group, this room, this community for a variety of reasons. Some of us seeking understanding of what does decolonization mean and look like for mathematics and the teaching and learning of mathematics. Others of us were seeking a place to start, a way to engage and share and to motivate others to do likewise. All of us came with experiences or stories that told us something was not right and with the bigger question of how can we reimagine mathematics and the teaching and learning of it; how can we decolonize?

We recognized that decolonization meant looking below the surface for the reasons that things are they way they are, and on a personal level, why 'I' believe, know, and do what 'I' do. We agonized over how to decolonize our own thinking, and how to discover our own biases and colonial beliefs.

We also recognized that 'waiting for the right time' was not a valid excuse for inaction. Now is the right time, yesterday was the right time, and tomorrow continues to be the right time.

SHAPING OUR UNDERSTANDINGS OF COLONIALISM

After break on the first day, we wanted to think deeply about what we mean when we say we are 'decolonizing' mathematics and/or mathematics education. To that end, we determined that we must begin by understanding what colonialism means and how it operates within our settler state, and subsequently within mathematics and mathematics education. Colonialism can play out in different ways, but here in the territories we now call Canada, it is settler colonialism that is operating on a daily basis. The Truth and Reconciliation Commission of Canada (2015) has explained that residential schools were a key part of implementing settler colonialism as these institutions were built to erase Indigenous Peoples from the Canadian landscape and destroy their relationship to the land. The TRC referred to these practices as cultural genocide, explaining:

> Cultural genocide is the destruction of those structures and practices that allow the group to continue as a group. States that engage in cultural genocide set out to destroy the political and social institutions of the targeted group. Land is seized, and populations are forcibly transferred and their movement is restricted. Languages are banned. Spiritual leaders are persecuted, spiritual practices are forbidden, and objects of spiritual value are confiscated and destroyed. And, most significantly to the issue at hand, families are disrupted to prevent the transmission of cultural values and identity from one generation to the next. (Truth and Reconciliation Commission, of Canada, 2015, p. 1)

The ideologies that bolstered this attempted cultural genocide continue to permeate public discourse today. Decolonizing education requires us to confront these ideologies.

To begin the second part of our discussions, we began with definitions of settler colonialism (Tuck & Yang, 2012) and Cognitive Imperialism (Battiste, 2005). These definitions are provided next to give the reader a sense of the introduction to the next section of our work together.

Settler colonialism:

> Settler colonialism is different from other forms of colonialism in that settlers come with the intention of making a new home on the land, a homemaking that insists on settler sovereignty over all things in their new domain. Thus, relying solely on postcolonial literatures or theories of coloniality that ignore settler colonialism will not help to envision the shape that decolonization must take in settler colonial contexts. Within settler colonialism, the most important concern is land/water/air/subterranean earth (land, for shorthand, in this article.) Land is what is most valuable, contested, required. This is both because the settlers make Indigenous land their new home and source of capital, and also because the disruption of Indigenous relationships to land represents a profound epistemic, ontological, cosmological violence. This violence is not temporally contained in the arrival of the settler but is reasserted each day of occupation. This is why Patrick Wolfe (1999) emphasizes that settler colonialism is a structure and not an event. In the process of settler colonialism, land is remade into property and human relationships to land are restricted to the relationship of the owner to his property. Epistemological, ontological, and cosmological relationships to land are interred, indeed made pre-modern and backward. Made savage.
>
> In order for the settlers to make a place their home, they must destroy and disappear the Indigenous peoples that live there. Indigenous peoples are those who have creation stories, not colonization stories, about how we/they came to be in a particular place—indeed how we/they came to be a place. Our/their relationships to land comprise our/their epistemologies, ontologies, and cosmologies. For the settlers, Indigenous peoples are in the way and, in the destruction of Indigenous peoples, Indigenous communities, and over time and through law and policy, Indigenous peoples' claims to land under settler regimes, land is recast as property and as a resource. Indigenous peoples must be erased, must be made into ghosts (Tuck and Ree, forthcoming). (Tuck & Yang, 2012, p. 5-6)

Cognitive Imperialism:

> Cognitive imperialism is a form of cognitive manipulation used to disclaim other knowledge bases and values. Validated through one's knowledge base and empowered through public education, it has been the means by which whole groups of people have been denied existence and have had their wealth confiscated. Cognitive imperialism denies people their language and cultural integrity by maintaining the legitimacy of only one language, one culture, and one frame of reference.
>
> As a result of cognitive imperialism, cultural minorities have been led to believe that their poverty and impotence is a result of their race. The modern solution to their despair has been to describe this causal connection in numerous reports. The gift of modern knowledge has been the ideology of oppression, which negates the process of knowledge as a process of inquiry to explore new solutions. This ideology seeks to change the consciousness of the oppressed, not change the situation that oppressed them. (Battiste, 2005, p. 12)

The larger working group split off into slightly smaller groups and found their own space and ways of discussing these two terms and how they relate to mathematics and the teaching and learning of mathematics. In particular, the groups were asked to consider, "*Are the definitions helpful in thinking and working toward decolonizing?*"

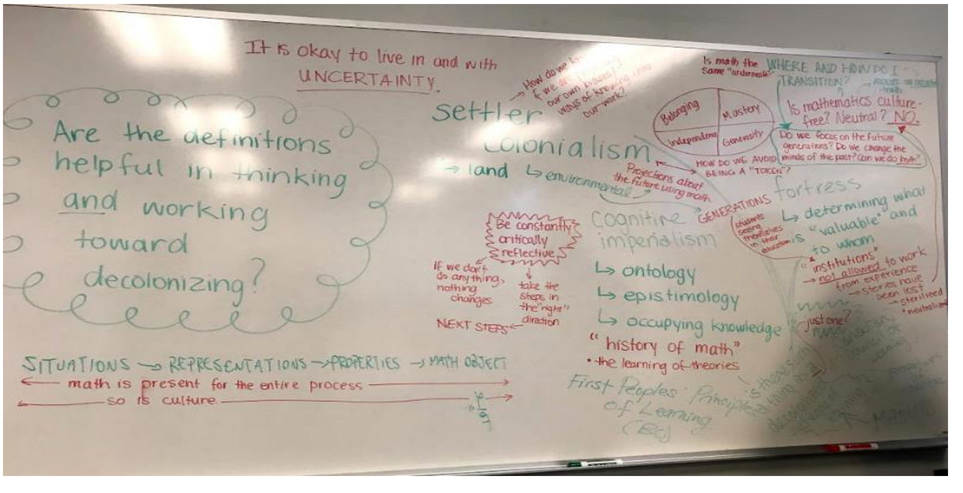

Figure 1. Whiteboard from group discussions.

Something that we experienced and learned many times during our work together was how, through dialogue and reflection, we were able to reveal successive layers of colonial thinking that impacts what we recognize and value as mathematics. As an example, nearing the end of the discussion by one of the groups (as determined by time, and not interest or engagement), the following progression of abstraction commonly associated with the term mathematization was recorded on the whiteboard (See Figure 1):

Situation → Representation → Properties → Math Object

As the group members contemplated what had been written, and discussed the relevance and importance of this progression, an insight began to emerge: math exists in all four of these stages, and not just at the end of the progression.

Situation → Representation → Properties → Math Object

← math is present in the entire process →

A deliberate discussion then ensued within this group, as a layer of colonial thought tied to singularity and correctness was peeled back to reveal a second truth of value. Discussion ensued about how, in mathematics classrooms and in society, this layer of thinking and understanding of mathematics most often remains hidden—an act of unconscious colonialism.

As the groups received their 'five minute' call to wrap up their discussion, this group took one final step in peeling back the layers of colonialism in their thinking about mathematics and mathematization as a final line was added:

Situation → Representation → Properties → Math Object
← math is present in the entire process →
← so is culture →

At first, this latest addition was contemplated in silence. Then, as the group packed up and started to return to our main meeting space, the discussion grew, recognizing that the math that lies unrecognized throughout the process of (Western, Academic, Abstract) mathematization was not just some abstract or isolated mathematics, but that it was inherently mathematics that is embedded with culture, in culture, and for culture. Over the course of their discussion, this group had acted in such a way as to "Be constantly critically reflective" and learning "to live in and with UNCERTAINTY", two ideas that had been proposed much earlier in their discussion of settler colonialism and cultural imperialism.

Another of the groups had a sparse recording of ideas that they had come to; however, the depth of what was written reflects the intensity and depth of their powerful conversations about the settler colonial and cultural imperialism nature of mathematics and how we teach and learn it. In this group, they came to question the prevalence of dichotomy within mathematics and the teaching and learning of mathematics, asking instead "*Is there another side?*" This simple yet powerful question asks us all to question our colonial positionings of right and wrong, black and white, good and evil. Can there be more than one right way and one or more wrong ways in mathematics? Can we trouble this notion of dichotomy and fixedness within our mathematics classrooms? Can "the math [belong] to whomever is doing it?" Rather than using "math to weed students out" can math be used to bring students in? Can students be part of doing all of these things?

Some of the groups also reflected upon the metaphor of the fortress that had also been briefly presented regarding school mathematics, the notion that mathematics as a subject and discipline is a heavily fortified and strongly constructed entity into which "those who can" may enter in, and "those who can't" remain outside the fortified walls and all of the opportunities that lie within. The question first pondered was "how do we make it so that all students can get into the fortress", which was seen as a problematic question because it is, in the nature of Western mathematics, that this fortress is accessible to only the very few (see the previous discussion of weeding students out). Thus, the group decided to ask a different question. Instead of trying to change the fortress in some 'acceptable' way for all students to be able to enter, why not build mathematics outside the fortress, and build it in such a way that those within the fortress may come out, but no one will have the power and authority once there to block others' entry?

One key idea that was beginning to emerge by the end of day 1 is that colonialism is learned and it can be unlearned. We ended our day with the following words from Richard Wagamese (2016):

> OLD WOMAN: You can't fix what's in your head using what's in your head.
> ME: What do I do, then?
> OLD WOMAN: Unlearn.
> ME: How do I do that?

> OLD WOMAN: Choose differently. (p. 118)

ENGAGING WITH DECOLONIZING PRACTICES

An opening reflection to begin our time together on Day 2:

> I've been considering the phrase "all my relations" for some time now. It's hugely important. It's our saving grace in the end. It points to the truth that we are all related, that we are all connected, that we all belong to each other. The most important word is "all." Not just those who look like me, sing like me, dance like me, speak like me, pray like me or behave like me. ALL my relations. That means every person, just as it means every rock, mineral, blade of grass, and creature. We live because everything else does. If we were to choose collectively to live that teaching, the energy of our change of consciousness would heal each of us—and heal the planet. (Wagamese, 2016, p. 36)

We began our second day by listening to the words of Aaron Prosper, the first Indigenous student union president of Dalhousie University, as he spoke about his experiences as a student participating in Show Me Your Math (SMYM). Aaron's story was recorded at a session he gave at a People for Education Conference in Toronto (https://youtu.be/YdvDkIbmOc4), and we soon realized that engaging in SMYM had helped Aaron learn a whole lot more than math.

Aaron shared his experience of studying the game of waltes and talked about the things he learned alongside his grandparents while playing the game. Waltes is a game played with a bowl and six dice, which are rounded on one side and flat on the other side. To play, you bang the bowl down, usually on a blanket, and try to get the dice to be 5 or 6 the same so you can score points and collect sticks. Aaron explained that the game has different rounds, and it can go quickly, or it can take a very long time to play. But Aaron learned more than just how to play the game; he heard from elders who shared with him stories of sitting on the front step of the house watching for Indian Agents while their parents played waltes inside. It is important to note that until the 1950s it was actually illegal for Indigenous peoples in Canada to gather and hold ceremony according to a clause in the Indian Act known as the Potlatch ban (Joseph, 2018). Thus children sat on porch steps watching for the dust of the car driving on the dirt roads knowing this would likely be the Indian agent.

Aaron noted that the waltes bowl was also used in a now forgotten traditional ceremony that involved filling the bowl with water. As a child, he noticed that his grandparents' waltes bowl had a hole drilled through the middle. He recounted a story of asking his grandmother about the reason for this hole and her response was that the Indian agents had done this, claiming it would make the bowl more aerodynamic. Humoursly, Aaron shared that by this point he had been to university and studied first-year physics and knew this was not factual. Rather, he argued, the holes prevented his ancestors from performing ceremony. Aaron also shared his knowledge of a different game, known as wapanaqan, that is similar to waltes, but today only two of this game remain—both in museums.

Finally Aaron spoke of his experience as a Mi'kmaw student in a Mi'kmaw Kina'matnewey (MK) school where he felt he could develop in a space that honoured his language and culture and worked to break down colonial barriers and hierarchies. He talked about how in this school system he was given the opportunity to learn about his own history, his family, his community, and the experiences of colonialism that have impacted Mi'kmaw life today. This approach to education highlighted the ideas of decolonization and prompted considerable points for discussion in the group.

One idea that stood out in the group was the loss of the wapanaqan game. This game has nearly disappeared, with the only two that still exist being held within Eurocentric institutions and not in the community. As some participants remarked, regaining and relearning these games is important as games are deeply connected to mathematics. Games can inspire mathematics and as such can be a great source of making mathematical connections within communities. Edward made the argument that we should repatriate many of the games that have been taken by churches and institutions. He also argued that there is a need to focus our efforts on regaining this knowledge rather than dwelling on what has happened, explaining, "*What we have we have gathered, we can regather it*." Edward's comments brought forward the idea that knowledge is not lost, but it may be forgotten, and we need to reconnect with the places and lands from which that knowledge first emerged to regather it. This harkens back to an idea that had been emerging throughout our time together, that the land can teach us many things if we can pay attention and learn from it.

From Aaron's video, we also were reminded of the power of colonialism and how it continues to shape society today. In particular, we were struck by the layers of colonialism at play in Aaron's story. The 'Indian agents' operating on behalf of the colonial government using a Western science physics concept to impose beliefs upon the community. We were struck by how these same ideologies play out today. The notion that laws were in place to keep people from participating in ceremony is an example of a form of coercion. Ceremony is not illegal today but coercion is still at play. The discourse of globalisation imposes western mathematics and holds it up as superior. As one participant remarked, "*Colonialism is within us and we should be respectful of that as we are challenged to do and to act. You might be making great changes, but we are always going to need to do more.*" We were reminded of the need to be always mindful of how the legacy of colonialism plays out in our taken-for-granted assumptions that have been so heavily influenced by the discourses that shaped this land we now call Canada. Many of us reflected on the need to question these assumptions and notice when what we are doing is being influenced by colonial ideologies.

A third idea that emerged in relation to this video is that language is a powerful resource for decolonizing education because it reflects how you see the world. Aaron's explanation of the verb-based nature of Mi'kmaq reminded the group of the significance of action and process embedded in Indigenous languages. Lisa was able to share some of her own insights in relation to these ideas from work she has done (Lunney Borden, 2011, 2013).

The notions of dichotomy and hierarchy also were prominent in our discussions of the video. Aaron's experiences and learnings from his community and elders made plain the colonial insistence on Western knowledge and ways of knowing as being superior; however, he also uncovered how this hierarchy of dominance was falsely imposed, and even demonstrated how "the master's house can be taken down using the very tools the master has given the slave." So too in mathematics, colonization has come with the valuing of western mathematics above all else, yet within its 'real-life applications' (when we attempt to attach it to place and the land) the potential frailty and limitations of this singular way of knowing quickly becomes evident. In decolonizing mathematics and the teaching and learning of mathematics, can we loosen its sole attachment to the domain of the abstract and let it regrow its chopped off roots in the land, place, and the community?

DAY 2: AFTER BREAK—THE QUILL BOX TASK

> Teachings come from everywhere when you open yourself to them. That's the trick of it, really. Open yourself to everything, and everything opens itself to you. (Wagamese, 2016, p. 58)

After the break on Day 2, Lisa lead the group through a task based on a story of making a quill box originally told to her and David Wagner by the late Dianne Toney, a Mi'kmaw craft person (See Wagner & Lunney Borden, 2015). The activity is attached in Appendix 1.

From this task two important points of discussion emerged: 1) the role of story in mathematics teaching and learning, and 2) the current predominance of pure rather than applied mathematics in school curricula. These both became key ideas as we thought about shaping a decolonized approach to mathematics teaching and learning.

Many participants remarked that stories are important in mathematics. Why is it that we tend not to tell the stories of mathematics? Why do we not let mathematics emerge from stories? This too seems to be a vestige of colonialism, the need to keep mathematics abstract, isolated from place and personal meaning, isolated from story. But stories can be generative in mathematics, and they have the power to humanize mathematics.

Our group discussed how stories of mathematical innovations, even those of Western mathematics, are far too often left out of mathematics teaching and learning experiences. Yet, those of us who have used stories of mathematics in our classrooms, expressed how these stories provided different ways for students to engage with mathematical ideas. The story of Dianne's Quill Box gives an example of community based stories that show the mathematical reasoning of the Mi'kmaw community. Lisa also shared a video of another elder using the "three and a little bit more" strategy to measure a piece of wood to make a frame for a hand drum, highlighting how the knowledge of this relationship is widely used within the community. This prompted us to wonder about other stories that could be generative for decolonizing mathematics. Stories of community may be an appropriate starting point for decolonizing mathematics teaching and learning, and as such, efforts should be made to reclaim some of these stories, while also recognizing appropriate protocols within community.

The mathematics that emerged from the story of the quill boxes led to a conversation about how this knowledge is rooted in a specific practice, an application of mathematical reasoning for a specific purpose. Applied mathematics, it was argued, could provide greater opportunities for making meaningful connections to community and cultural practices. Yet, school mathematics tends to marginalize applied mathematics and privilege instead pure mathematics. We wondered, do we have it wrong? Should we be ensuring a greater focus on applied mathematics in schools? Would this allow more stories to emerge? Could pure mathematics be seen more as an optional pursuit? Could we recognize that pure mathematics as currently taught in school is embedded in a certain philosophical tradition and be upfront about that tradition? Could it be that other philosophical traditions might generate other types of pure mathematics?

While we did not answer these questions we raised, we did begin to articulate an idea that Western mathematics and Indigenous mathematics might live alongside each other, co-existing, each rooted in specific ways of knowing, being, and doing. We recognized that a two-eyed seeing (REF) approach might enable us to see how each body of knowledge exists separately but how the two can also inform one another.

COMING TO THE METAPHOR OF THE TWO TREES

> ME: When are things going to get easier?
> OLD WOMAN: They already are.
> ME: Doesn't feel like it. I keep waiting for Creator to step in.
> OLD WOMAN: She already has. She always will. Keep faith burning in your heart.
> ME: I have. I've been waiting for things to change.

> OLD WOMAN: Faith isn't about waiting for things to change. Faith is the constant effort to keep pushing through.
> ME: What's on the other side?
> OLD WOMAN: You. (Wagamese, 2016, p.134)

Our morning circle on day 3 was fraught with tensions. We had generated so many questions in our first two days. We had grappled with so many ideas. We wondered when it would get easier but found wisdom in the words of Old Woman who reminds us to keep pushing through these tensions. We were mostly settlers in this room, having had various depths of interaction with Indigenous peoples. We chose to be present in this group because we see value in these efforts to decolonize education and we are committed to doing what we can from our own places and spaces, yet we also recognize that this work is hard, deeply emotional, and along the way we must constantly recognize and confront our own colonial assumptions. But this is how we push through. This is how we get to the other side.

Our circle conversation brought us back to the idea that the land can teach us as Malgorzata shared a story of trees she had seen on her excursion. The two trees, different species, grew tall in the same space, intertwined, interdependent, supporting one another to grow strong and healthy. See the photos in Figure 2.

Figure 2. The two trees.

These trees provided our group with a metaphor for our emerging understandings. Western mathematics and Indigenous mathematics could grow together, alongside one another, each supporting and informing the other but both being strong and healthy. As we broke into small groups for the final time, this metaphor was guiding our thinking. The pictures in Figure 3 show some of the ideas that emerged in these discussions.

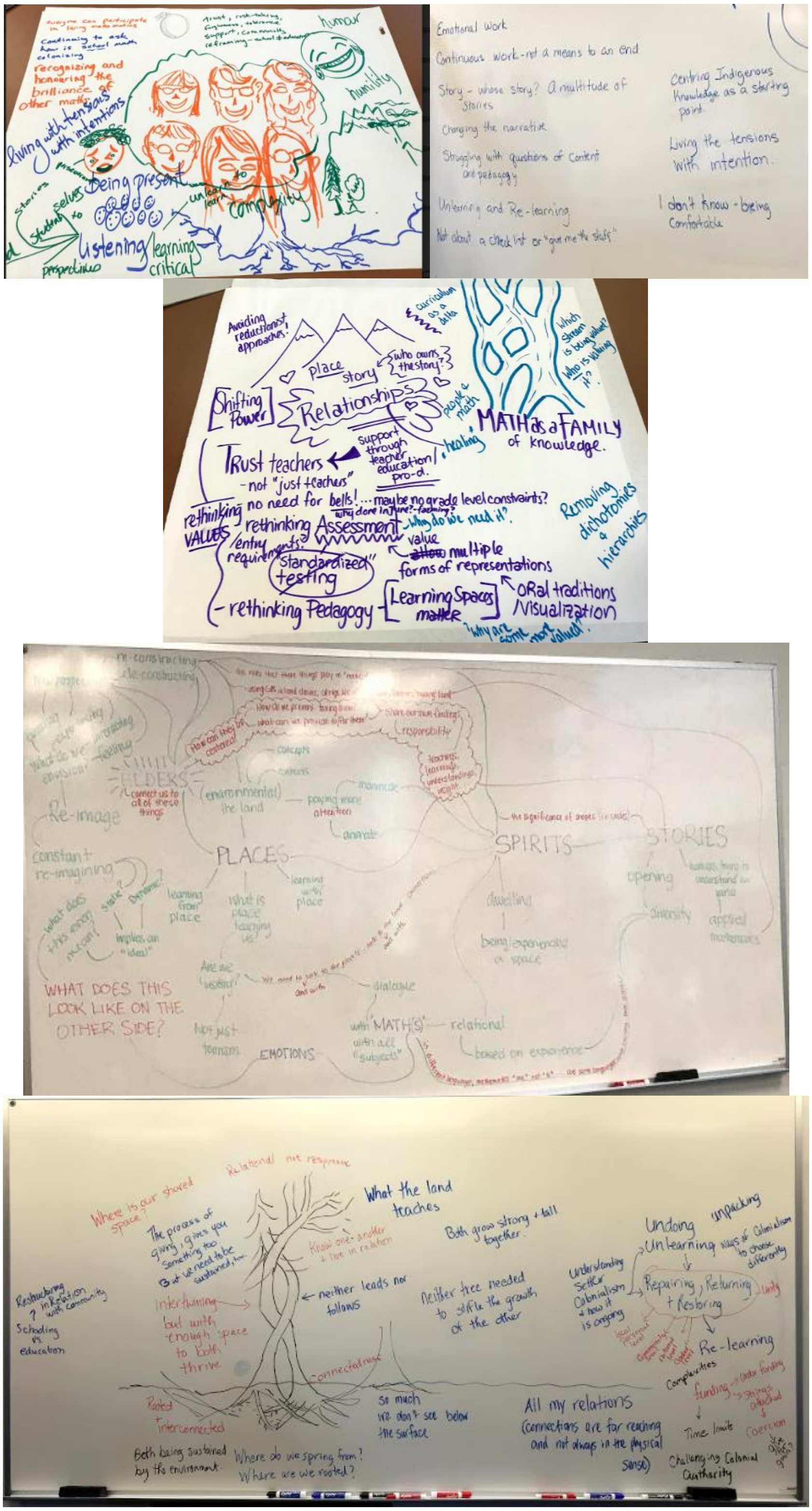

Figure 3. Whiteboards from final small groups.

What emerged were ideas related to learning from place, honouring the stories, and valuing community knowledge systems that are rooted in Indigenous languages. We recognized the need to unlearn colonial ways and relearn new ways of engaging with mathematical ideas. We recognized that a considerable amount of Indigenous knowledges have been suppressed by colonial forces that attempted to eradicate these knowledges, and as such there is a real need to repair, return, and restore (TRC, 2015) these ways of knowing being and doing. We see that we need to remove dichotomies and hierarchies within our existing educational practices and instead build relationships that will allow knowledge systems to thrive alongside each other and learn from each other. We recognize that it is our responsibility as settlers to do the learning and to not place the burden of decolonization on the colonized. We recognize that we must do this work in a good way, in a relational way. This is how we will begin the journey to decolonizing mathematics teaching and learning at all levels.

We recognize that our stories need to change: we can no longer privilege only the Western ways of mathematical thinking. We also recognize that this work is filled with tensions and we must learn to live in the tensions for they are generative. Tensions cause us to feel uncomfortable and this discomfort forces us to shift how we do things. When we recognize the colonial narratives at play, we can choose to act differently, to do things differently, to honour community knowledges and pay attention to what this land is teaching us. We need to be intentional about these changes. Thus, we end as we began, in dialogue with Old Woman and recognizing the need to go forward with hopefulness.

Old woman: Are you changing the narrative? **Group**: We are living the tensions. **Old Woman**: With intention.	**Vieille femme**: Changez-vous le récit? **Groupe**: Nous vivons les tensions. **Vieille femme**: Avec intention.

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APPENDIX 1

DIANNE'S QUILL BOXES

The late Dianne Toney was a Mi'kmaw elder who made quill boxes like the one in the picture. She always made them by starting with a circle of bark for the top on which she made her pattern. She would always start at the centre of the circle to make her pattern. After that she would make the ring for the top from strips of wood. To ensure the ring was the right size, Dianne said she would measure three times across the circular top and add a thumb. She claimed this would make a perfect ring every time.

WHY DOES THIS WORK?

It is time for you to investigate some circles to determine if three and a thumb is a good measure for a ring to fit around the circle. Why do you suppose this always worked?

Your task:

- Draw some circles of different sizes using the bullseye compass.
- Use the rolls of paper to make rings by measuring three times across and adding a thumb.
- Do your rings fit your circles? Will they always fit?
- Use a ruler to measure the diameter of each circle and record this in a table.
- Use string to measure the exact distance around your circle (the circumference) and record this as well.

EXPLORING RATIOS

Create a table to show the ratio of the Circumference to the diameter like the one below:

Circumference (cm)	Diameter (cm)	$\frac{\text{Circumference}}{\text{Diameter}}$

What do you notice about the ratio in column three? How does this relate to three and a thumb?

If you knew that a circle had a diameter of 15cm, what would you estimate the circumference to be? Explain your predictions.

EXTENDING TO THREE AND A HAND…

Another elder from the Mi'kmaw community, after hearing the story of three and a thumb shared that when making hamper size baskets the ring is measured using three and a hand. Why might this be? Explain why this would make sense for larger circles.

Search for information on line about the number pi (π). How does this relate to three and a thumb?

PLAYING WITH MATHEMATICS / LEARNING MATHEMATICS THROUGH PLAY (K-12)

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INTRODUCTION

Predominantly in today's discourses, both doing mathematics and learning mathematics is described as 'hard work'. The only reason that we call the activity that we do 'work' is that it was the dominant word of the 19th century. We wanted to show that formulas *work*ed, that our children did their home*work*. But what if the nature of mathematics was much more playful than the nineteenth-century work ethic of the last 200 years has portrayed it? Maria Montessori said, "play is a child's work" (Child Development Institute, 2018).

Many theorists advocate the use of games, and their effectiveness has been demonstrated by several studies on the subject. When used well in class, play-based practice makes it possible to orient pupils towards a motivating, pleasure generating activity in which they engage freely with interest. Through play, the teacher has more time and opportunities to observe the students' different learning processes. Yet, despite the evidence, many teachers remain skeptical about the relevance of play to facilitating learning in the classroom.

This report is an account of the activities, discussions, and playing that took place over the three days of the working group (oops—make that 'playing group'!). The primary aim of this report is to provide two things: (1) A collection of activities that can be introduced in a learning environment in which people are engaging in mathematical thinking (see Appendix), and (2)

The thoughts and analyses of play and playfulness in mathematics by a group of mathematics educators.

ACTIVITIES: PLAYIN' WITH MATHEMATICS

Two activities led our initial foray into playing with mathematics:

ACTIVITY: Folding a Square[1]

- Take a weird-shaped piece of paper and fold it to make a square.

ACTIVITY: Mobius strip

1. Take a strip of paper, and bring it around and tape it to make a loop.
2. Take a new strip of paper and tape it to make a Mobius strip. This means turning one end half way around.
3. Walk your fingers around each. Take your time. Walk your fingers around the edges of the papers too.
4. Draw the loop. Draw the Mobius strip.

The tone of the working group—playful activity with a mathematical medium—was prompted by an offering: that participants take care when engaging with the activities and with one another in thoughtful and collaborative ways.

> *"There may be a lot of ideas that come up as we are thinking about (1) playing, (2) what we are doing while we are playing, (3) how we might use this in a classroom, and (4) what value each activity has. Make sure you have something out to write your ideas on—to free up that working memory to be present in the play."*
>
> *"Furthermore, as we work, you may come across a task that you understand intuitively. Make sure that you allow others the space to think and play and get messy, and wait until they ask you for help. One thing you might do is to write down any questions that may extend the problem itself into different mathematical spaces."*

We jumped right into the two tasks above and then reflected on them afterward. Participants played at tables in groups of approximately six. The dialogue at the tables included both the mathematical relationships present or discoverable in the activity, as well as pedagogical ideas. We (RM and AM) followed this up with a question for the whole group:

> QUESTION: *What key moments arose during your activity? Write down key moments and what you found was significant about these moments.*

Unsurprisingly, our group discussion revolved around two major themes: pedagogical and mathematical. The mathematical moments were comprised of mainly questions and conjectures.

MATHEMATICAL MOMENTS

Initially, many of the questions were aimed at understanding the problem and the solution space, such as,

> *"How can we mathematically prove that it is in fact a square? Are there different ways? What if we don't have a ruler, or a square angle, or other tools? What if we just have paper?"*
> *"Can it be a small square, a big square, etc.?"*

---

[1] Each activity presented in this document is described in more detail in an appended list.

Questions such as this one help us to make rules which guide the play. Participants commented on *how* the mathematical moments altered the course of the activity, or 'propelled the play'. According to the participants, clarifying questions, such as "*What assumptions are being made?*" and "*What were you thinking?*" made up only a small portion of the important moments, though participants mentioned asking what each other were doing. The most significant moments appeared to be investigatory questions directed at the mathematical object themselves, such as, "*What is the construction?*" "*How can you know/test/prove that it is a square?*" "*How did we go from one half-twist to two full twists?*" Additionally, some enumeration and optimization questions were posed, "*How many different ways can you construct a square?*" "*How few folds?*" "*How do we make the largest?*" "*Is there a better strategy?*" There were some questions about orientation, space, and symmetry: "*What if you flipped it around?*" "*Can you make it symmetrical?*" and "*What are the physical limits?*" Furthermore, the question "*How can I extend this?*" appeared to be significant, as, for example in the Folding a Square activity, participants mentioned considering different shapes of paper, cutting, repeating patterns of squares to make fractals, turning the activity into a puzzle or game, and finding the center of this piece of paper. Finally, a few participants cited metacognitive questions as key moments in the activities, including "*What does it mean?*" "*Does it matter?*" and "*What do we take away from this?*" One participant explained to the group that she realized the shape is not actually a square on paper, rather, it is a square in her mind. Many conjectures were made along the way, though very few of these were identified as key moments.

This group of mathematics educators was, of course, keen to do mathematics as well as to analyze the merits of playful mathematical activity in a mathematics classroom. Our discussions included what the players actually do, what we, as educators, want them to get out of it, and how we can facilitate these goals.

> *"Everything that is on our table is inert. It's all just stuff. What is it that made it move from being inert to play, and from play to mathematics? I think it's certainly more than the instructions that we were given. What were the actions that turned these things into activities?"*

<u>Actions of the Players</u>

A good deal of the pedagogical discussion of the day centered on the actions of the players, and whether or not our students would participate in ways similar to the ways that we engaged. In particular, much of our inquiry was guided only by the statement of the problem and prompt for the collaborative tone of the activities. But our prior experiences and habits of mind prompted us to engage deeply with the mathematical content. This activity was highly collective in that we played off one another a great deal. "*How did we inspire each other?*" asked one participant, clearly demonstrating that the dialogue amongst this particular group was mathematically rich. As a group, we never did decide whether or not our students would similarly engage with the activities. How we moved forward was by asking what it was that we got out of the activities, and what we would want for students to get out of the activities. Namely, we discussed the goals of such playful activities.

<u>Pedagogical Goals</u>

A participant noted, "*the square is deductive reasoning, and the mobius strip is inductive reasoning.*" While the tasks themselves offer a lot of fodder for discussion, the core of the WG conversation was aimed at the pedagogical reasons for bringing such tasks into a mathematics classroom. The central questions that were asked included what the goals might be, and if they are achievable or even important. Complementary to the mathematical content goals, participants identified some other mathematical behaviours to be important elements of the playful process. These include making space for mathematical discussion (such as the mathematical moments described above) and facilitating an experience in which solutions are not immediately available and where resilience is needed in the approach.

> *"There is nothing special about a piece of paper, but there is something interesting about trying to bring the people back to a discussion, asking a single question that motivates so much…"*

While these content goals (tools and topics) and practice goals (mathematical thinking, discussion, curiosity, resilience) were being discussed in small groups, some folks wondered about the importance of the instructor setting goals and about the merits of allowing students to set their own goals.

> *"Maybe you don't need a single goal to come back to. So many different ideas, it can be hard to bring things back to a goal/end place. Is that a problem? Shall we try to be more flexible with our goals?"*

For example, an instructor might want students to practice their formal mathematical writing, and so might re-word the task to include the statement 'prove that…' What is the value of that act, versus providing the space for students to play rule-less, or to make their own rules?

> *"Assumptions people make, and individual ideas are interesting, but deciding on a collective goal and trying to stick to it is another way to work in a group."*

In the end, playing with mathematics need not be oriented to a specific content goal. While the fundamental goal of the Folding a Square activity is to construct a square with a piece of paper, in fact, the participants in this WG identified many different experiences, including having the freedom to ask different kinds of questions (e.g., clarifying or extending), which are valuable to the mathematical learning process. While the apparent goal of the Mobius Strip activity is to discover interesting properties of surfaces, the experiences in the group included an embodied experience, as well as conjecturing, and free exploration. One participant noted that comparing the Mobius Strip to the loop is a simple jumping off point to explore surface area and topology. Embodied experiences, the mathematical thinking and actions, and fun and interesting ideas that emerged during these activities were identified to be valuable pedagogical moments of these playful mathematical activities.

Figure 1.

Of course, the question, "*How do we assess students' work?*" was raised, but we did not spend time discussing this. What we did spend the remainder of our time discussing was how valuable participants found being actively given the freedom to play.

<u>Learning Environment</u>

As the participants reflected on the significant mathematical moments of the activities and discussed the pedagogical implications of playing with mathematics, one final and profound theme emerged as participants shared the satisfaction they felt from setting the tone of freedom with no judgement:

> *"Everyone shares their ideas and their sense of the activity, but it was crucial that there was no judgement in sharing and ideas."*
>
> *"Liking the lack of judgement."*
>
> *"Facilitating safe fun free."*
>
> *"Being heard when saying what you are thinking."*
>
> *"If you already know what's going on, allowing others to have the space to figure things out."*

One participant wondered about recognizing the differences between working alone, giving space for others to figure out, group work, and the *marche silencieuse*. In the end, there was a collective appreciation for the role of play and playfulness in creating a judgement-free learning environment. Furthermore, there is a need to overtly recognize and promote it.

After some deeper analysis in small groups about these arising themes, RM asked small groups to consider the following question:

> QUESTION: *Is this [Holding up paper artifact from Mobius strip activity] piece of paper play? Is this mathematics? Is this mathematical play? Is this play mathematics?*

The majority of the groups said that the piece of paper is not play because it is an inert object, but that it is mathematics because it has inherent mathematical properties and questions which involve mathematical thinking can be asked about the object. The natural question, which followed this discussion, was What is play? This was quickly followed by a whole slew of other questions, very few of which were ever answered. These questions carried us over into the next day's activities.

ACTIVITIES: PLAYING GAMES

To start the second day we played games in six groups. As an introduction, we all played the same game, and then each group was given a new game.

ACTIVITY: 4x4 towers

- You will need a four by four grid, with room on the left side and the bottom for some numbers. A piece of grid paper will do. Using square blocks, make towers of heights 1, 2, 3, and 4 blocks, and place them on the 4 x 4 grid in such a way that there are no towers of the same height in each row, and there are no towers of the same height in each column.

Figure 2.

- Looking down the rows, we will now write down the number of towers we can see from in front of each row. For example, if a row has a front tower of height 2, then a tower of height 3, then height 4, then height 1, we can see parts of three towers (the tower of height 1 is hiding) and so into the front of this row we write '3'.

- Write all these line-of-sight numbers for your grid. What sets of numbers are possible for these line-of-sight numbers? For example, can you have 2-2-2-2 along rows and 2-2-2-2 along columns? How about 1-2-3-4, 1-2-3-4?
- Now start a new piece of 4x4 grid paper and start with numbers along the sides. Can you make the towers inside to match?

Figure 3.

ACTIVITY: Play games

1. Factor taxes
2. Juniper green
3. NIM 5-4-3
4. Game on a Graph
5. Race to 30
6. Take-away game (up to 3)
7. Penny trips

We (RM & AM) ended up giving out six of the eight different games shown above (Excluding 6 and 7), each of which was prepared previously in written format. Two groups received the same game (Factor Taxes), but each group had the prompt worded in a different way (see Appendix).

The didactic elements present in these games include addition, multiples, multiplication, factors, divisors, primes, squares of primes, averages, probability, combinations, characteristics of numbers and number concept practice and reinforcement, as well as a creative place to make links between primes and other numbers. Some games also included problem solving moves, such as simplifying and game strategies. The Game-on-a-graph, NIM, and the Mobius strip are not obviously connected to the traditional curriculum, instead they are more easily connected to 'competencies', or mathematical thinking processes.

The participants played their game in small groups for about 20 minutes, after which we choose one group and asked each member to act as a 'pioneer', to travel to one of the remaining groups and once there, to

1. Explain the rules of the game.
2. Give every person who had just been introduced to the game an opportunity to say something about the pedagogical possibilities of the game.

After this activity, the participants were asked to discuss the following question in their small groups:

> QUESTION: *What are the pedagogical possibilities of playing the game you have just been introduced to?*

Many of the same topics that arose from the question about key moments in the two original activities also arose here. Problem design was debated first. We cannot take for granted the pedagogical choices involved in presenting a task: there are different ways that games, problems, playful activities are presented, and it is possible to present them poorly. For instance, a bad pedagogical move is to spend 10 minutes reading the rules. Beyond the rules themselves, the way that the instructions for the game are written informs how to play the game, and this affects what happens next. Participants discussed their pedagogical criteria for 'good' games, which include the following:

- easy to start (little prior knowledge needed),
- ability to adapt the introduction/game to the reality of the players/learners, and
- generalizable, or ability to extend or modify in multiple ways; flexibility of rules.

They also noted that soft skills and mathematical content should both be observable and should be clearly available (listed) for teachers

ACTIONS OF THE PLAYERS—SUPPORT DIFFERENT APPROACHES TO PLAYING GAMES

There are some of us who buy a game and take it home and open up the rules, and there are some of us who buy a game and take it home and break out the game pieces and play for a while going forward with what we think makes sense. We see this in our students' actions: some take a task and immediately begin to consider, usually verbally, what would change if the rules were different; others become adamant that any investigation must stick to the specific parameters provided. Allowing players to play games *in the way that they want to* opens the door for a discussion about two fundamentally important mathematical actions: solving a problem, and inventing new problems. The dreamers engage in mathematical practices (e.g., What if? What if not? questions) while the orthodox develop their deductive reasoning (e.g., offensive manipulation of opponent into losing position). All such actions can be valuable, and this is reinforced if there is an opportunity for students to show off their own thinking to the rest of the class. If we allow students the autonomy to pursue their own interests, then it is possible for them to present their own, different knowledge to one another. This carries all sorts of pedagogical value.

With games, in particular, there is a tension between collaborative and competitive play. Every competitive game can lead to the topic of fairness, and this motivates an evaluation of strategies, which changes the activity from competitive to cooperative. Playing cooperatively changes the conversation around the game and provides an opportunity for deeper analysis. During this analysis, the players have opportunities to practice problem-solving strategies, such as reducing to a smaller problem. There is, in addition, the opportunity to generalize further—to go beyond the analysis of a particular game into topic explorations including learning about the category 'games' and strategies.

LEARNING ENVIRONMENT—PLAYING GAMES TAKES TIME

Participants made clear the great importance of having time enough to truly buy-in to a game. "*Only after playing for a long time and many iterations did I begin to get an understanding of the game. That time was valuable.*" Having enough time to play allowed some participants to move beyond playing and into analysis. "*Playing is the vehicle for understanding—We had to start playing the game in order to really understand it, but eventually reached a critical point* *where we stopped playing and started analyzing*." In order for play to be pedagogically valuable, it is of the utmost importance to create a learning environment that affords learners a lot of time for play and analysis. Only time can allow for the development of resiliency/persistence/perseverance/grit. And play should be enjoyable!

We finished up the day by giving the participants the choice to continue their small group discussion or to be introduced to a new activity.

ACTIVITY: Triangles! Paper cutting and folding.

1. Fold an equilateral triangle from a square piece of paper.
2. Make an equilateral triangle on the piece of paper using compass. Fold your triangle into a pyramid.
3. Make something else that you want with circles and lines and folding… explore your tools!
4. What kinds of triangle can the shadows of an equilateral triangle be? What triangles can make an equilateral shadow triangle?
5. Draw a triangle on a piece of paper, then fold-and-cut (fold the paper so that all the lines lie on one line, then take scissors and make one straight cut). Do you get your triangle back?

The groups that chose this new activity got through parts (1) and (2).

ACTIVITIES: MATHEMATICS PLAYFULLY

We began our third day in different rooms with two simultaneous activities chosen from a wealth of tasks developed by RM. The chosen tasks represent two formal mathematical topics: counting by sevens, and powers of 2 and multiples of root 2, and are examples of foundational activities that can be playful.

ACTIVITY: Penny Flowers. Creating flowers out of pennies.

1. Begin with a bag of pennies for each pair of people.
2. Placing one penny in front of you, surround this penny with more pennies. [It takes exactly 6!]
3. Now how many penny-flowers can you make with the pennies in your bag? For example, 23 pennies make three penny-flowers, with two pennies left over.
4. Each person is given a unique number of pennies and invited to make a penny-flower garden. They are then invited to find their 'penny-flower garden sibling' (a person with the same number of left-over pennies), or 'penny-flower garden friend' (remaining pennies in each person's garden combine to make exactly one more penny-flower).
5. Imagine having a garden of 23 pennies. Which person, one with 47 pennies, or one with 16 pennies, would be your penny-flower sibling? Which would be your penny-flower friend?

ACTIVITY: Radical Rose

- Here is some grid paper (printed paper with 8 x 11 grid (squares) with margins. Here are some scissors. You could cut out an 8x8 grid, or you could do whatever else you like.

AM had some idea about constructing the Radical Rose, as explained by RM, but she felt that her novice presentation of this task would contain very little of the richness that RM had developed around the activity. So instead she presented the activity as a challenge to the participants:

> *"See what you can come up with."*

Two tables began cutting out squares that did not follow the grid lines. In this way they were able to make squares of area 10 and 13, among many others.

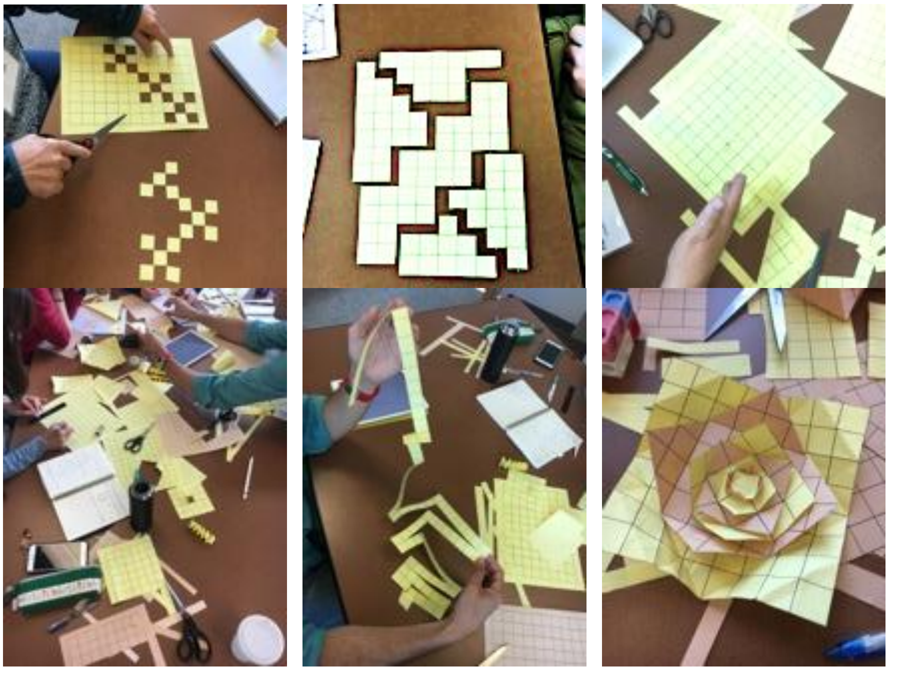

Figure 4.

When she noticed a few people watching others, AM asked if they were interested in an idea to follow, and if they agreed, she walked them through the first few prompts from Ralph's Radical Rose. Participants suggested extensions such as continuing the process to get areas smaller than one and also discussed the total area present in the rose, which can be represented as a geometric series.

REFLECTION: WORKING GROUP / PLAYING GROUP

In our days together, we found ourselves repeatedly attending to the same questions.

<u>What are play and playfulness?</u>

What do play and playfulness look like in a mathematics learning situation? What are the essential conditions that you can use in order to allow for play to happen? Is there a difference between play and playfulness? We had a major debate about whether goals or structure was needed for play in mathematics learning.

<u>What is the value of play and playfulness in learning mathematics? What actions do players take to deepen that value?</u>

Mathematics has a long place to go to be seen as playful. Yet *we* know, from this experience and others, that playing with and being playful with mathematics is a worthwhile activity because it creates opportunities for

- authentic practice, for example, analysis and generalization (rather than, say, just computation),
- less anxiety, more resiliency/persistence/perseverance/grit, enjoyment
- being okay with making mistakes: it's okay to not know all the rules!
- introduction and exploration of mathematical topics.

Practical and playful do not need to exclude one another. Furthermore, as the WG itself demonstrates, play and playfulness can change the course of activity form passive to active learning: "*We, as participants, changed our activity from Day 1—being led to do things, to Day 3—making our own directions.*"

<u>What playgrounds are the exemplars of our interactive learning opportunities?</u>

> *"Activities which have a great deal of adjustability of constraint."*
>
> *"Inviting play. We had lots of instances when we could make choices, in our materials, in the types of mathematical questions we asked, and who to do it with."*
>
> *"We did little right away about setting THE pedagogical context/moves, had space to play with our own pedagogical ideas."*
>
> *"Openness of time in contrast to where we often find ourselves."*
>
> *"Possibility to work together, work alone. Multiple sets of resources were available and helped this."*
>
> *"Lots of continuous and repeated connections between doing and thinking."*
>
> *"Table of colleagues with whom I share interest."*

NOTEWORTHY TEACHING MOVES

Although we did not talk about 'teaching moves' in the mathematically playful learning environment, some of the actions taken by the WG leaders stood out to some participants, including the following.

AM reminded everyone: There may be a lot of ideas that come up as we are thinking about playing, what we are doing while we are playing, how we might use this in a classroom, and what value it has. So make sure you have something out to write your ideas on, to free up that working memory to be present in the play.

Before starting the first task, AM reminded people that we are all different humans and asked people to be mindful of two things: (1) To write ideas down to free up working memory, and (2) To allow others the space to think and get messy on their own.

RM asked people in a group to have a discussion about a moment that they found interesting or memorable. They are to choose one person from their group to share with the whole class, and also to choose an appreciator from the group, who listens, adds on to the statement, and then asks a new person to share.

RM asked one participant to 'do a teacher move' and get people to sit in new groups.

RM talked to a group who were all legitimately surprised at a result (Mobius strip). He then asked them to each go to a different group and 'infect' others.

At one point during the day there was lots of talking in groups and a lot of noise. RM asked everyone to spend some time writing something that they were trying to explain.

When RM tried to get everyone to listen, a few people kept talking. He reminded the rest of us that she was doing the right thing. She has something she needs to say, and that is the right thing to do.

After RM introduced the block towers activity (deductive reasoning), he said, "What is possible? What is impossible? Now go to your tables and do this… and ask yourself, what if?"

In one group of six, a few people were left out of a fast-paced discussion. AM suggested that they as a group identify three different versions of the game, and then split up in pairs to each play a different version of the game and then come back together and discuss the similarities and differences.

IN CLOSING

We, as a larger group, come together to think about doing and learning mathematics. We gather to think and talk about what we can do as mathematics teachers, parents, citizens, researchers. In this group we did and learned mathematics though play.

We played with paper, we played with blocks, we played with games. Examples:

1. From a weird shaped piece of paper, fold a square.
2. From some grid paper, and some scissors… what can you do/make/say?

We participated in, and talked about the value of, directed and undirected play. We talked about doing mathematical things playfully. We played with mathematical ideas, pedagogical ideas, and some philosophical ideas. For example, [Hold up piece of paper] Is this play? Is this mathematics?

We did not talk about what teachers can do, e.g., if some students are playing and others are not. We did talk about what play and playfulness can look like.

Some valuable things that we will take away from this experience include

- Being allowed *freedom to make choices* in our materials, in the types of mathematical questions we asked, or if we wanted to work together, or take time to work alone for a bit.
- Having an *openness of time*, to let us find depth in the mathematical ideas, but also to think about the pedagogical moves we might make. Having time changed our expectations as learners! From waiting to be told what to do, to making our own direction.
- Lots of explicit and repeated *connections between doing and thinking*.

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Child Development Institute. (2018). *"Play is the work of the child" Maria Montessori* [website]. Retrieved from https://childdevelopmentinfo.com/child-development/play-work-of-children/#.XKZgh4hKiUk

APPENDIX 1

FOLDING A SQUARE

Take a weird-shaped piece of paper and fold it to make a square.

Extensions: Can you find the center of gravity of this piece of paper?

MOBIUS STRIP

Take a strip of paper and bring it around and tape it to make a loop. Take a new strip of paper and tape it to make a Mobius strip. This means turning one end half way around.

Walk your fingers around each. Take your time. Walk your fingers around the edges of the papers too.

Draw the loop. Draw the Mobius strip.

Extensions:

With scissors, cut long-ways all the way around the loop, halfway between the edges. What are you left with? [Answers for loop will likely be that we have two loops, nearly the same as the first, just thinner. For Mobius strip, many conjectures!] Now actually do it. [Extension questions that may arise: What if we turn one end of the strip 2 half ways around and do the cutting thing? What will we get?

QUEST (4 X 4) TOWERS

Game board: You will need a four by four grid, with room on the left side and the bottom for some numbers. A piece of grid paper will do.
Write these numbers on the left, in front of the four 'avenues': 2 2 1 3
Write these numbers on the bottom, below the four 'streets': 3 2 4 1

Playing pieces: You will need four towers of each of these heights: a height of one, a height of two, a height of three, a height of four.

To win: Put the towers in the sixteen cells so that they match the numbers.
Here's the idea: each number shows how many towers are visible to you, if you gaze from that position (from the left for an avenue, from the bottom for a row).

Then: Make a new puzzle.

To learn: What can you say about the possibilities? Some things might be possible. Some things might be necessary. Some things might not be possible.

Something to think about: What is your team doing well together?

FACTOR TAXES (A DIFFERENT VERSION OF TAX MAN OR WOMAN)

Game board: Use a hundreds board, with a piece of paper to limit the size.
Try the first three rows of the board, to start.

Pieces: One tile for every cell on your game board. Half should be one color, half another.

To win: Cooperate. Your team wins if your business makes more money than it pays to the tax collectors.

One move: Select a number. Put a tile on that number.
Now use the other color to give the tax collectors their share. The tax collector gets the factors of the numbers you selected.
The tax collector always gets a share, or, the game ends.
The number you chose, and all numbers the tax collector received are now out of play.

When you have no more moves left: There's some disappointing information I'll save for the right moment. When you have no more moves left, it's the right moment. Ask.

To win: Add up the tiles that your business covered. Add up the tiles that the tax collector received. If your team made more money than you paid in taxes, your team has won!

Hint: **Winning this game is hard!** Keep track of your moves, and the tax payment for each move. Then you can talk about your strategy before playing again.

Challenge: If another team wins on the same game you played (e.g., Factor Taxes 30), the team that scored highest gets bragging rights if the other team agrees that their moves are legitimate.

TAX MAN OR WOMAN (A DIFFERENT VERSION OF FACTOR TAXES)

The game is played like this: Start with a collection of paychecks, from $1 to $12. You can choose any paycheck to keep. Once you choose, the tax collector gets all paychecks remaining that are factors of the number you chose. The tax collector must receive payment after every move. If you have no moves that give the tax collector a paycheck, then the game is over and the tax collector gets all the remaining paychecks. The goal is to beat the tax collector. For example,

Turn 1: Take $8. The tax collector gets $1, $2 and $4.
Turn 2: Take $12. The tax collector gets $3 and $6 (the other factors have already been taken).
Turn 3: Take $10. The tax collector gets $5. You have no more legal moves, so the game is over, and the tax collector gets $7, $9 and $11, the remaining paychecks.

Total Scores:

You: $8 + $12 + $10 = $30.

Tax Collector: \$1 + \$2 + \$3 + \$4 + \$5 + \$6 + \$7 + \$9 + \$11= \$48.

Questions: Is it possible to beat the tax collector in this \$12 game? If so, how? What is the maximum score you can get?

Bonus: What if you played the game with paychecks from \$1 to \$24? How about \$1 to \$48?

JUNIPER GREEN

Game board: Use a hundreds board, with a piece of paper to limit the size. Try the first three rows of the board, to start.

Pieces: One tile for every cell on your game board. Color doesn't matter.

To win: You win if your opponent cannot make a turn.

First move: Select an even number. Put a tile on that number. That number is now out of play.

Every move after: Select a **factor** or a **multiple** of the number just covered. Put a tile on it. That number is now out of play.

To win: Some numbers have many factors. Some numbers have only a few. With multiples, the small numbers on the board have many multiples. What about the big ones?

Question: What is the reason for the special limitation for the first rule?

Challenge: What is the shortest game possible, if you cooperate?

THREE FOUR FIVE

Game board: Twelve positions in three rows: three positions, four positions, five positions.

Players: Two.

Playing pieces: Twelve pieces of three kinds: three of one kind, four of another, five of a third. Place the pieces in the positions, one kind per row.

To win: Force your opponent to take the last piece.

Moves: Take one or more pieces, from any one row.

To learn: Talk about the possibilities, and talk about the consequences of each possibility.

Something good learners do: There are some great positions to leave for your opponent, near the end of the game. When you find one, *name* it.

GAME-ON-A-GRAPH

Here is a game for two players. First draw a graph. Not like a line on a cartesian grid, but a graph with nodes and edges, like in a network. Now, can you develop a strategy for the following game-on-a-graph?

Two people play a game with a finite graph. They take turns coloring the uncolored vertices of the graph, one vertex at a time. Say Player 1 uses red and Player 2 uses blue. Two adjacent vertices (nodes joined by an edge) cannot be colored in the same color. The person who loses the game is the first person who is unable to color a vertex.

RACE TO 30

Game board: Use a hundreds board, with a piece of paper to limit the size to the first three rows.
Or— Use a calendar page.

Playing pieces: Six tiles of one color, six tiles of another. Two dice.

To win a game: Your team wins if you get to 30 (or beyond) before I do. We tie if we both get to 30 (or beyond) in the same number of moves.
I win if I get to 30 (or beyond) before your team does.

For the class to win: We will have figured out if this is a fair game. At first it seems that it obviously isn't.

The first move: If your team goes first, roll two dice. Put one of your tiles on the position that the dice show.
If you let me go first, put one of my tiles on **seven.**

Every move after: For your team's move, leave the original tile in its position, and count on whatever you roll. Put a tile of your color there.
For my move, well, I always go seven. Count on from my last move, and place another tile of my color.

When someone gets to 30 (or beyond):
Have we both had the same number of turns? If yes, then the game ends, and someone has won!
If no, then let the person who went second have one last turn, to try to tie by getting to 30 or beyond.

Make a record! Write down the footprints of your moves, and mine. Write down if your team won, I won, or we tied.

To learn: You will learn about dice rolls. You will improve your adding, especially if you figure out before you roll what you will need to roll to win. And you will get to know the sevens numbers.

For us all to learn: If we have reliable results, we will be able to see if the game is fair or not. Maybe we will figure out a way to change the finish line, to make it fair.

Questions for later: What number do you roll most often? What numbers almost never come up?

TAKE-AWAY GAME

There are 21 pebbles in a pile on a table.

Two players take turns removing 1, 2, or 3 pebbles from the pile.

PENNY TRIPS

Material: Two pages of grid paper of different colors.
Scissors.
A penny.

Game board: From one piece of grid paper, cut a square of a size of your choice. How about five by five?

Playing pieces: Along with the penny, you need some triples (three by one pieces), enough to almost cover your game board. Make them from the second color of grid paper. Actually, they're easier to use if your trim them down by a couple of millimeters.

To win: Your team wins if it can find all the places that the penny can be placed.

First move: Put the penny down.

Every move after: Now you should take turns, putting your triples down. Can you cover the whole board?

To learn: This game is a chance for you and your team-mates to improve at least three things:

- your interpersonal tactics
- your strategic reasoning and
- your strategic communication.

Start with the interpersonal tactics. How can you help your team develop, so the team can figure out stuff more powerfully than you could alone?

Question for later: How many different positions are there for the penny, when symmetry is taken into account?

TRIANGLES!

1. Fold an equilateral triangle from a square piece of paper.
2. Make an equilateral triangle on the piece of paper using compass. Fold your triangle into a pyramid.

3. Make something else that you want with circles and lines and folding… explore your tools!
4. What kinds of triangle can the shadows of an equilateral triangle be? What triangles can make an equilateral shadow triangle?
5. Draw a triangle on a piece of paper, then fold-and-cut (fold the paper so that all the lines lie on one line, then take scissors and make one straight cut). Do you get your triangle back?

PENNY FLOWERS

Creating flowers out of pennies.

1. Begin with a bag of pennies for each pair of people.
2. Placing one penny in front of you, surround this penny with more pennies. [It takes exactly 6!]
3. Now how many penny-flowers can you make with the pennies in your bag? For example, 23 pennies make three penny-flowers, with two pennies left over.
4. Each person is given a unique number of pennies and invited to make a penny-flower garden. They are then invited to find their 'penny-flower garden sibling' (a person with the same number of left-over pennies), or 'penny-flower garden friend' (remaining pennies in each person's garden combine to make exactly one more penny-flower).
5. Imagine having a garden of 23 pennies: Which person, the one with 47 or the one with 16, would be your 'penny-flower sibling' or 'penny-flower friend'?

RADICAL ROSE

1. Begin with an 8x8 grid.
2. Next, count the number of squares that you have on your grid. You know you have 64, but count them, and show with your fingers how you are counting. You can skip count, if you like. [Some people may count by rows (multiples of 8), some by doubling the number of rows (8, 16, 32, 64), or even by 20s (20, 40, 60, 4).]

3. Now, fold each of the four corners of the square in to the center. The corners are the petals of a flower. They fold perfectly onto a square base. How many squares in the base? Open up the petals of this flower so that you can see the base which has area 32. Count, again with your fingers, the number of squares. What is the side length of this square?
4. Okay, now cut out from another square of grid paper (another colour is nice too) that has area 32.
5. Repeat Step 2 (fold the corners into the center). [Half of 32 is 16.]
6. We repeat this process. Near the end we have a 2x2 grid of area 4, then a grid of area 2, then area 1.

Now we can go back to look at the relationships between the length of each side of each square and the area of the square. For example, the square of area 32 has four diagonals on each side, that is, 4 root 2s.

ROBOTICS IN MATHEMATICS EDUCATION

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INTRODUCTION

Robots and robotics have spread out of research laboratories, industrial and commercial settings to a variety of new locations including living rooms and classrooms. This incursion has afforded different learning opportunities for children and adults. In the tradition of Papert (1980), who identified educational robots as 'objects-to-think-with', our working group set out to explore some of the potential for using robots to think about mathematics and other powerful ideas through engaging with building, programming, testing, mathematising, playing and discussing emergent ethical issues.

Participants came with a variety of previous experiences and expertise in mathematics education and coding across diverse settings and with different goals. These included making stronger connections between coding and robotics, or computational thinking and mathematical thinking; examining potential for inclusion in courses for pre-service teachers given jurisdictional pushes or emerging curriculum emphases; seeking active hands-on experiences for applied mathematics courses and modelling; and also because of the 'that's cool!' factor.

DAY 1: NECESSARY FOUNDATIONAL EXPERIENCES

Getting started with robots in the classroom for learning requires some necessary foundational experiences to learn how the robot is put together and how the programming works.

BUILDING THE ROBOT

Figure 1. Participants building the EV3 robot.

First we built the robot as per the instructions in the manual provided with the EV3 educational robot kit. The design in the manual provides a solid, stable robot that can easily be adapted to suit a variety of tasks. This robot is suitable for beginners and intermediates. Building a robot from the manual also provides opportunities for using and developing spatial reasoning skills. "[S]patial reasoning is the ability to recognize and (mentally) manipulate the spatial properties of objects and the spatial relations among objects" (Bruce et al., 2017, p. 146). A child's spatial skills at three years of age are strong predictors of how well a child will do in kindergarten. Spatial skills are a better predictor of later mathematics than vocabulary or mathematics (Newcombe, 2010). Spatial ability and mathematics ability are strongly correlated (Mix et al., 2016; Mix & Cheng, 2012). All STEM careers demand strong spatial reasoning (Newcombe & Shipley, 2015; Wai, Lubinski, & Benbow, 2009)—so does architecture, art, drafting, carpentry. Some spatial skills described in Davis et al. (2015) that were engaged when building the robot from the instruction booklet included the following:

- shifting back and forth between dimensions of the 2-D booklet and the 3D emerging robot;
- scaling to match the size of the represented and actual shapes;
- assembling the pieces together requires rotating, arranging, fitting, balancing, comparing, interpreting, visualising, locating and orientating for accurate building.

ROBOT DANCE

Once the robot was built, a robot dance provided opportunities to learn how to instruct the robot to move forward, backwards, and turn repeatedly with a loop. Very quickly, everyone learned how to add programming blocks to their chain, connect their robot to the computer or tablet and download the program to the robot. One group was particularly fast building their robot, so they had already attached many of the sensors. They played with the programming blocks and figured out how to dance using their ultrasonic sensor. The sensor kept them away from walls and other robots. Visit https://vimeo.com/291510578 to see our dance. The dance added an element of fun and set the tone for playful exploration throughout the workshop.

These first tasks established the group dynamics and how the teams worked together. Some groups split roles and responsibilities. For example, while building often one person was in charge of assembling the pieces, while another was in charge of finding pieces for each step. When it came to programming, often one person did most of the on-screen coding. We wondered: How does this splitting of roles build or prevent development or skills? and How could we structure the activities differently to ensure everyone has equal and meaningful opportunities?

RACE TO THE WALL

A race to the wall challenge provided the groups a chance to understand how the power option in the steering block works. The challenge was to program the robot to get as close to the wall as possible without touching it, as quickly as possible, while having a Lego person (mini-figure) remain standing on top of the robot for the entire race. The competitive nature of our working group and this cultural aspect of robotics became apparent in this race. Watch https://vimeo.com/292340498 to see our race.

Figure 2. Ready, set…robot race to the wall.

After the race, we discussed the strategies for how the groups attempted to win. Trial and error was a dominant strategy for approaching this challenge. We discussed how limiting the number of trials might encourage more strategic mathematical thinking with the use of measurement tools. One group found the highest speed that the robot could have without the Lego mini-figure falling off, and then computed the number of rotations for the right distance. Another group found a programming example on the internet that made the robots' speed increase steadily without stopping.

Our discussion of the elements of mathematics in the race to the wall generated the following:

- The feedback from the environment (e.g., the robot and peers) lent itself to developing habits of mind: iterating, adjusting, problem solving, striving for accuracy, flexibility of thinking and innovating.
- Calculating the relationship between wheel circumference and distance travelled required proportional reasoning.
- Programming distances required applying knowledge of integers, decimals, and negative numbers.
- The group that found the fastest speed the robot could go while balancing the Lego mini-figure, used a stepwise linear function for the velocity. We questioned if a constant value for the power translates into a constant value for the speed: if the robot starts at fully stopped, there must be acceleration at first. What function should be used for modelling the velocity of the robot? The group that found a program that used steady acceleration applied continuity and derivatives. These could be explored mathematically.
- Tension between satisfaction and dissatisfaction: there was satisfaction in getting the robot to race to the wall, but there was a dissatisfaction knowing that by fine-tuning the model/program to be more sophisticated, the robot could perform the task better. This created the need for using more variables and/or more advanced mathematics concepts such as continuity.

EMBODIED NOTIONS OF NUMBER

During the working group questions arose around whether activities/tasks like the race to the wall challenge should only be used for knowledge integration and whether new mathematics could be learned with it. In this section we draw on ideas from embodied cognition to address these questions in a more direct way.

Conceptual metaphors are one of the ways we understand mathematics (Lakoff & Núñez, 2000). With regard to the concept of number, Lakoff and Núñez describe four fundamental metaphors of arithmetic: arithmetic as object collection, arithmetic as object construction, arithmetic as measurement, and arithmetic as object along a path (Figures 3-6). The metaphor of arithmetic as an object collection is based on a one-to-one correspondence of numbers to physical objects (see Figure 3). With this metaphor a greater quantity of objects corresponds to a bigger number. For instance, five is greater than two because it is quantitatively more objects. The metaphor of arithmetic as object construction is based on fitting objects/parts and arithmetic operations (see Figure 4). For instance, five is greater than two because an object comprising five units is larger than one comprising two. The measuring stick metaphor maps numbers onto distances, whereby five is greater than two because it is longer (see Figure 5). The metaphor of arithmetic as an object along the path is based on arithmetic as motion by which five is greater than two because it entails moving further from a common starting point (e.g., zero; see Figure 6). Programming robots provides opportunities for illustrating, experiencing and connecting these multiple interpretations of arithmetic metaphors.

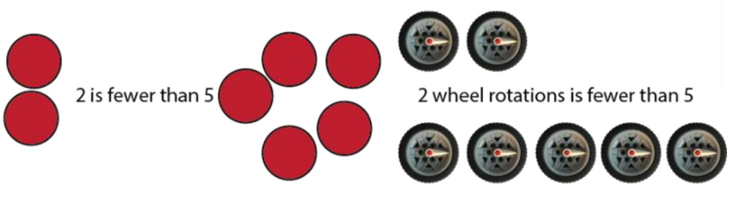

Figure 3. Arithmetic (number) as object collection.

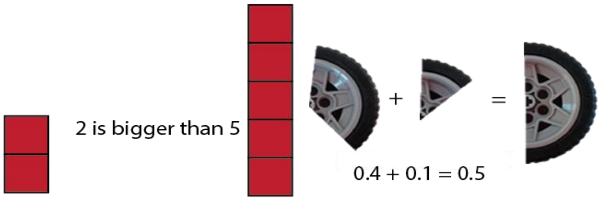

Figure 4. Arithmetic (Number) as object construction.

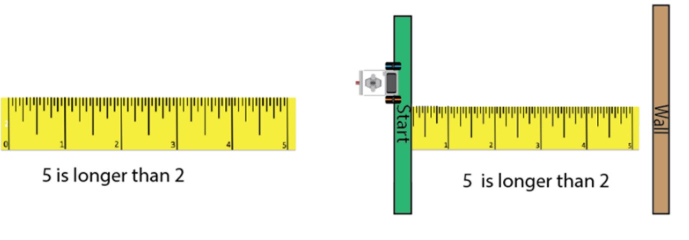

Figure 5. Arithmetic (Number) as Measuring.

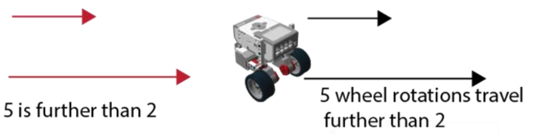

Figure 6. Arithmetic (number) as distance.

DAY 2: UNPACKING THE STEERING BLACK BOX

EXPLORING THE EFFECT OF THE SINGLE PARAMETER FOR STEERING

The activity for understanding the role of the single parameter ($-100 \leq s \leq 100$) for steering (fixed wheels) revealed the richness of the mathematics that could be done with the integration of robots. As a low floor and high ceiling activity, it opened up many approaches for exploring and modelling.

1. Looking at what each wheel does for different values of s, when the (maximum) number of rotations (r) for the two wheels is set to 1: idle, full rotation, half rotation, etc. This experimental approach prompted a systematic data collection activity in a table of values.

Figure 7. Data collection during an inquiry task.

If one decides to use negative numbers to account for the direction of a wheel (positive as going forward and negative as going backward), such data can be summarised in one or two graphs to get a better feel of the coordination of the two wheels with respect to the steering parameter. This brings an opportunity for meeting and using piecewise linear functions.

Using a reciprocal approach, one may also want to find the function that uses as input the number of rotations of each of the wheels (LR and RR) and computes as output the steering parameter. This would typically invoke inductive reasoning to come up with a rather simple expression such as $s = (LR - RR)/2$.

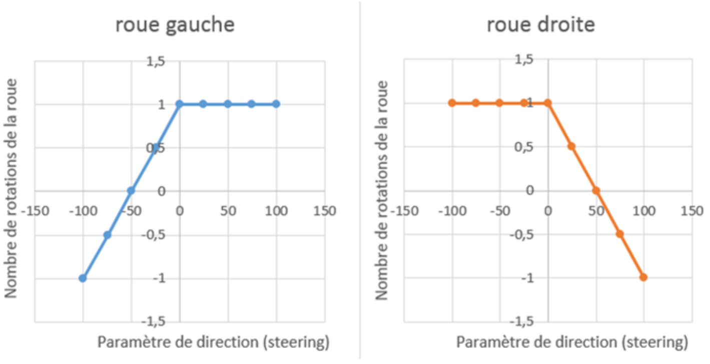

Figure 8. Graph of steering parameter versus number of rotations for each wheel.

The connection between the steering parameter and the number of rotations on each of the wheels was perceived as a true 'black box', as it had an *arbitrary* component (how that steering parameter had been defined and programmed in the design of the robot) that only exploration and data collection could manage to unveil.

2. Looking at the trajectory of the robot

After having looked at the connection between the steering parameter and wheel rotations, teams started looking at the trajectory of the robot.

One team mapped that trajectory on paper: for different values of the steering parameter and with one as the maximum number of rotations for the two wheels.

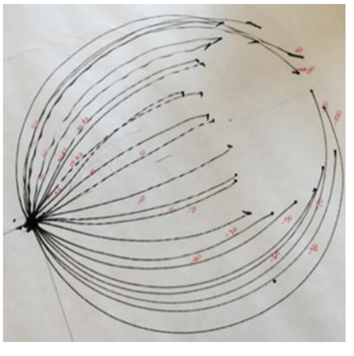

Figure 9. Trajectories of the robot for different steering parameters.

From this empirical investigation, an interesting (and open) question emerged: Could the locus of the end points of the trajectory be an ellipse? And if so, why?

To support their exploration, teams were provided with a mat that showed different concentric circles along which the robot could be made to move (see https://www.ucalgary.ca/IOSTEM/files/IOSTEM/steering-mat.pdf for a downloadable version of the mat). Team members started considering the different arcs of circle that different values of the steering parameter would generate as trajectories for the center of the robot and, eventually, for each of the two wheels.

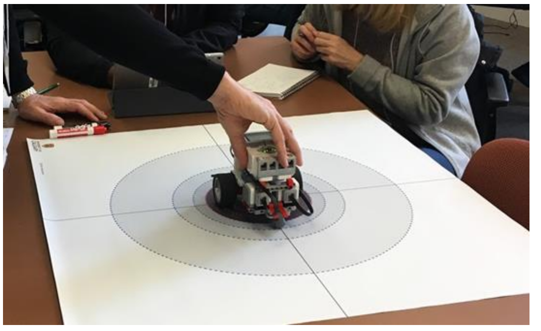

Figure 10. Exploring the trajectory more systematically.

This led to associating the number of rotations performed by each wheel to the distance covered by the wheel and, consequently to

- the corresponding radius for each of the arcs
- the angle of the rotation made by the robot
- the center of that rotation

Contrary to the initial relationship between the steering parameter and the wheel rotation, these relationships did not have an arbitrary aspect but came as consequences of the specific geometry of the system.

A team combined some of these relationships into a single function.

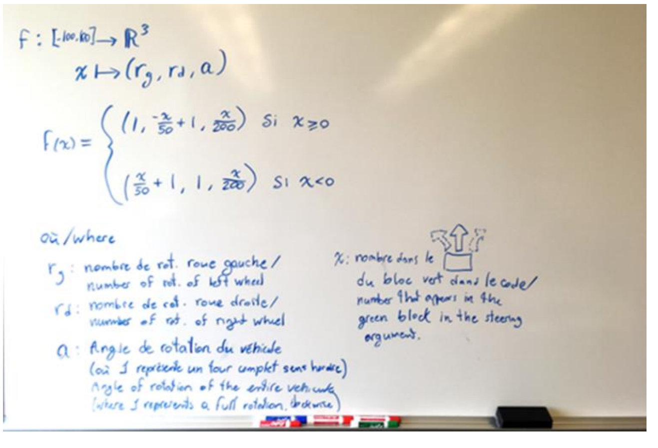

Figure 11. Defining a function for mapping the steering parameter to the movement of the robot.

Their functional model was very close to the geometric model we had designed with GeoGebra and with which participants were later invited to play:

- Steering Wheels Power (https://goo.gl/6e16Py)
- Steering Wheels Rotation (https://goo.gl/VKypNj)
- Steering Wheels Rotation Power (https://goo.gl/CMSYvv)

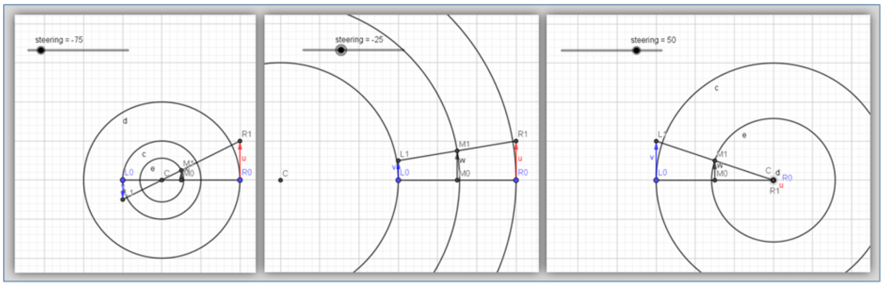

Figure 12. The velocity of each of the wheels modelled as vectors.

Participants spontaneously expressed their appreciation of exploring, all possible angles in the steering of the robot.

- Samuel: *Je suis impressionné et surpris du plaisir qu'on a eu à explorer la boîte noire du robot*.
- James: *It was interesting to see how each team focused on different aspects of the situation, and how the combination of these explorations or models enriched the understanding of all*.
- Frédéric: *Moving from modelling the distance to modelling the velocity is a step up in the abstraction*.

LA TÂCHE DU POLYGONE RÉGULIER

En tirant parti de ce qu'avaient appris les participants avec l'étude du paramètre de direction, la tâche suivante leur demandait de programmer leur robot de façon à ce qu'il se promène le long des côtés d'un polygone régulier. Des tapis avec le dessin d'un tel polygone servaient à la fois à définir la tâche et à fournir un environnement de validation. Il fallait donc construire le modèle mathématique, le traduire en modèle informatique sous forme de programme, l'implémenter dans le robot et en valider l'exécution.

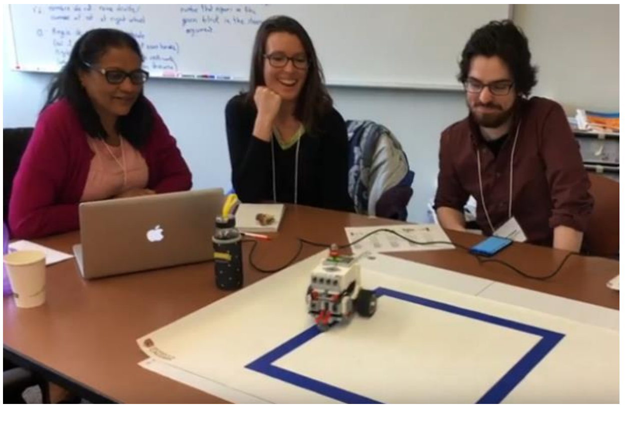

Figure 13. Programmer une trajectoire sur un polygone régulier.

Cette activité donna lieu à deux découvertes surprenantes.

1. On semblait obtenir un meilleur respect du parcours si l'on choisissait de ne pas recourir au paramètre unique pour la direction, et d'y aller plutôt pour un contrôle indépendant de chacune des deux roues, comme le permet aussi le robot utilisé.

2. Si l'activité paraissait *a priori* l'occasion tout indiquée d'utiliser une boucle dans les instructions de déplacement (pour chacun des côtés du polygone), l'utilisation d'une telle boucle semblait créer un moment d'hésitation dans le changement de direction qui se traduisait par un décalage au regard de la trajectoire visée. Le trajet était plus précis si l'on recopiait la séquence des instructions, autant de fois que l'on avait de côtés.

Bien que ces différentes approches soient théoriquement équivalentes et qu'il paraisse légitime de penser *a priori* qu'elles donnent lieu à une même trajectoire, les différences observées constituaient un rappel qu'on était ici avec des objets réels, autant du côté mécanique que du côté informatique, avec la matérialisation de l'information et son traitement (Bertrandias, 1992).

Sur le plan informatique, des hypothèses ont été avancées pour expliquer les écarts observes : erreurs d'arrondi dans la conversion du paramètre unique pour la direction en commandes pour chacune des deux roues; délai lié à la vérification de la condition associée à une boucle.

Sur le plan mécanique, en raison du fait que la direction dépend de la différence entre les vitesses des deux roues, le parcours en ligne droite est extrêmement sensible au moindre écart entre ces deux vitesses. Et quand il s'agit en plus de faire un virage sur une ligne polygonale, les arcs de cercle sur lesquels se réalise un tel virage rendent l'exercice toujours un peu approximatif.

L'importance accordée à la validation, dans cette activité comme celles qui ont précédé, permet d'apprendre à cerner autant les limites que les apports des modèles mathématiques et informatiques qu'on utilise pour atteindre un but au regard d'une situation réelle. Cela peut donner lieu à certains moments de frustration, quand on semble loin du but, mais aussi à de grandes joies quand on y arrive finalement!

Dans ce travail itératif, le robot participe au milieu didactique (au sens de Brousseau, 1990), dans la mesure où il renvoie l'effet des actions posées, même si cet effet fait parfois intervenir des éléments qui n'étaient pas considérés au départ. Un tel travail nous paraît s'inscrire non seulement dans le développement de compétences de modélisation, mais aussi dans le développement d'une pensée informatique.

DAY 3: LOOKING AT THE BIGGER PICTURE

CONTRIBUTION TO THE DEVELOPMENT OF COMPUTATIONAL THINKING

Weintrop et al. (2016) have proposed a taxonomy aimed at providing a clearer and more operational definition of computational thinking for mathematics and science (Figure 14).

Using this taxonomy, the robot tasks that the group experienced (robot dance, race to the wall, steering & polygon tracing) seem to address mainly Computational Problem Solving Practices, and Modeling & Simulation Practices: participants had to assess, design and construct computational models, which were more or less tied to mathematical models. Most of these computational models were implemented with some amount of programming accompanied with some troubleshooting and debugging.

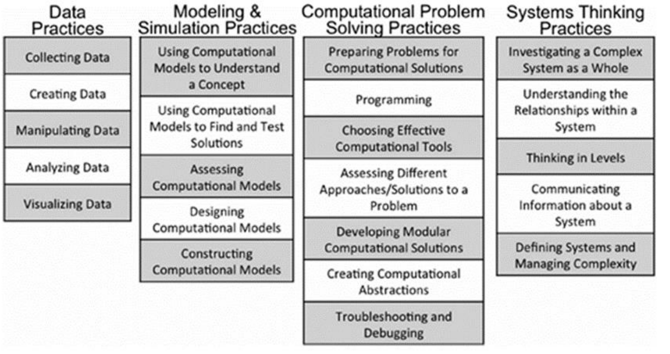

Figure 14. Weintrop et al. (2016) taxonomy for computational thinking in mathematics and science

Through the different tasks, we also saw participants express the need to collect, organise and analyse data: this was mainly done with paper and pen. A noteworthy exception was when sensors were used by a team to direct their robot eliminating the need to record the specifics of each experiment. This gave rise to a very different approach to the polygon task where the robot was instructed to follow the line as it would 'see' it.

The point could be made that some thinking in levels also occurred when participants had to reconcile the moves a robot makes with the moves its wheels make, and later used the steering box that they had just opened to define a more elaborate trajectory.

ETHICAL AND SOCIOPOLITICAL ISSUES

In our third day, we did not get to spend as much time on the ethical and sociopolitical issues as we had originally planned. Therefore,we feel that the discussion we had should be expanded in the report, recognizing that it is an area that we often do not address when working with robotics in classrooms.

Overview

- Ethical and sociopolitical concerns and themes are woven into human engagement with technologies like robotics that promise large scale social transformation.
- These themes are not often engaged with in mathematics education (school, research).
- Learning mathematics with robots should include such themes and discussions.

Sociopolitical concerns in mathematics education have moved from the margins to the mainstream over the last four decades (see Jurdak & Vithal, 2018). This shift includes a clearer relationship between sociopolitical issues and sociocultural aspects of teaching and learning and a concern with ethics. Political and ethical issues have been woven into human engagement with technologies, including mathematics, across time and space. Technologies like robotics promise rapid large scale social and economic transformation. Such transformations raise important issues of concern to mathematics educators beyond 'the mathematics' such as those related to equity, access, identity development, and social justice. The discussion of ethics and responsibility in mathematics education in the context of learning robotics was framed by D'Ambrosio (1998) is still a pointed reminder to our field that, "particularly in mathematics, there is an acceptance that we are fulfilling our responsibilities if we do our mathematics well…But this is not enough. This must be subordinated to a much broader attitude towards life" (p.71) and his recommendation to engage learners in inquiring how mathematics is implicated in the problems as well as the solutions we face today and tomorrow.

The contest/competition dynamic

Some elements related to this theme emerged from participants' questions and concerns over the three days. For example, questions were raised about the dynamics of the 'contest' format during the 'race to the wall' challenge and possible differential engagement within school-age populations. More explicitly, might the framing as a 'contest' with 'winners' and by extension, 'losers' make the learning, and the learning of mathematics through robotics a more fraught experience for some learners? We note that robotics and informatics contests (e.g., Bebras, https://www.bebras.org/) and competitions are a current part of the modern milieu. This ranges across a variety of school level contests and competitions (e.g., FIRST Robotics Competition, VEX Robotics Competitions, World Robot Olympiad), some of which have been running for over a decade. In a similar vein there currently exist many varieties of school-level mathematics competitions, some of which have been running for over a century, some of which have been the focus of research. The form of engagement in math competitions versus robotics competitions however differs in that the latter is more visible and collaborative (in contrast to individual achievements) and sometimes has a higher cost of entry for participation. That competitions and contests are part of the landscape should prompt reflection and inquiry however as to the influences on robotics education in schools and how this has been and is being shaped. This remains an open question.

Changes to work and workforce

The McKinsey Global Institute Report (2017) on workforce transitions in a time of automation notes that very few occupations, less than 5%, consist of activities that can be fully automated, though in about 60% of occupations at least one-third of activities could be. Those jobs most susceptible to automation are physical ones in predictable (stable) environments and jobs involving the collecting and processing of data. This will likely result in shifts in task allocations as robots and automated processes are not yet better at managing people and social interactions. As many as 800 million people may need to find new jobs by 2030. Though some job displacement may be offset by the creation of new occupational categories that do not currently exist, the challenge will be in ensuring that workers have the right skill-sets to transition quickly. The report also notes that, "We will all need creative visions for how our lives are organized and valued in the future, in a world where the role and meaning of work start to shift" (p. 20).

Given the strong association that has been made between mathematics and labour/workplace knowledge and skills in the political and public spheres, it is relevant to draw learners' attention to the potential disruptive impact of robots on labour and employment.

Robot racism & sexism

Although robots are playing increasingly larger roles in the social sphere with children, the diversity of robots remains limited. Most robots are either stylised as white or metallic (Beswick, 2018). Researchers demonstrated a shooter bias for robots racialised as black. Several examples also demonstrate that some decision algorithms used for artificial intelligence systems produce behaviour that resembles racist and sexist human ones. The biases of the person programming algorithms matter as well as the training data, sets of experiences that one draws upon. There are good analogues for mathematics teaching.

Kyriakidou, Padda, and Parry (2017) note that there do not currently exist guidelines on robot ethics in ethical frameworks and there is limited discussion in research reports. They suggest teachers and researchers have a duty to explain a robot's operational nature to children and to report on this in their methodology. Other issues around preferential attachment for robots over humans and emotional distress due to perceived harm and not understanding the nature of a robot is also an emerging concern. With voice-assistants and A.I., features of social interaction and communicative norms of young children are being re-shaped (Rosenwald, 2017). Papert (1980) too had concerns about the influence of mechanized thinking machines on the development of children's values and self-image. This is a dimension to which we need also attend and draw learners' attention to.

CONCLUSION

We believe we are still at the beginning stages of studying the impacts of robots deployed in mathematics education towards an end of learning mathematics. We acknowledge that there is much still to be learned about teaching and learning with robots, in particular the specific mathematical opportunities that can be afforded. We see one productive avenue as to explicitly create opportunities to open the many black boxes encountered, which involves using mathematical ideas, tools and representations with others. We acknowledge as well that the increasing presence of robots in classrooms, homes and in communities will have transformative effects on human social relationships—a dimension of education we cannot ignore.

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RELATION, RITUAL AND ROMANCE: RETHINKING INTEREST IN MATHEMATICS LEARNING

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In this working group, we had as our background concern the challenge of evoking and sustaining interest in the mathematics classroom, interest that is mathematical in its focus (rather than a lure into mathematics). We approached this concern by working with three concepts that have some historical connections to the topic of mathematical interest but that also provide new and productive ways of engaging it. These concepts were ritual, romance and relation. By ritual, we wanted to foreground the classroom-wide establishment of interest that arises from working on mathematical objects in a communal and sometimes even chant-based manner. By romance, we were drawing on Whitehead's (1967) argument that it is in the stage of romance that we can develop interest, which is necessary for attention and apprehension. Finally, by relation, we wanted to re-think the assumption that student interest begins in the concrete/real-life and investigate ways in which a relational approach to mathematics concepts may also provide students with opportunities for form their own relations with mathematics.

DAY 1: 2 JUNE 2018

We began by watching a Nicolet film involving relations between circles, an activity that Alf led using the modes of working with film described in Gattegno (2007/1951). J. L. Nicolet was a Swiss teacher of mathematics who suggested that the mind does not spontaneously adopt a logical approach to the study of a subject but rather acts intuitively on the material presented to it. 'Intuition' is a dangerous word to use without further explanation: what Nicolet meant by it is the direct apprehension of the situation and the unconscious associations with it which together leave traces in the mind for future analysis and rational consolidation. In this sense, intuition is the ready grasping of a situation in terms of knowledge already existing in the mind, which carries with it a high degree of conviction. Nicolet himself believed that we cannot know unless we use this intuitive process, though it is not sufficient by itself: it must be consolidated logically.

When applied to geometry, this proposition means that in each geometrical fact or set of facts (theorems and riders), there will be, for a given class of students, some intuitive basis which it is the job of the teacher to discover and use. Nicolet's approach was to animate the figures involved in a theorem and to try to view the result as a case standing out from an infinite number of possibilities; for example, the right angle when compared with acute and obtuse angles.

Nicolet's films are not merely illustrations: they are tools for teaching and research. As tools for teaching they cannot replace the teacher, she must use them; and she can use them in many ways. The explanation is left to her and so is the follow up work on the problems. All the films are intended to do is to enable the child to grasp a geometrical fact and be convinced of its truth—to enable them to see the internal structure of a set of results. The child is able to assimilate the total situation. Questions will arise, the consideration of which will give an opportunity for training her power of analysis and foster a demand for a mathematical proof for his conviction. The films are also useful tools for research. Indeed, because their content is selected for their simplicity and deals with the fundamentals of geometry and because their method appeals to deeper mental processes than syllogistic and abstract reasoning, they can be used to investigate the processes of the analytic mind. The films reveal the meaning of geometry in a way that enables pupils to talk about the facts as living reality.

Alf began by asking the group to imagine a series of transformations of circles that was not always difficult to follow. After slowly working through these images, we watched the Nicolet film which extended beyond the mental visualising that Alf had led us through. The transformations of the circles produced different types of loci (parabola, ellipse and hyperbola) but in ways that most participants were not familiar. As we worked on the visualising, several people commented on how it was difficult to see motion and sometimes they would take a snapshot of what they were imagining in order to be able to hold on to it. Others remarked on attention to velocity and to colour and to their own bodies in relation to the circles. There was a feeling of recognition when watching the parts of the film that Alf had asked the participants to imagine and there was a feeling of mystery around the last part of the film (in which the locus of a hyperbola is produced).

After a discussion, we invited the participants to engage in the following questions:

- Conics are a group of curves, which include the ellipse, the parabola and the hyperbola (as well as the circle!). Having watched the film *Common generation of conics,* there may be some questions that arise for you and we would like you to work on them!
- For example, where is the moving circle smallest? Where is it biggest?
- Are the curves, generated in the film, actually conics?

- To answer this last question, it may be interesting to know some properties of conics and consider them, arguing from the way the curves are generated in the film.

Distance properties

- For any point on an ellipse, the sum of the distances to the two foci is a constant.
- For any points on a parabola, the distance to the focus is the same as the distance to the directrix.
- For any points on a hyperbola, the difference between the distances to the foci is a constant

Reflection properties

- Any ray that leaves one focus of an ellipse, will 'bounce' off the curve and hit the other focus.
- Any ray that leaves the focus of a parabola will bounce off the curve and travel parallel to the line of symmetry of the parabola.
- Any ray that is heading, through the curve, towards a focus of a hyperbola, will bounce off the curve (before it hits the focus) and meet the other focus.

Algebraic properties

- An ellipse has an equation of the form: $x^2/\, a^2 + y^2/b^2 = 1$
- A parabola has an equation of the form: $y^2 = 4ax$
- A hyperbola has an equation of the form: $x^2/\, a^2 - y^2/\, b^2 = 1$
- (Proving algebraic properties, from the film, comes with a health warning however, particularly for the ellipse and hyperbola).

This activity worked as a backdrop for thinking through the idea of romance. Before delving into how Whitehead used this concept, participants were asked to reflect on how they saw their experiences in the mental visualisation and in watching the film as being related to romance.

- Some talked about romance not being comfortable, coming apart, being lost, involving a struggle—therefore romance was not like interest which is always framed as a positive thing.
- Others evoked the sense of romance as a space for collective intuition.
- Some evoked the sense in which the romance was in the big open space that was originally produced in the imagining and the film (the circle could be any size, anywhere on the screen, moving around), and there was progressive precision as the circle took on certain roles, as in a narrative, and finally, developed participate relationships.
- Some evoked the psycho-analytic third space

We closed this first day by inviting the participants to think back to the main activities of the morning and to write about which ones seemed to be related to romance.

DAY 2: 3 JUNE 2018

On day 2 the idea of romance was introduced and, in particular, Whitehead's notion of romance:

> The stage of romance is the stage of first apprehension. The subject-matter has the vividness of novelty; it holds within itself unexplored connections with possibilities half-disclosed by glimpses and half-concealed by the wealth of material. In this stage knowledge is not dominated by systematic procedure. Such system as there must be

> is created piecemeal ad hoc. We are in the presence of immediate cognizance of fact, only intermittently subjecting fact to systematic dissection. Romantic emotion is essentially the excitement consequent on the transition from the bare facts to the first realizations of the import of their unexplored relationships. (Whitehead, 1929, pp. 10-11)

Sean invited participants to work on what this might mean for them through watching film (as in Day 1) but this time, films of a teacher and students in a classroom. A notion that emerged through the course of the day was of romance as involving something *half-revealed and half-hidden*. Sean showed the group a film of Caleb Gattegno, working with primary age children, making use of Cuisenaire® rods (https://www.youtube.com/watch?v=Kw94gmzRrOY&t=7s; Sean showed sections 0:00 to 1:55 and 15:40 to 19:19). In the clip, the class (a French language classroom) are working on doubling and halving and there is a way of working in which Gattegno asks for responses from the class in unison.

Gattegno developed a way of working with Nicolet's films in the classroom, which began with the class reconstructing what took place. (Alf had used a variation of this way of working on Day 1, getting participants to evoke a mind image before showing the film). In the 1980s, John Mason (personal communication with Alf Coles) had adapted Gattegno's way of working for use at the Open University with teachers and films of classrooms. In this Open University way of working (described in Jaworksi, 1990 and recapitulated in Coles, 2013), there is a discipline of using short clips of films and starting by getting participants only offering 'accounts of' what they saw, before later moving to interpretation. It was this way of working that Sean used for work on the clips of Gattegno.

Having worked on the two selections of the Gattegno film, Sean also showed a clip of Dave Hewitt teaching 12 to 13 year olds, also making use of chanted, choral responses from students. Hewitt made the following comment at the six-minute mark, which Sean projected on the screen for the group:

> I use the idea of the class speaking in unison in a sort of chorus way on a number of occasions. In this occasion I wanted to use it because you get carried along, with… You get soaked up in what's going on and also you contribute, because of that. You are committing yourself to saying that next number and there's a self-checking element in that there were times during that lesson when people said a different number and they can be clearly heard that there are two or three numbers being said and that means that everyone has an opportunity to reflect on…, hang on, did I say the right number, did I say the wrong number and they have to work on that and reflect on that. (Hewitt, 1991)

Some of the comments from participants, when we moved to interpretation were as follows:

- Moments of pause, there is romance in the apprehension of 'what next?'
- In chanting, students are in a rhythm of ritualistic behaviour, the teacher moves (e.g., disrupting a sequence of questions, or asking an inverse question) change ritual to a romantic situation
- There is a dynamic between romance and precision
- Just as the class get used to a ritual the teacher shifts, adding uncertainty, fear, wondering
- Students have to be paying attention, perhaps there is a power in this way of working for students who can be 'slow processors'
- Ritual automates the details and allows your mind to wander; metacognition needs this
- Ritual behaviour gives space for students to see patterns develop but does not push that

- Shared experiences become a resource for teachers
- Participating in a ritual makes you part of a community or group but ritual without thinking about what you are doing means you are only being part of a community (and not learning)
- In contrast to the point above, we do not need to privilege ‘thinking’ over ‘acting’. What if we do not split thought and action in the first place? This is not advocating rote learning; but rote practices can look like rituals. What are the boundaries or shifts between rote and ritual?
- ‘Story’ is a feature of both romance and ritual. An engaging story half reveals, half discloses, perhaps; a ritual has a familiar story-like feel of knowing what will happen next but also there being surprises.

Part of the interest generated during this day was around Sean’s explicit metacommentary on using the Open University way of working on film and, for instance, the difficulties of keeping to ‘accounts of’ phenomena on the film, before moving to interpretation. There was a ritual in the way of working and the ban on starting with interpretations provoked a sense of the half-revealed, half-hidden that could be characteristic of romance. In other words, there was ritual and romance present in the way of working as well as in the ways of working in the classroom that were evident in the films themselves.

DAY 3: 4 JUNE 2018

On day 3, we wanted to explore the relations that form in the doing of mathematics. We wanted to shift from the belief that mathematics is something that one has to be motivated to do, instead, rather, to potentially see that mathematics is grounded in the relations that are forged through doing mathematical activities. This goal was explored by engaging in the topic of projective geometry led by Nathalie.

Projective geometry is a geometry of lines and points. It can be thought of as a geometry that does away with most of what concerns Euclidean geometry: there is no circle; there is no measure—that means, no right angles, no congruent sides, etc. At first, thinking in terms of only points and lines might seem a little boring, but projective geometry has some amazing results.

One of the purposes of introducing projective geometry was simply to explore a topic that most participants were not familiar with. This was a bit of an assumption but with the heavy emphasis on Euclidean geometry in most curricula across Canada, we thought this was reasonable. Also, we thought mathematical thinking would emerge through the process of rethinking previous definitions and assumptions of geometrical objects and phenomena found in Euclidean geometry. For example, projective geometry is based on lines of perspective or lines of sight rather than a perspective from ‘above’. In Euclidean geometry, there is no perspective or difference in terms of how you look at things. Angles and other such notions do not change depending on how you look at them. Think of an angle of 28 degrees; does it change if you tilt your head?

One of the tenets of projective geometry is that parallel lines meet at infinity. Think of when you have looked at railway tracks and noticed that they meet at the horizon. This line of sight, towards the horizon, presents a different property than if you look straight down at the tracks, in which case the tracks look parallel in the classical sense. This makes projective geometry a less objective perspective and more about how one positions themselves. In one of the activities on this third day, we constructed a harmonic net of quadrilaterals, namely quadrangles. To begin this construction, one starts with a straight line, which acts as the horizon line and two points are plotted on the line (see P and Q in Figure 1). These two points act as points at infinity (the end of the railway tracks). One then draws multiple lines with various slopes through these two points and multiple, tessellated, yet different sized quadrangles begin to emerge (for example, see quadrangle ABCD). Since parallel lines meet at infinity, AB and CD (as well as BC and AD) are parallel. It is also evident from Figure 1 that the quadrangles vary in size (see the yellow, green shaded quandrangles in Figure 1b). They are smaller closer to the horizon line and get larger as you move away from the horizon line, reflecting once again a phenomenon that we experience every day in the world in terms of our own lines of sight. From the ground, the lower windows on a skyscraper will appear larger than the windows higher up.

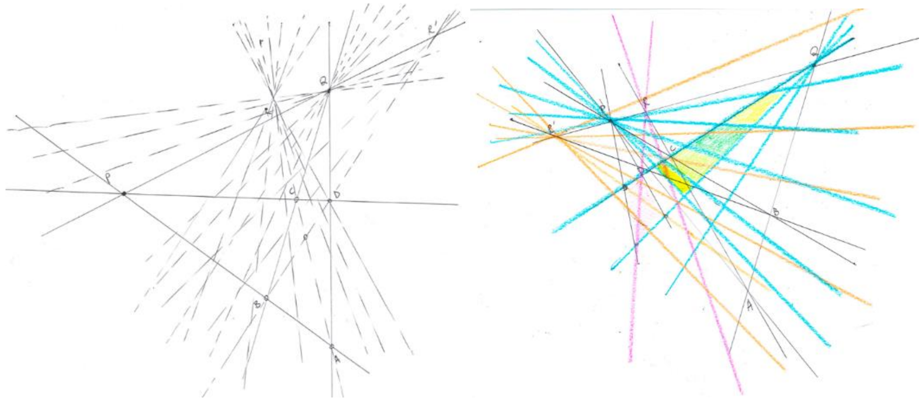

Figure 1. Participants' constructions of a net of quadrangles.

One of the interesting findings in this activity is that all quadrangles with a diagonal passing through R will have its other diagonal pass through the same point, R' (see Figure 2).

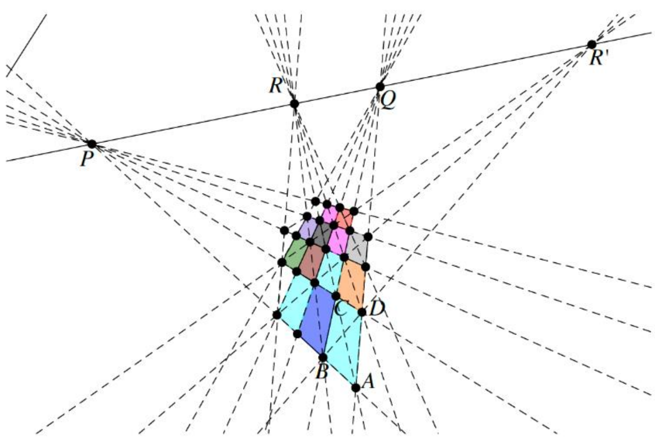

Figure 2. The lines passing through opposite vertices of the quadrangle all meet at R'.

The personal connections to the mathematics on this day was vivid. Participants drew lines, erased, redrew lines, looked for quadrangles, chatted, quit. These actions forged relations between group participants and also between participants and the mathematics of projective geometry. At the end of day 3, a reflection from one of the participants drew on activities from both day 2 and day 3:

> *Interest can come from engaging in a problematic situation or a ritual that is purely mathematical in nature. The students in the [Hewitt] videos were not interested because what was happening relates to some real-world context happening outside of the current one; or some skill they need to develop to participate in that Real [sic] world outside. They seemed interested because they were engaging in some shared activity…with moments of feeling as though one has mastered something and can participate with moments of feeling unsure, but perhaps (or hopefully) eager, excited, or something of that sort due to suspense.*

The projective geometry activity provided an opportunity for participants to suspend understanding but yet still engage in mathematical activity. One of the ways this suspension was collectively voiced was through the construction of points of infinity. Projective geometry is more than different perspectives or lines of sight. It also draws on the notion of infinity. Points on the horizon represent a point at infinity. Caleb Gattegno is once said to have described mathematics as 'shot through with infinity'. Creating infinity through construction materialized mathematics in a very different way than many of us were accustomed. The reaction in our group was profound. How can we position a point at infinity? When Nathalie projected the quadrangle net using *The Geometer's Sketchpad* (Key Curriculum Press, 2001) on the front screen, we all watched intently. Nathalie dragged the points of infinity along the horizon line and the properties of the quadrangles remained (for example, the lines through R and R' retained their common points). Other points were dragged and the quadrangle net remained mathematically cohesive and connected. The points on the horizon line, even when dragged off the screen retained a mathematical tangibility whereby infinity came into being.

One of the participants in their concluding reflection wrote, "*Interest is achieved in the balance between the: known and new; ritual and unpredicatable [sic]; ability to participate and observe safely. The new, unpredictable and safety was materialized in projective geometry*."

Projective geometry is interesting for the simple fact that while on the one hand it is not connected in a direct way to the real world (when was the last time you created a quadrangle?), it does create a positional perspective in terms of seeing the world. We drew some practical and accessible advice from Dick Tahta (1989) in his article "Is there a geometric imperative?". In it, he writes,

> It is not good enough to offer adolescents merely more of the same experience they have had when they were younger. It is not good enough to use something as important and pervasive as geometry merely as a convenient medium for public examination and the selection of suitable candi¬dates for higher education. Nor is it good enough always to algebracise geometrical experience, or to do so prematurely. What we do need to do, is to think in terms of the concerns of students themselves at this stage, and to see in what ways geometry can offer them something pertinent to their intellectual, emotional, social and spiritual needs. Of course, this is easier said than done…But it is the right of adolescents to explore complexity; and it is the duty of teachers to help them maintain it. (p. 28)

We also drew on Anne Watson's remark in her plenary talk at the 2008 CMESG about the fact that the particular needs to adolescents do not require that abstract mathematics be discarded in favour of relevance or 'real world' mathematics—indeed, some kind of escape is sometimes just when they want (see https://files.eric.ed.gov/fulltext/ED529561.pdf).

That was an interesting point for us in relation to the choice of activities and to people's assumptions that you give meaning to the teaching of mathematics by showing how it can help you balance your budget, for example. The engagement with a topic that was 'shot through with infinity' and that mattered in terms of how one can choose to see, created a space for participants to simply engage mathematically and create ways of talking about and sharing their experiences. As one participant wrote, we need to offer opportunities for students to experience mathematics. And another concluded: "*Perhaps it's time to embrace/integrate/blend seemingly controversy of perspectives and develop a balance in teaching math in the classroom*".

SUMMARY

In reporting back to our colleagues, we chose to re-enact a mental imagery exercise with the whole group. Then pairs of participants stood up and articulated specific ways in which this visualization activity related to romance, ritual and relation.

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Topic Sessions

Séances thématiques

WHAT MY GRANDSON TAUGHT ME ABOUT LEARNING MATHEMATICS

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ABSTRACT

My grandson Liam is three years old. Since he was born, I have been observing him as he struggles to learn the skills he needs and as he discovers the world around him. And, while as a mathematician I know that I should not make generalizations based on a sample of one, nevertheless these observations do merit reflection.

It seems that we are biologically programmed to learn. We like to practice, and to keep practicing until we master things. Then we like to proudly show our accomplishments (and enjoy the praise). And then we look for new things, new challenges.

We are not concerned with our limitations, real or imagined. And we definitely do not feel any math anxiety.

Can we retain this excitement and drive to learn throughout our lives? Can we help those who have lost the desire to learn—or possibly the faith that they can, which is not that uncommon in relation to mathematics—to regain it?

Having grandchildren is a wonderful experience and much easier than having children of your own. You are not as tired and as stressed out, so it is much easier to observe their development and reflect on your observations.

A disclaimer: I am not a child psychologist and only an amateur math educator—I did not have any formal training in either psychology or mathematics education, except reading books and articles, and talking to (or listening to) educated colleagues at conferences. What I say here therefore are just my observations and reflections, and they will probably not be new to many of you. In fact, while working on this talk, I found an article by Susan Engel, titled "Joy: A subject schools lack", published in *The Atlantic* on January 26, 2015, that contained many ideas I had 'discovered' while observing Liam.

The main thought from Susan Engel's paper that I felt describes just what I feel is the following: "The thing that sets children apart from adults is not their ignorance, nor their lack of skills. It's their enormous capacity for joy" (para. 5). I would say 'joy and wonder' as it is both I see in Liam's eyes, and now also in Izzy's, his little sister.

My grandson Liam lives in Chilliwack, and I see him on average once a week. Once a week is often enough to get to know him well, and, at the same time, leaves enough time in between visits for reflections on our interactions.

My observations really started when Liam started to show awareness of his surroundings. I quickly noticed his enormous curiosity and determination. Whenever he sees something new, he watches it so intently that it often seems he is eating it up with his eyes. He loves walks in the park; watching trees, ducks and geese on the lake; and throwing rocks from the bridge into the water. For months, he was trying to find where the baby on the other side of the mirror, who was looking at him so intently, was hiding. He got it eventually.

Figure 1. Watching ducks at Como Lake park.

On a walk a couple of weeks ago, Liam suddenly started to play the following game: he would say, “With my eyes, what do I see?”, and he would look around and look for something interesting. Sometimes he would say, “With my ears, what do I hear?”

The game lasted all the way home. Where did he learn this? Maybe his daycare, maybe one of the recent books: I remember one with words “Baby Bear, Baby Bear, what do you see?” (Martin & Carle, 2009, p. 1). No matter—he took it very seriously.

He loves books and asks that we read them to him again and again. Then, from time to time, he takes a well-read book and says, “I will read it now”—and re ‘reads’ it to himself, since has memorized it already—but the familiarity does not diminish the joy!

He enjoys learning the alphabet and finding opportunities to practice it. He would notice the alphabet next to a slide in the playground, and read the letters aloud, as much as he could. One day, after a bath, I gave his father a towel for him, which had some words on it. Liam exclaimed joyously, “There is ABC on it” and tried to find all the letters.

Liam likes counting. Just as other children are, he was taught to count and likes to count everything, especially if somebody says “how many”: ducks in the park, cats on the windowsill, strawberries on a plate. But sometimes he still gets the order wrong: one, two three, four, two, five, … At the same time, there have been situations where he demonstrated a deep understanding of numbers.

One day last April, we went to a park together. Liam found a nice pine cone and decided to take it with him. When we were walking back, he said, "At home, I have another pine cone." I replied, "So now you have two cones, because one and one makes two", and showed him this again on my fingers. We did not discuss this further, but I told the story to his parents when we reached home. I thought that, maybe next time, I will tell him that two and one makes three.

It turned out that there was no need for this. A few days later my son called and told me that my arithmetic lesson paid off. While Liam's mother was giving him a bath that day, she said, "Liam, I bathed you yesterday, and I am bathing you today, so tomorrow your father will give you a bath." Liam thought for a while and then said, "No Mummy, I want you to bath me tomorrow. I want you to bath me three times."

Several months ago, when he was about two and a half, I realized that he is quite capable of abstract reasoning. He has a cardboard house, and he likes to move his animal 'friends' into the house. That day, he started to move his favourite wooden cats into the house. After a while, he would say, "I need one more cat in my house", bring one more, and say again, "I need one more cat in my house." After he moved all the cats, he said, "Now I need cows in my house." After a while, he came out and said, "I need more animals in my house." Then he said, "At home, I have two whales, Douglas and Buttercup."

Figure 2. Favourite cats and other friends.

So, what have these, and many other discussions with Liam, taught me?

First, probably the obvious one: that a desire to learn, to explore our surroundings is something we are born with.

Second (which I knew for many years and which follows from the first), that play is learning, and learning is play.

Third, that practice is an essential part of learning, and that children love to practice. They want to master the skills they are learning, and they find their own opportunities to practice: counting cats, or finding the ABC in a playground or on a towel.

Fourth, that recognition (by others) of newly learned skills, and the work that went into acquiring them, is important and encourages further learning. Liam is proud of what he has learned, and when he shows you, he looks at you and waits for your praise and lights up when the praise is given.

Fifth, that children are smarter than we think. They are more observant, they remember everything, and they start to generalize and are able to think abstractly much earlier than we expect them to (definitely earlier than I expected!).

Sixth, children need to be able to explore what they want at a given moment—or, occasionally, nothing at all.

Seventh, that there is always something more to explore joyously—as long as we do not kill that joy.

Figure 3.

What happens to wonder and joy when we grow up? Do you see wonder and joy in your students' eyes? Where and when does it disappear?

Quoting Susan Engel (2015) again:

> "Becoming educated should not require giving up joy but rather lead to finding joy in new kinds of things" (para. 7).
> "A child's ability to become deeply absorbed in something, and derive intense pleasure from that absorption, is something adults spend the rest of their lives trying to return to" (para. 5).

So, how can we help our students rediscover the joy of learning mathematics?

First, we need to create a safe environment: one cannot feel joy if one feels fear. We need to respect and like our students, and we need to expect them to succeed.

Second, we need to learn from our students. We need to listen to them and try to understand the reasons for what we perceive as their 'lack of motivation': these are often much more complex that we think.

One of my colleagues from Simon Fraser University, Veselin Jungic, came up with the idea of asking his students to write (anonymous) reflections titled "I wish my instructor knew that …". I have started using this in my courses, too, and it has made me look at my students very differently.

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PROBLEMS WITH NUMBERS: AN ONTARIO PERSPECTIVE

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The context for teaching (and thus learning) foundations mathematics in a community college setting presents unique challenges to faculty who work there. In this topic session report, I will provide a brief outline of my search for guidance to help improve my teaching of those who come to our colleges with many gaps in their mathematical knowledge, and who have no desire to become mathematicians. It is a process through which I begin to re-imagine the teaching of what is often called remedial or foundations mathematics in light of that context and what we know about numeracy.

In particular, I will:

- describe the unique challenges that teachers of foundations mathematics in Ontario colleges face.
- critique Ontario colleges' response to an identified numeracy gap (Orpwood & Brown, 2015).
- compare results of an informal survey of Ontario College Math Association (OCMA) 2018 conference attendees to attendees of this topic session. Topics in a typical college foundations math course were presented and respondents decided whether they should be a part of a course in numeracy or not.
- present a conceptualization of numeracy that allows math educators to think about what it could mean to teach numeracy as separate from mathematics in a foundations mathematics course, and clarify and address the numeracy gap that is of concern to many.

PERSPECTIVE

I do not have, nor have I had any formal role in shaping the way mathematics is taught throughout the college system. This report is a part of a personal exploration not as a representative of the college where I work.

I have been a teacher of mathematics at the high school level (15 years) and college level for another 15 years and have struggled with making both high school mathematics and foundations college mathematics courses meaningful to those students, by playing around with how I teach, but more often being frustrated at my lack of a clear grasp of what it is that I am teaching.

ONTARIO COLLEGE CONTEXT

WE ARE SPECIAL

The Ontario Colleges of Applied Arts and Technology (CAAT) are a product of government policy and have a well described mandate that differentiates them from universities. Four core principles include the embrace of total education; development of curricula that meets cultural aspirations and occupational needs; work with business and industry as well as social and public agencies; dedication to research in education and administration (Ontario Department of Education, 1967)

The 4 core principles are clear; the bureaucratic structure of colleges is less so. Each college in Ontario has a president who is (more or less) responsible to a board of a mixture of appointed and elected, internal and external governors. Sheridan College is unique in that it has a 72+ member faculty senate which has a say in academic policies and programming (Sheridan Senate, 2017). In the other 23 colleges, it is the exclusive function of the College management to make academic and programming decisions, and this is enshrined in faculty contracts (OPSEU/SEFPO & College Employer Council, 2017). Ontario colleges have the lowest funding per Full Time Equivalent (FTE) in Canada, with Saskatchewan having FTE funding roughly double that of Ontario's at just under $9,000 (Colleges Ontario, 2018).

In 2015-2016, 67.7% of Ontario College applications were from students who did not come directly from high school (Colleges Ontario, 2016). Strong secure employment, and avoidance of precarious and poorly paid work, is the draw for many individuals including those who have partially or fully completed Ontario University degrees. Ontario colleges are playing to that need in their marketing campaigns, while simultaneously supporting government bills that undermine strong secure employment (College Employer Council, 2018).

MATH EDUCATION AT THE ONTARIO COLLEGES

TERRA INCOGNITA—NUMERACY DIMENTICATA

College mathematics education and the faculty working there remain an under-researched area of math education in Canada. Even those of us who attend OCMA events have a weak understanding of what our colleagues do and what it is that they teach. The only comprehensive research that I am aware of in which college math faculty were of interest was funded by the now disbanded ministry of Human Resources and Skills Development Canada (HRSDC) as part of a project titled "Understanding individual numeracy: How are we doing? Does it matter?" (HRSDC, 2009). It spawned a few reports that introduce readers to the uniqueness of college faculty in Ontario and Canada. In their survey of 99 college math faculty, Maciejewski & Matthews (2010), HRSDC (2013) found out that only 49 had at least one math degree. Who knew?

Teaching mathematics courses at a college can be a lonely endeavor as most colleges do not have a mathematics department (only eight of 24 are known to have one). This means that faculty teaching mathematics rarely interact and have no formal role in decisions about the teaching and learning of mathematics at the colleges. This has led to strange phenomena: inappropriate textbook choices, disappearing courses, designating math course to have solely multiple-choice testing. Many students, as well as fellow faculty and chairs (who are managers), see mathematics as peripheral to the core courses that the students are required to take. My spidey senses tell me that this may be a sign of avoidance of difficult material (i.e., one must keep failure rates low), but I must admit that seeing math as peripheral (though compulsory) could also be a sign that some of the course content is truly unrelated to the program of study.

Due to the structural constraints and disconnect between the classroom teachers and decision makers, numeracy has become no more than a tick box on our course outlines—an 'essential employability skill' indicating that numeracy is a priority and being learned, practiced and/or evaluated. Nevertheless, the only direct reference to numeracy like tasks, *execute mathematical operations accurately*, is perhaps accurate but inappropriate.

There is hope however: our current collective agreement includes, for the first time, a clause that explicitly grants faculty academic freedom (OPSEU/SEFPO & College Employer Council, 2017). This may spur many of us into action.

FOUNDATIONS MATHEMATICS AT ONTARIO COLLEGES

DASHED HOPES

A strong case has been made that colleges can and should be doing a much better job in helping students strengthen their mathematics and/or numeracy skills. After over five years of data collection, the College Math Project (CMP) renamed the College Student Achievement Project (CSAP) launched an initiative called the Numeracy Gap (Orpwood & Brown, 2015), which after some media attention including a panel discussion at the Economic Club of Canada in the spring of 2016, has not translated into any system wide initiatives to rethink how foundations mathematics is taught in the various college contexts, nor guidance in what it would take to narrow the gap. Instead, the ministry, college leadership, and researchers focused on developing an assessment tool, the Ontario College Math Test (OCMT), which I am afraid, perpetuates the status quo of our current teaching approach in foundations math.

As described in the Assessment Development Project: Final Report (Orpwood & Brown 2014), the OCMT assessment framework, and the 72 items in this test came from 9 topics gleaned from a collection of course outlines, assessment tools, and commonly used textbooks for foundation mathematics courses, with a nod to Trends in International Mathematics and Science Study (TIMMS; International Association for the Evaluation of Educational Achievement, 2019). The steering committee of the OCMT, comprised of two college faculty, nine college managers, two high school teachers, and a publisher/technology partner, reports to a Coordinating Committee of College Vice Presidents Academic (CCVPA). I took a quick look at the sample five question pre-test on the OCMT site and did not see anything beyond calculation and recall questions (see Figure 1, OCMT, 2017).

Evaluate

$$21 \div 7 \times 11^2 + (12 - 5)^2 \times 2 = \boxed{}$$

Figure 1. Sample question from OCMT (2017) website.

It would be hard to argue that competence at evaluating the string of operations with numbers in the copied example is necessary for a person coming into or graduating from a large majority of college programs. You may disagree. The topic 'order of operations' (nemesis to many foundations math students) may be important to building mathematical skills (like the soccer player doing fancy kick-ups and demonstrating ball control) and an important step to what I like to call algebraic gymnastics, but is not one that we need to insist on in the typical college setting.

The OCMT purports to test numeracy separate from mathematics, but the numeracy items in the test include 36 questions systematically selected out of the full 72 item pool (grouped into nine topics similar to the 20 topics reported on in the survey below). This is surprising, as the CSAP report had a much more nuanced view of numeracy.

It is notable that despite the seemingly unanimous sense that we are doing poorly at helping students become more numerate, and given that the typical college foundations math course focuses on the procedural and computational aspects of arithmetic and algebraic procedures, there seems to be no interest in examining the content of the courses (and assessments) as to their relevance, yet we are consistently unhappy with student results.

WE ARE TEACHING, BUT DO THEY LEARN?

HOW MUCH NON-LEARNING IS ACCEPTABLE?

Two old friends meet after a long absence, and one remarks to the other, “I didn’t know you had a dog?” “I not only have a dog,” responds the second friend, “I have been teaching it how to whistle.” The first friend crouches down: “Come on boy, give me a whistle.” Tail wagging… no whistle. Looking up incredulously he hears his friend say, “I didn’t say that he learned.” (Thanks to Paul Balog, from whom I first heard this.)

As part of the aforementioned HRSDC funded research, I led a study that examined the efficacy of an adaptation of the Jump math program to the college setting, through a randomized field trial (Gula, Hoessler, & Maciejewski, 2015). We used the Weschler test of adult numeracy (Wechsler, 2009), whose content focuses on the execution of mathematical operations and mirrors what is currently the focus of many foundations mathematics courses, as a proxy for numeracy. Though Jump math student respondents’ mean improvements were higher than non-Jump math students, what became apparent was that 29% of non-Jump respondents, and 20% of Jump respondents had improvement scores ≤ zero (post-test - pre-test). The teachers taught them, but they did not learn.

SURVEY SAYS

CONTENT IS KING

Though many education theorists may cringe at the practitioner focus on learning content, nevertheless, I will forge ahead with the survey results. I chose 20 topics (from a typical first year foundations math course) and solicited reactions of an unscientific sample of OCMA 2018 attendees versus CMESG 2018 topic session attendees to the question:

Would you include the topic in Table 1 ‘for sure’ in a course in numeracy?

	parabolas	quadratic equations	working with variables (x and y)	short cuts with positive exponents	negative exponents	linear equations	arithmetic with algebraic terms	solving equations	basic logic of sets	scientific notation
% OCMA	11.54	15.38	26.92	30.77	30.77	30.77	34.62	38.46	42.31	42.31
% CMESG	25.81	14.81	63.33	29.63	31.03	51.72	60.00	48.28	20.69	34.48
OCMA - CMESG	-14.27	0.57	-36.41	1.14	-0.27	-20.95	-25.38	-9.81	21.62	7.82

	Multiplication & division of negative integers	perimeter and area of basic shapes	Order of operations	placing numbers close to zero on a number line	division of fractions	Add and subtract negative integers	conversion between fraction/decimal/percent	percent of	ordering numbers close to zero	addition of fractions
% OCMA	69.23	69.23	80.77	88.46	92.00	92.31	92.31	92.31	92.31	96.15
% CMESG	51.85	93.10	58.82	82.14	70.37	77.78	100.00	100.00	92.59	96.55
OCMA - CMESG	17.38	-23.87	21.95	6.32	21.63	14.53	-7.69	-7.69	-0.28	-0.40

Table 1.

Results showed that despite the fact that there was general agreement (Pearson's r = 0.838), a notably higher rate of CMESG attendees would include algebraic topics like working with variables and linear equations in a course in numeracy, while on the other hand, a notably higher rate of OCMA attendee respondents were prepared to include topics like division of fractions, order of operations, and basic logic of sets.

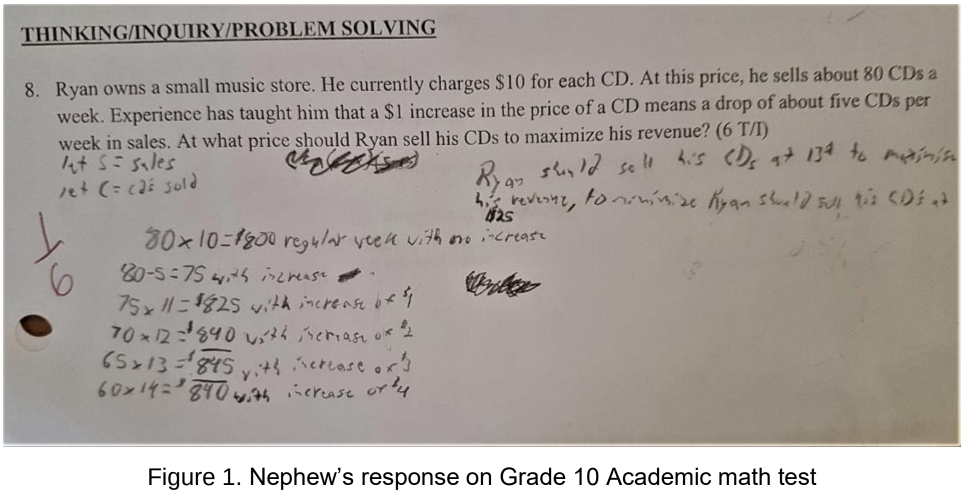
MATHEMATICS VS NUMERACY: WELL DEFINED SETS?

> "The question is," said Alice, "whether you can make words mean so many different things." (Carroll, 1872, p. 124)

BUILDING A FOUNDATION:

Based on the many words written about numeracy, the answer to Alice's question is unequivocally, "Yes!" In a recently published book, *Teaching Adult Numeracy: Principles and Practice* (Griffiths & Stone; 2013), David Kaye took on the tension between mathematics and numeracy directly. Though Kaye makes clear that numeracy is more than just mathematics 'lite', he goes on to conclude that "not having a precise definition of numeracy is a strength and not a weakness" (Kaye, 2013, p. 75). Unfortunately, neither Kaye, nor any of the other authors seem to have read a book published 12 years earlier *Mathematics and democracy: The case for quantitative literacy* (Steen, 2001), which was referred to in the CMESG 2012 working group on numeracy (Caron & Liljedahl, 2013), and which finally provided this practitioner the kind of foundation on which I could begin to design teaching and learning for those who are struggling at the high school or the college setting. (Note: quantitative literacy and numeracy are used interchangeably—perhaps they should not be.)

> In quantitative literacy, numbers describe features of concrete situations that enhance our understanding. In mathematics, numbers are themselves the object of study and lead to the discovery and exploration of even more abstract objects. (Manaster, 2001, p. 69)



THINKING/INQUIRY/PROBLEM SOLVING

8. Ryan owns a small music store. He currently charges \$10 for each CD. At this price, he sells about 80 CDs a week. Experience has taught him that a \$1 increase in the price of a CD means a drop of about five CDs per week in sales. At what price should Ryan sell his CDs to maximize his revenue? (6 T/I)

let S = sales
let C = CDs sold

Ryan should sell his CDs at 13\$ to maximize his revenue, to maximize Ryan should sell his CDs at 825

80 × 10 = \$800 regular week with no increase
80 − 5 = 75 with increase
75 × 11 = \$825 with increase of \$1
70 × 12 = \$840 with increase of \$2
65 × 13 = \$845 with increase of \$3
60 × 14 = \$840 with increase of \$4

6

Figure 1. Nephew's response on Grade 10 Academic math test

As an illustration of the usefulness of the definition, let's take a look at my nephew's response to a Grade 10 Academic math test question (parabola unit) in an Ontario high school using Manaster's conceptualization. My nephew described features of the concrete situation at hand and enhanced his understanding of it using numbers, and he got the answer right! His response showed clever numeracy skills, though after discussions with him after the fact I cannot figure out why he did not just use 'the formula' to get full marks. He was penalized for not applying the newly presented algebraic model (the more abstract mathematics) to solve the problem. One could argue about whether the penalty was appropriate in the particular context but that would be a digression. Using Manaster's conceptualization to think about the student solution, the goal of the teacher, etc. opens up a discussion in which it may become clearer to the teacher and student what it is that we are expecting students to learn in our classrooms, and what kind of thinking that we are trying to evoke from our students.

Solving the problem using the parabola as model is as much an act of quantitative literacy as much as the nephew's response. The teacher simply wanted him to use different mathematics. Using Manaster's conceptualization as a lens provides a fresh perspective on analyzing the student response and teacher expectations and could help all of us clarify our presentation of problems to students, in order that we elicit the approach that we want.

Though I see great potential for the adoption of Manaster's conceptualization, it has not been yet fully stress-tested as an effective way of thinking about what it is that those of us teaching first-year math students at the colleges want our students to learn. It is a project I believe will yield better teaching and possibly even learning.

NUMBER SENSE IS THE GLUE

NUMBER SENSE IS NOT SO COMMON

My bet is that if you write out the following challenge to a typical Canadian college student: "Find a number about half-way between 0.017 and 0.05", you will at worst get a blank stare and at best see the student reach for their calculator (or pencil and paper) in order to do a calculation. These habits are rewarded in our classrooms and the college context, and they work against the reinforcement of a strong number sense.

Number sense requires having an understanding of numbers and their relations (as abstract objects). It is at the root of both mathematics and numeracy. Stanislas Dehaene in a book called *The number sense: How the mind creates mathematics* makes it clear that numbers come from thinking quantitatively, that biologically sapiens have evolved to be able to distinguish one, two, and three objects, but not beyond that, and that by the age of three children know that number words are a distinct category, like colour words, connected to concrete objects. He is describing the beginning of the development of number sense (Dehaene, 2011). Using Manaster's conceptualization, one can see that developmentally quantitative literacy precedes mathematics and that the learning of mathematics requires a child (or adult) at some point to separate out numbers from quantities and see them as objects (as written symbols) themselves, which can have properties and relations, etc. Very cool: I highly recommend it!

MANASTER IN A FOUNDATIONS MATHEMATICS CLASSROOM

USEFULNESS

What will a foundations math class look like if the above conceptualization is taken as a starting point in developing both the content and the format of teaching? I do not know. What I do know is that the status quo will have to change.

If we take Manaster's conceptualization as the starting point we will have to change what it is that we are expecting students to learn and to clarify this for ourselves as teachers first.

In rethinking the mathematics content of a foundations course, we need to consider just how much abstraction is needed, but we should not shy away from abstraction. Remember that in a first-year college foundations math class, it is safe to expect that all students will have all had sufficient dose of mathematics in previous school experiences. Which topics need to be brought back? Let's start with number sense. Since it is at the core of mathematics and numeracy we need a shift to ensure that number sense is explicitly at the core of what we do with strengthening student understanding of mathematics.

Number sense goes well beyond the natural numbers. Complex numbers? Perhaps not. What about zero? How deeply do we want students to think about zero? Infinity? Multiple infinities?

What about integers? I still have operations with negative numbers included in a foundations math course that I teach, but it would not take much of a push for me to remove the topic from the course. Why remove it? Let's flip the question around: Why keep it? If the course goal is to help strengthen numeracy, and if the kinds of concrete situations that we are preparing students for can be understood without the subtraction of a negative number from another negative number, then why keep it?

The numeracy piece will be much more of a challenge in part because of the expectations that students bring to a mathematics class. Manaster's conceptualization of quantitative literacy compels us to design content that aims to enhance the student's ability to use numbers (and other abstract objects) to describe features of concrete situations that enhance our understanding (of the situation). This is an inversion of the aims of a course in mathematics where we use concrete situations to enhance the understanding of mathematics (to ever higher abstractions). Ouch!

There are a myriad of numeracy activities out there in the Google world. Manaster's conceptualization can act as a filter for us to classify them as serving mathematics or serving the concrete situation at hand. It is important that we have both, and I hope I have shown that in the college context, at least, it important that they be separated out.

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APPRENDRE DANS UN ENVIRONNEMENT RICHE EN TECHNOLOGIES NUMÉRIQUES : COMMENT LES COMPÉTENCES DISCIPLINAIRES INTERAGISSENT AVEC CELLES APPELÉES « DOUCES »

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RÉSUMÉ (EN ANGLAIS)—SUMMARY (IN ENGLISH)

Since the beginning of the 21st century, new learning spaces, rich in digital technologies, have given a boost to numerous initiatives in schools. For example in New Brunswick, several studies were conducted: one-to-one access to laptops (Freiman et al., 2011), robotics-based learning (Savard & Freiman, 2016), mathematical problem solving in a virtual learning community (Freiman & DeBlois, 2014), and, more recently, makerspaces (digital manufacturing labs; Freiman et al., 2017) and computer programming (Djambong, Freiman, Gauvin, Paquet, & Chiasson, 2018). In addition to creating new opportunities to enrich and eventually transform student learning, these innovative practices highlight a complex dynamic of interactions between disciplinary learning (in mathematics) and new types of skills called 'non-technical' (21st century, or soft-skills) which have rather transdisciplinary flavour. In my presentation, I discussed theoretical perspectives and shared some research findings that sheds light on both the benefits and challenges of these changes regarding their appropriation by the educational community.

INTRODUCTION : BESOIN DE COMPÉTENCES NOUVELLES AU 21E SIÈCLE

La séance thématique a invité les participantes et les participants à réfléchir sur la complexité des enjeux nouveaux que les environnements riches en technologies apportent à l'enseignement et l'apprentissage de mathématiques et surtout sur le rôle de nouveaux types de compétences qui sont de plus en plus mises en valeur par la société et, par ricochet, par nos systèmes éducatifs.

J'ai commencé ma présentation par deux mises en scène. La première (Figure 1) montre un collage à l'entrée d'une exposition provinciale de laboratoires de fabrication numérique (fablabs) connus sous le nom de labos créatifs (Brilliant Labs, 2017) qui sont en croissance dans les écoles du Nouveau-Brunswick et ailleurs au Canada Atlantique depuis 2014. On y retrouve plusieurs photos de génies de sciences de tout temps. On y voit également les photos de quelques élèves qui présentaient leurs projets réalisés dans ces laboratoires. En plus de permettre aux élèves d'explorer différentes technologies de pointe dans le domaine des sciences, des technologies, de l'ingénierie et des mathématiques (STIM), ces projets visent également à développer diverses compétences dites « nouvelles », aussi connues sous les noms « globales », « douces », « transversales », « essentielles », ou celles « du 21^{e} siècle »; les appellations ne manquent pas sans toutefois donner un sens clair au phénomène qui risque de changer le profil de l'école et des matières « traditionnelles » dont les mathématiques font partie, de quoi provient l'intérêt des didacticiens pour le sujet.

Figure 1. Un collage à l'entrée d'une Expo STIM—Labos créatifs.

Certaines écoles tentent d'introduire ces compétences tôt dans le parcours scolaire de l'élève, voire de la maternelle. L'une des écoles primaires que nous avons visitées a amené ses élèves au labo STIM pour les faire travailler sur des tâches complexes. Par exemple, les élèves devaient fabriquer un refuge pour un animal en peluche de façon qu'il s'y trouve suffisamment d'espace pour se protéger contre le vent et la pluie. Les élèves se sont mis à travailler dans des équipes de deux en utilisant des grandes feuilles de carton qu'ils pouvaient découper pour ensuite construire leur refuge. Les élèves devaient estimer, mesurer, à leur manière, réorganiser les parties découpées pour finalement trouver une forme plus solide. En commençant leur travail, en petits groupes, les élèves tentaient de bâtir leur refuge en forme de prisme rectangulaire. Or, il était difficile pour eux d'assembler les faces pour que la construction se tienne. Par essai-erreur, en persévérant, de manière collaborative, la plupart des équipes se sont rendu compte que la forme de prisme triangulaire leur donnera une option plus plausible. Ils devaient aussi s'ajuster à la taille de leur animal qui était parfois très grande de sorte qu'ils devaient être plusieurs pour transporter leur œuvre vers leur salle de classe comme le montre la Figure 2.

Figure 2. Les élèves de la maternelle transportent leur modèle vers leur salle de classe.

Cet exemple met bien en évidence plusieurs facettes de nouvelles compétences qui sont largement présentes, depuis 2000, dans nos programmes d'études en mathématiques sous forme de compétences transversales, comme c'est le cas du *Programme de formation de l'école québécoise* ou des résultats d'apprentissage transdisciplinaires introduites en même temps, dans le cadre théorique de tous les programmes d'études au Nouveau-Brunswick M-12, secteur francophone. On y inclut, notamment, des habiletés de communiquer, de s'approprier de la culture du patrimoine, de développer des méthodes de travail efficaces permettant de résoudre des problèmes de façon créative en exerçant la pensée critique bien articulée et, finalement, on veut développer chez tous les élèves des habiletés à utiliser, de façon judicieuse, des technologies de l'information et de la communication (TIC) dans des situations variées.

Les compétences, définies par Scallon (2003) en tant que « la capacité, pour un individu, de mobiliser savoirs, savoir-faire, stratégies et savoir-être dans le but de traiter de situations complexes appartenant à une même famille » (p. 2) demeurent toutefois un concept flou et complexe dont le sens peut varier selon le contexte dans lequel elles se manifestent. À l'école, selon l'Éducation Alberta (2016), « les compétences sont essentielles pour permettre aux élèves de se doter des connaissances, des habiletés et des attitudes dont ils auront besoin pour réussir leur parcours personnel dans les apprentissages, le monde du travail et la vie quotidienne » (p. 1). À la pensée critique, la communication, la résolution de problèmes et la collaboration, le document albertain ajoute la gestion de l'information, la citoyenneté culturelle et mondiale, la créativité et l'innovation, le développement et le bien-être personnel tout en citant les situations inconnues aux élèves qui leur posent des défis comme contexte de mobilisation de ces compétences.

INDUSTRIE 4.0 ET ÉDUCATION STIM : VERS UNE APPROCHE INTÉGRALE DE L'ENSEIGNEMENT ET DE L'APPRENTISSAGE DANS LES ENVIRONNEMENTS RICHES EN TECHNOLOGIES NUMÉRIQUES

En amplifiant la valeur des compétences de 21^{e} siècle (ou bien, non-techniques, douces, globales, transversales, transdisciplinaires, essentielles), on fait souvent référence au contexte de changements dans l'économie mondiale qui serait basée sur ce qu'on appelle l'Industrie 4.0 et qui émerge comme concept clé de la soi-disant « 4^{e} révolution industrielle ». De façon très simplifiée, on pourrait décrire ce phénomène comme usine numérique dans un monde numérique global. Un monde que Wallner et Wagner (2016), en citant Bennet et Lemoine (2014), décrivent comme étant volatile, incertain, complexe et ambigu. Selon Roblek et al. (2016), l'Industrie 4.0 est axée sur trois éléments principaux, soit l'intelligence artificielle, le big data et la connectivité. Allant au-delà de processus d'automatisation (robotisation) de production dans une *usine intelligente* (« smart factory »), cette industrie exploite Internet (connectivité) sous forme *d'Internet d'objets* (« Internet of Things ») et *d'Internet de services* (« Internet of Services »), ce qui crée une structure très différente et très complexe du monde de travail en modifiant profondément la structure de compétences requises pour s'y retrouver et s'y contribuer pleinement.

La compréhension de ce changement et de la nature des compétences qui s'y rattachent, bien perçues par le monde de l'éducation, est toutefois dans un stade embryonnaire tant sur le plan de la recherche et que de la pratique éducative. À titre d'exemple de changements perçus, citons une étude belge auprès de 964 ingénieurs qui démontre qu'au niveau de compétences actuelles, on retrouve la maîtrise des outils de gestion de données (51 %), les connaissances en TI (50 %) et l'analyse de données (48 %). Dans un avenir (proche), on anticipe plutôt la capacité de résolution de problèmes (55 %), les compétences multidisciplinaires (51 %), la gestion de la complexité (48 %), la volonté de changement (47 %) (Docendi, 2018). Ces données semblent refléter justement une tendance vers des compétences nouvelles (douces, du 21^{e} siècle, etc.). Or, projetées sur un parcours éducatif, ces compétences ne feront probablement pas partie d'un niveau scolaire ou postsecondaire particulier. Elles ne seront pas enseignées dans un cours particulier, mais feront plutôt l'objet d'un développement continu, la vie durant, dans un contexte inter- et transdisciplinaire axé sur la résolution de problèmes complexes et mal définis.

Ce contexte introduit également de nouveaux types d'environnements : dynamiques, flexibles, mobiles, riches en technologies qui permettent justement de résoudre des problèmes complexes. En fait, la résolution de problèmes dans ces environnements (riches en technologies) a fait l'objet d'une étude internationale d'OCDE PEICA[1] qui ciblait les compétences des adultes âgés de 16 à 65 ans, dont la littératie et la numératie. En mesurant la capacité d'utiliser la technologie pour résoudre des problèmes et accomplir des tâches complexes, le cadre d'OCDE (repris par Statistique Canada, 2014) « n'est pas une mesure des connaissances en informatique, mais plutôt des capacités cognitives nécessaires dans notre société de l'information—une société où, puisque nous avons accès à une quantité illimitée d'information, il est essentiel de pouvoir comprendre l'information dont nous avons besoin, de l'évaluer de façon critique et de l'utiliser pour résoudre des problèmes » (para. 9).

Depuis quelques années, une combinaison de lettres S, T, I, M commence à prendre une place d'importance dans la « brochette » de disciplines scolaires. Introduit par la NSF (National Science Fondation) dans les années 1990, l'acronyme contenait les mêmes quatre lettres, mais dans un ordre différent, soit S, M, E, T. Sanders (2008) mentionne que le changement d'ordre des lettres a été effectué probablement pour que ça sonne mieux (essentiellement, selon l'auteur, c'est plutôt que « SMET » sonnait mal en anglais). De toute manière, selon l'auteur, le sens original se perd dans la foulée de débats et de pratiques actuelles (par exemple, ce sens pourrait être lié, en anglais, au « stem cell research »). Sans entrer dans des débats linguistiques, on peut assez facilement trouver un terrain d'entente entre les chercheurs qui semblent se situer plutôt au niveau d'un sentiment d'urgence pour mieux éduquer les jeunes dans les disciplines qu'on croit cruciales pour le leadership dans l'économie globale en étiquetant le mouvement éducatif STIM comme Sputnik-2 faisant allusion à une compétition entre les Soviétiques et l'Occident pour la conquête de l'espace dans les années 1950-1960. Le Canada s'invite, lui aussi, « à la fête » avec une initiative stratégique STIM2067 (Canada 2067, 2017) liant le mouvement STIM au développement de compétences. En effet, selon le document, partout au pays, on vise une pleine intégration dans les programmes pédagogiques et dans la formation initiale des enseignants des approches axées sur les compétences, des approches multidisciplinaires et axées sur des enjeux, ainsi que de nouvelles technologies favorisant l'apprentissage et la culture numérique.

Pour le milieu scolaire francophone du N.-B., ces objectifs ne sont pas tous complètement nouveaux. Or, depuis, le début des années 2000, on a développé une Communauté d'Apprentissage Multidisciplinaires Interactifs (CAMI, www.umoncton.ca/cami) qui a mis en évidence l'apport des environnements virtuels au développement de la créativité des élèves en résolution de problèmes mathématiques complexes qui demandaient une construction, par l'élève, d'une démarche authentique (Manuel, 2018; Freiman & DeBlois, 2014). D'autres initiatives visaient un accès direct à l'ordinateur portable pour les élèves de 7^e^-8^e^ années leur permettant, entre autres, de vivre des scénarios interdisciplinaires (mathématiques, sciences, français) dans une *approche par problèmes* (Blain et al., 2007; Freiman et al., 2011). Freiman et Lirette-Pitre (2007) ont expérimenté l'utilisation un *espace collaboratif wiki* pour leurs cours de didactiques en mathématiques et en sciences pour créer des occasions de co-construction et de partage de connaissances didactiques pour les futurs enseignants au secondaire, et ce, dans une perspective interdisciplinaire. Citons également une initiative de Fonds d'innovation en apprentissage lancée en 2008 qui a donné naissance à un grand nombre de pratiques innovantes initiées par les enseignants dont le projet RoboMaTIC qui a mis en évidence des liens entre les mathématiques et les TIC dans un contexte tout aussi signifiant que complexe *d'apprentissage par la robotique* (Blanchard, Freiman, & Lirette-Pitre, 2010; Savard & Freiman, 2016).

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[1] PEICA, Programme pour l'évaluation internationale des compétences des adultes (Statistique Canada, https://www150.statcan.gc.ca/n1/pub/89-555-x/89-555-x2013001-fra.pdf)

Toutes ces initiatives étaient précurseurs d'une nouvelle vague d'intérêt pour l'éducation STIM dans la province qui a abouti, au cours de dernières années, à l'émergence de labos créatifs et d'introduction de programmation informatique et de codage dans plusieurs écoles au niveau primaire, intermédiaire et secondaire (Djambong, Freiman, Gauvin, Paquet, & Chiasson, 2018; Chiasson & Freiman, 2017). Ces pratiques ont permis à notre équipe du Réseau des partenaires CompeTI.CA (Compétences en TIC en Atlantique) d'étudier l'apprentissage dans les environnements riches en technologies qui sont potentiellement propices au développement de compétences non techniques.

Ces pratiques ont permis aux élèves de résoudre des tâches complexes (présentant un problème réel), interdisciplinaires (STIM—STIAM—STIEAM…, si on ajoutait un « A » pour les arts et un « E » pour entrepreneuriat), mettant en valeur l'apprentissage expérientiel (axé sur le design : le processus avant le produit). Concernant les compétences non techniques (« douces », soft-skills), ces pratiques rendent explicites des habiletés de résolution de problèmes, de communication, de pensée critique, en créativité, de travail d'équipe, de self-management, de gestion de conflits, tout comme la capacité de négociation, l'ouverture à la diversité, les connaissances générales, l'estime de soi, le sens de responsabilité, l'éthique, l'empathie, la curiosité, la persévérance, la détermination, et bien d'autres (Freiman et al., 2017).

En résumant nos analyses, on constate une multitude d'enjeux qui accompagnent les problématiques de développement de compétences non techniques. Tout d'abord, la liste de compétence est longue reflétant une grande *disparité*. On peut y retrouver un vecteur commun, aussi difficile à définir (« melting pot »). Souvent dans la littérature, ces compétences sont associées aux *dichotomies* de toute sorte (soft-hard, ouvert-fermé, mal défini-bien défini, complexe-simple, vie réelle-secteur académique, …). Elles sont *complexes* et *mal définies* (difficile d'opérationnaliser-développer-enseigner-mesurer). Elles sont jugées très *importantes* pour la société d'aujourd'hui (et encore plus de celle de demain); souvent associées au secteur Hi-Tech (Industrie 4.0); font partie de la littératie numérique (Habilo-Média, 2016). Elles sont bien présentes dans nos *curriculums* M-20 sous différentes formes ('compétences' en Alberta, M-12; 'transversales' au Québec, 1-11; 'transdisciplinaires' au N.-B., M-12; commencent à faire leur place au postsecondaire, 13-16—compétences essentielles—Collège communautaire du N.-B.). Finalement, on se pose la question sur leur(s) *lien(s)* avec les disciplines scolaires, dont les mathématiques. La 2e partie de la présentation fait part de quelques observations à cet effet.

ÉTUDE DE CAS : MATHÉMATIQUES

La 2e partie de la séance thématique a été dédiée à un exemple de mise en évidence de changements en enseignement et en apprentissage de mathématiques, à la lumière de nouvelles compétences et de l'émergence de nouveaux liens inter- et transdisciplinaires dans le contexte d'éducation STIM (avec un ajout plausible des arts et de l'entrepreneuriat, selon le cas). Notamment, on se demande comment évoluent des liens de mathématiques avec les sciences et les technologies, et/ou avec les arts (en faisant référence au quadrivium, aux sept arts libéraux). Puis, les mathématiques mêmes, ne jouent-elles pas, elles-mêmes, le rôle d'une (nouvelle?) compétence transversale?

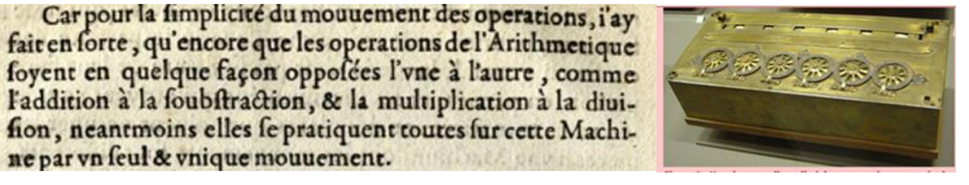
Commençons par citation provenant d'Adecco Canada (2014) : « Cela va probablement de soi, mais un emploi dans un domaine technique, comme l'ingénierie ou la technologie de l'information, exige l'exécution d'une très grande quantité de calculs élaborés. Et les mathématiques constituent la seule langue réellement universelle. Existe-t-il une compétence plus transversale que celle-là? » (para. 8). En effet, des liens entre les mathématiques et les technologies existent depuis très longtemps en marquant, de façon fascinante, l'histoire de l'humanité.

MATHÉMATIQUES ET TECHNOLOGIES—UNE RELATION COMPLEXE

Les auteurs en informatique citent plusieurs artéfacts présentant une évolution des outils de calcul[2], à commencer par l'Os *d'Ishango* qui est vu, par certains auteurs, comme l'une des premières calculatrices (Figure 3).

Figure 3. Bâton d'Ishango : quelles compétences ont mené nos ancêtres d'il y a plusieurs milliers d'années à concevoir cet outil? https://ww.radiookapi.net/actualite/2015/07/12/los-dishango-ete-decouvert-en-1950-en-rdc.

Le boulier (une sorte d'abaque) est un autre outil ingénieux qui a survécu, du moins comme outil d'apprentissage de calculs, jusqu'à l'ère d'Internet et d'appareils numériques mobiles (Volkov, 2018). Plusieurs mathématiciens ont été impliqués dans un long processus de création de calculateurs mécaniques étant confrontés aux multiples obstacles, dont, par exemple, le problème de traitement « automatique » de retenues. Par exemple, le génie de Pascal a été bien mis en évidence dans avec son système de sautoirs en cascade (Temam, 2009; Figure 4).



Car pour la ſimplicité du mouuement des operations, i'ay fait en ſorte, qu'encore que les operations de l'Arithmetique ſoyent en quelque façon oppoſées l'vne à l'autre, comme l'addition à la ſoubſtraction, & la multiplication à la diuiſion, neantmoins elles ſe pratiquent toutes ſur cette Machine par vn ſeul & vnique mouuement.

Figure 4. La machine de Pascal (1645) : Car pour la simplicité du mouvement des opérations, j'ai fait en sorte qu'encore que les opérations de l'Arithmétique soient en quelques façons opposées l'une à l'autre, comme l'addition à la soustraction, et la multiplication à la division, néanmoins elles se pratiquent toutes sur cette Machine par un seul et unique mouvement.

PLURALITÉ DES LITTÉRATIES, EN LIEN AVEC LES MATHÉMATIQUES

De contextes nouveaux liés à l'Industrie 4.0 font appel à une réflexion profonde sur une variété de mathématiques qui y sont associées et de (nouvelles) littératies qui en découlent. Entre autres, on insiste fortement sur de nouvelles pratiques en *numératie* qui devient de plus en plus numérique. On commence déjà à introduire des bases de *littératie financière* dans nos programmes d'études. Le contexte de *littératie numérique* met en valeur l'apprentissage de la programmation et de codage pour tous (« computational thinking »). Le contexte d'analyse de données massives (« big data ») et le besoin de faire des prédictions par rapport aux relations entre différents composants de systèmes (modélisation mathématique; Pilgrim & Dick, 2017), à son tour, mettent en évidence un nouveau type de *littératie*, celle *de données* (« data literacy »). En somme, le processus de création, de transmission, d'accès et de traitement de données de plus en plus nombreuses et de nature de plus en plus diversifiée mène à l'émergence

[2] Pour plus d'information sur différents types de technologies et leur évolution, voir Freiman, V. (2019). Types of technology in mathematics education. In S. Lerman (Eds), *Encyclopedia of mathematics education*. Cham, Switzerland: Springer. doi:10.1007/978-3-319-77487-9_158-4. Voir aussi Freiman V., & Robichaud, X. (2018). *A short history of computing devices* …, Récupéré de https://link.springer.com/chapter/10.1007%2F978-3-319-73396-8_13

de nouvelles formes de citoyenneté dans un monde numérique global (Freiman et al., 2017). Ces phénomènes méritent une étude plus approfondie par les didacticiens de mathématiques afin d'y déceler de nouveaux enjeux sur le plan de l'enseignement et de l'apprentissage de mathématiques.

QUELQUES EXEMPLES DE PRATIQUES STIM QUI REFLÈTENT LES CHANGEMENTS

Comme discuté au début de notre présentation, le contexte de labos créatifs met l'accent sur l'apprentissage STIM auquel les mathématiques et les compétences douces sont intégrées (LeBlanc, Freiman, & Furlong, 2018). Selon les élèves qu'on a interviewés, ils apprennent par *essai-erreur* : « Je fais beaucoup d'erreurs et j'apprends de ces erreurs-là. » « Tout de suite on est en train de mesurer les volts puis les ampères pour voir si qu'on peut avoir un switch. […] On avait mesuré avec juste lui, puis là on essaie avec les deux. ». La *pensée critique* est aussi au rendez-vous : « On a utilisé un USBC puis on l'a coupé. Puis là, il y avait 4 fils. Il y avait le blanc que j'ai coupé, le jaune que j'ai coupé parce que ce n'était pas eux… le le rouge puis le noir, c'est eux qu'on utilise. En total, ils ont cinq volts, puis quand ça passe à travers ça se divise. C'est pour ça qu'on a besoin d'une grosse batterie pour que ça puisse charger. […] Ça ne va pas durer cinq secondes, ça va durer 1h, 2h. On n'a pas encore testé comment longtemps que c'était ».

Un autre élève nous a partagé sa réflexion sur ses propres actions : « J'ai été un ti-brin trop loin » (en couture). En expliquant leurs choix de projets, plusieurs élèves évoquent leur *préoccupation par les enjeux de société* : « Notre maquette, c'est montrer si on arrête de polluer, à quoi la Terre va ressembler. », ou bien pour répondre à un besoin concret « C'est pour réchauffer des guidons comme sur ton vélo pour qu'à l'hiver, tu ne vas pas avoir de frostbite ». Le *plaisir d'apprendre* fait aussi souvent partie de leur discours : « Quand tu finis par comprendre comment résoudre le problème. C'est probablement le meilleur feeling! ». Entre autres, ils expliquent leur plaisir par des occasions de *résoudre de vrais problèmes* : « Ça nous permet de comprendre les différentes possibilités et ça nous aide avec des vrais problèmes ».

EMPHASE SUR LA CRÉATIVITÉ (FREIMAN & TASSELL, 2018)

En examinant différentes contributions au volume sur la créativité et la technologie en enseignement de mathématiques (Creativity and Technology in Mathematics Education), Freiman et Tassell (2018) ont ressorti plusieurs leviers que les environnements riches en technologies apportent pour faire émerger des conditions d'un apprentissage dynamique en mathématiques qui peut être *enrichi* (« enriching ») par des activités et des approches *engageantes* (« engaging ») permettant aux apprenants de maximiser leur *potentiel créatif* (« empowering ») par le biais de mécanismes efficaces *d'étayage et de feedback constructif* (« enabling et encouraging »). L'un des chapitres de ce volume (deChamplain, DeBlois, Robichaud, & Freiman, 2018) dit que l'usage de TIC pourrait transformer la relation de l'apprenant avec le savoir par *ses* pratiques de *créativité émancipatoire* du statut social de connaissances.

EMPHASE SUR LA PENSÉE INFORMATIQUE (GADANIDIS, CLEMENTS, & YIN, 2018)

En plaidant pour le retour en force d'activités de programmation et de codage informatique, on évoque le concept de la pensée informatique (une autre compétence clé du 21e siècle?) en faisant référence aux travaux de Papert qui voyait dans un environnement LOGO « un objet pour penser avec » dont l'utilisation permet à l'élève de s'engager dans « des activités mathématiques riches en mathématiques qui pourraient, en principe, être bénéfiques à la fois aux novices et aux experts, jeunes et vieux ». Déjà, dans les années 1980, Papert envisageait l'intégration de la « pensée informatique dans la vie quotidienne » en vivant des expériences « les plus attrayantes et des plus partageables » (Papert, 1980, p. 182).

En revanche, Wing (2006) a défini le terme comme « un ensemble d'attitudes et de compétences universellement applicables et que tout le monde devrait apprendre, pas seulement les professionnels de l'informatique » (p. 33). Or, étant « universellement applicables », ces attitudes et ces compétences sont encore mal définies, tout comme la problématique de leur développement et l'évaluation. (Gretter & Yadav, 2016). Djambong et al. (2018) mettent également en évidence l'ambiguïté épistémique et didactique de cette pratique innovatrice pour nos milieux scolaires, qui ne permet pas encore de distinguer clairement, du moins au niveau macro de conceptualisation et de réflexion au sujet de la programmation, de codage et de science informatique qui peuvent pourtant bien être exploités dans le contexte de l'enseignement et d'apprentissage de mathématiques plus avancées comme le démontrent les travaux de l'équipe de recherche *Computational thinking in mathematics education* (Gadanidis et al., 2018).

QUELQUES QUESTIONS OUVERTES À TITRE DE CONCLUSION

Un monde nouveau qui marque le début du 21e siècle apporte de nouveaux défis pour le système éducatif en général et pour l'enseignement de mathématiques, en particulier. À la lumière du développement accéléré de nouveaux types d'environnements riches en technologies qui mettent en évidence une tendance révolutionnaire de faire des choses dans l'industrie et dans la société qui deviennent de plus en plus numériques. Ce développement augmente également la pression mise sur nos systèmes éducatifs, en lien avec de nouvelles compétences dites « non-techniques » (ou, douces, globales, essentielles, du 21e siècle, …). Par conséquent y émergent des questionnements nouveaux sur la place des mathématiques dans ce contexte nouveau, sur leurs interactions avec ces compétences, ainsi qu'avec d'autres disciplines STIM qui pivotent les changements paradigmatiques dans la façon d'enseigner et d'apprendre, de plus en plus inter- et transdisciplinaire.

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USING COMPLEXITY THINKING IN MATHEMATICS EDUCATION RESEARCH

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ABSTRACT

Comment la pensée complexe peut-elle contribuer à comprendre l'enseignement et l'apprentissage des mathématiques? Dans une recherche effectuée en salle de classe il y a de cela plusieurs années, je me suis intéressée aux implications d'activités de haut niveau en mathématiques dans une classe de septième année. La théorie de la complexité nous a permis à moi et mes collègues de réfléchir à cette question sous différents angles. Dans un premier temps, nous avons utilisé des concepts issus des théories sur la complexité pour décrire la dynamique de la salle de classe à partir de laquelle les mathématiques émergent (Davis et Simmt, 2003). Par la suite, nous avons discuté et utilisé de manière intentionnelle la pensée complexe lorsque nous travaillions avec des enseignantes et des enseignants dans des contextes de développement professionnel (Davis et Simmt, 2006). Plus récemment, la complexité a soutenu deux autres volets de notre travail : une tentative intentionnelle de créer des outils et des méthodologies pour observer l'apprentissage dans des systèmes d'apprentissage collectifs et complexes (Simmt, 2015; Mc Garvey et al., 2015; Mc Garvey et al., 2017; Mgombelo, 2018) et le renforcement de capacités dans le cadre de projets de développement (Simmt et al., 2018).

How can complexity thinking contribute to understanding the teaching and learning of mathematics? In classroom based research I conducted many years ago, I was interested in the implications of high activity mathematics in a grade 7 class. Complexity theory provided my colleagues and I ways to think about that question. We began with using complexity concepts to describe the classroom dynamics from which mathematics emerged (Davis & Simmt, 2003). Then we deliberately discussed and used complexity thinking when working with teachers in professional development contexts (Davis & Simmt, 2006). In more recent years complexity has underpinned two other areas of our work: a deliberate attempt to create tools and methodologies for observing learning in collective and complex learning systems (Simmt, 2015; McGarvey et al., 2015; McGarvey et al., 2017; Mgombelo, 2018) and capacity building work in the context of development projects (Simmt et al., 2018). But this is telling a story without its beginning.

FRACTAL GEOMETRY

Let me begin In 1985. I was an undergraduate student taking a course for pre-service teachers and was expected to do an essay on a geometry topic. I selected fractal geometry as the focus of my paper. It was a new topic at the time, with Benoit Mandelbrot's book *The fractal geometry of nature* having been published just eight years earlier. I had come across the topic in the "Mathematical games" column written by Martin Gardner in a 1976 issue of *Scientific American*. In that paper, he wrote about monster curves. It was my introduction to fractals. The pathological curves he wrote about were very interesting to me, and I was strongly attracted to the visual imagery of fractals. By studying them, I found my way of observing the world included new distinctions I could make about geometric forms. Forms that, as Mandlebrot noted, were found in nature.

In that 1985 essay I drew the images by hand (Figure 1). I had not yet taken Tom Kieren's computers in mathematics course that included 'turtle geometry'.

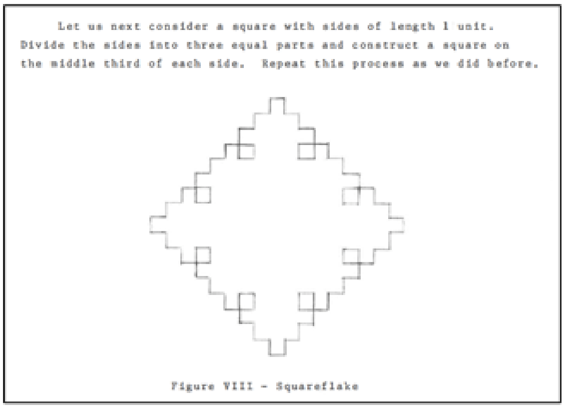


Figure 1. Hand-drawn fractal from undergraduate project.

Fractals have continued to be a source of inspiration (metaphorically and aesthetically) ever since I encountered them as an undergraduate. Fractal 'like' objects seem to emerge when I visualize my own research processes, when I analyse data, and when I think about recursive teaching practices. Fractals have been a constant in my research and scholarship, and I have enjoyed introducing them to students in mathematics classes.

DYNAMICAL SYSTEMS AND CHAOS

After completing my undergraduate studies I began teaching mathematics and chemistry in secondary school. Unfortunately, for me—and my students—I lost touch with fractals until I returned to university to do my master studies. While auditing a course in dynamical systems I encountered fractals again, this time in relation to chaos theory. I will not focus on dynamical systems or chaos here, except to say that the study of them raised my awareness of the significance of initial conditions in complex systems, and it taught me to look for emergence of order in chaos: particularly, bifurcations and other forms of attractors (Gleik, 1987).

Studying about the same time as I was doing my M.Ed. (and then Ph.D.) were Brent Davis, Ralph Mason, David Reid, Lynn McGarvey and Joyce Mgombelo. Tom Kieren led a seminar year after year in which topics such as chaos and fractals intersected with the study of cognition and curriculum. My classmates and I participated in that seminar year after year. From those seminars, I came to see how the study of human learners and classrooms were at least as complex as the chaotic systems I was learning about in the study of dynamical systems (weather, stock market, waves). This led me to new ways to think about mathematics teaching, learning and curriculum.

In the early 1990s, Sandy Dawson, a graduate of the University of Alberta, visited our cognition and curriculum group. He brought with him Varela, Thompson and Rosch's (1992) The Embodied Mind. Tom Kieren was already familiar with Maturana and Varela's work, Autopoesis and Cognition (1972) and Tree of Knowledge (1987). We had been reading those texts, but VTR (as we affectionately refer to the Embodied Mind) was a significant piece of work that elaborated and tied together many of the ideas about knowing and being we had been grappling with. Indeed most of us would go on to frame our research with enactivism. For me personally, Varela's notion of laying a path in walking began to permeate my way of knowing and being.

ENACTIVISM

In Varela, Thompson and Rosch's (1992) *Embodied Mind* the notion of laying a path in walking was explicated in their discussion of embodied cognition. They theorized cognition *as perceptually guided action that brings forth a world of significance*. I used their concept of cognition to frame my doctoral research. In it I used enactivism to study the interactions between parents and children as they did mathematics together, brought forth mathematics, and were brought forth as mathematics knowers (Simmt, 1996, 2001). As I attempted to understand their interactions and the mathematics that was brought forth, I came to see how knower and known co-emerge (Simmt & Kieren, 1999).

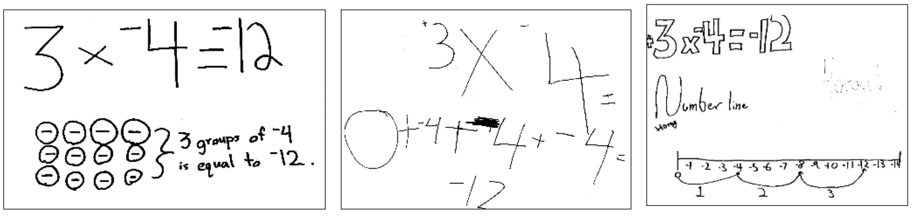
During my doctoral studies Tom Kieren introduced me and my colleagues to Susan Pirie and her students, Jo Towers and Lyndon Martin. (Some time later we would meet Jennifer Thom.) As we completed our doctoral studies and went on to academic to positions in mathematics education, the Pirie-Kieren students began to collaborate.

My first SSHRC study as a junior faculty member included Lynn McGarvey and Jo Towers as co-applicants. In that project we studied the implications of high activity mathematics in classroom contexts. My work focused on a grade 7 mathematics class, which I taught over the course of the year. One exploration out of that work involved thinking about the implications of moving the responsibility of explaining from the teacher to the learner (Towers & Simmt, 2007). This subsequently led us to ask about the ethics that are enacted in the day to day and moment by moment interactions between teachers and learners, and among learners. It also pointed us towards the teachers' mathematics and the teachers' orientation to mathematics. We recognized the importance of the teacher's curiosity about learners, mathematics and the learner's mathematics (Simmt, Davis, Gordon, & Towers, 2003).

Now in a faculty position, I was supervising graduate students: Immaculate Namkusa, Helena Miranda, and Jerome Proulx also took up an enactivist frame for their doctoral studies. Proulx has gone on to do extensive research about enactivism and using it to understand mathematics knowing in action.

RESEARCHING IN THE CLASSROOM

As I noted earlier, I was strongly influenced by chaos theory and recognized that the classroom itself could be understood as a dynamical system. The work in the grade 7 classroom provided me insight into those complex dynamical systems that I had been introduced to in my graduate studies. It was with this study that we first used complexity concepts to understand the classroom dynamics from which mathematics emerged (Davis & Simmt, 2003). The simple prompt "3 x -4 = -12, show how you know" led the students to articulate a diversity of thoughts and interactions.



A. Groups B. Repeated Addition C. Number Line Hopping

Figure 2. Student representations of multiplication of integers.

Working in pairs on the prompt, the students produced posters of their representations and were asked to explain their thinking in a plenary discussion. There were a diversity of ideas, representations and explanations: some students thought about integers as groups of negative objects (Figure 2A); others thought about it as repeated addition (Figure 2B); and yet other students thought about it as jumps on a number line (Figure 2C). It was in the plenary discussion that I observed evidence of collective understanding. I had been inviting students to offer their explanations, poster by poster, when I asked the students to turn their attention to a poster illustrating repeated addition (Figure 2B). (This episode is documented in Davis & Simmt, 2003, p. 158-159).

> "Negative four plus negative four plus negative four was equal to negative twelve."
>
> My attention had been drawn to the zero placed in front of the repeated negative fours. "Why did you put the zero there?"
>
> He pointed to another students and said, "He said you have to start somewhere."
>
> "Actually it doesn't matter where you start from." Tim said as though he was thinking this idea for the first time. "It will be the same."
>
> I asked for elaboration.
>
> Now Tim spoke with more conviction. "Well, negative four, three times, is always negative twelve, even if you start somewhere else on the number line".

In this classroom research, we were able to identify conditions of complex systems present in the group. By offering the prompt as the introduction to multiplication of integers, there was an opportunity to solicit a diversity of ideas. By having students present their representations and explain them in the plenary discussion, space was created for neighboring and random interactions. By removing the responsibility for explaining from the teacher and placing it on the students control over the ideas was decentralized. This decentralization of control enabled the students to act as specialists, bringing their ways of knowing to the class discussion. Some of the students understood multiplication as repeated addition and were able to offer that idea in the group, others specialized by using a number line. The conditions for complexity within the collective and the dynamics of interaction created space for one student, Tim, to articulate a brand new idea. I assert this was an idea that would not have emerged in the classroom without the affordances described above. I label Tim's ideas about multiplication as an example of collective understanding. Some time later, I brought Tim's utterance about it not mattering "where you start" back to the class. Not even Tim recalled it. But, after a brief pause he confirmed that it "makes sense". Using this particular lesson, Brent Davis and I (Davis & Simmt, 2003) wrote about the classroom as a dynamical system, one that is self-organizing, self-maintaining and adaptive—that is to say a learning system.

METHODS FOR OBSERVING COMPLEXITY

Since that work in the early 2000s, I have become more and more interested in understanding the dynamics of the class as a collective learning system. My colleagues and I have turned our attention to use what we know about complexity and about mathematics knowing to develop research methods and tools to study the collective learning system that emerges in the mathematics classroom. In 2014, McGarvey led a SSHRC proposal that would bring together the Kieren-Pirie students and some of their students to seek out new tools and methods for observing collective knowing in the mathematics classroom context. As we describe in a recent paper, "Our overall purpose is to consider the value of observing classrooms as collectives as well as potential methodological tools that may serve as vital signs of collective classroom life" (McGarvey et al., 2018, p. 156).

The notion of vital signs comes from the medical field: heart rate, blood pressure, temperature, respiration rate, and oxygen level (among others) are assessed all at once to establish base-lines, to observe for anomalies—in short, to identify potential problems with the body (system). Any one vital sign tells the medical practitioner something; for example, if a person's oxygen level is low, there might be an assumption that the pulmonary system not functioning well. The key is that any one vital sign does not indicate what precisely is wrong, it only suggests that something could be wrong. Further, vital signs are read in relation to one another, and in relation to the individual in a particular context or state of health.

We "propose that a suite of 'classroom vital signs' could be used to distinguish between different forms of collective classroom activity while pointing to critical elements of dynamic engagement" (McGarvey et al, 2018, 156). At PME 42, we shared a particular suite of classroom vital signs that included utterance length and distribution where the frequency and number of words spoken by teachers and students in a lesson were analysed for insights into classroom discourse patterns (McGarvey & Simmt; (non)actions on the whiteboard, where attention is focused the writing on, pointing to, and erasing of marker on the board [Mgombelo, Thom, Glanfield, & McGarvey]; an emergent ideational network that displays the structure of ideas emerging over the course of a lesson [Proulx & his students]; the Pirie-Kieren model to explore collective understanding by modelling the emergence of ideas within a classroom [Glanfield & Thom]. Individually and together, we propose these four vital signs can offer insight into classroom collectives. (McGarvey et al., 2018). This work is ongoing. We have just begun to apply the vital signs to a variety of video-taped classes to see what they might reveal.

DESIGNING A DEVELOPMENT PROJECT USING COMPLEXITY

Complexity thinking has been valuable to me, as I have been able to use it: first as a theoretical tool for research, and second as a pragmatic theory for informing teaching and educational endeavours. Having spoken to the first use above, here I will share how my colleagues, Florence Glanfield, Joyce Mgombelo and Andrew Binde, and I used the features of complexity and indigenous perspectives to design a development project to build capacity for mathematics teaching in rural and remote communities in Tanzania. For that project we conceptualized a highly networked structure (Figure 3) and activities in which we could apply conditions for a learning system to emerge. Over its duration, we paid attention to human relationships, valued traditional and community wisdom, and leveraged emergent ideas and phenomena.

The project was conceptualized as a connected graph in which all branches contribute to enhancing mathematics teaching in rural and remote communities. The branches reflect three primary areas of activity: a policy network that would bring high level education officials together to discuss the development, communication and implementation of policy related to gender equity in mathematics education, and professional development for in-service rural and remote primary mathematics teachers. A second branch of the network involved those who work directly (or could work directly) with teachers to facilitate their professional development in relation to mathematics teaching. This group consisted of teacher college tutors, school inspectors, master primary school teachers, district academic officers and adult educators. These people were offered training (with an emphasis on mathematics for teaching and facilitating professional development) in a series of nine short courses over four years. Invitations were extended to members of the short courses to develop and facilitate teacher professional development in 27 rural wards for more than 400 teachers. In addition to building this capacity to deliver professional development through the short courses, another 25 people were sponsored to do graduate studies (22 M.Ed and three Ph.D). The third branch of activity involved enhancing the awareness of community members about the importance of mathematics education for the children in their villages. They were offered two workshops, which focused on the value of mathematics education, gender equity, and the mathematics they use in their day to day lives and to operate their schools.

The three primary branches of the network were deliberately established to create redundancy in the network and to engage a diverse group of agents who could have impact on mathematics teaching in a number of different ways. In order to ensure neighboring interactions among the participants, their activities, and their ideas, the roles of the participants in the various branches were connected. For example, college tutors participated in the short courses and facilitated professional development workshops for teachers, and the principals from the tutors' colleges participated in the network. Other examples of paired roles across the activities included school superintendents (short courses) / district superintendents (policy network); master teachers (short course) / head teachers (policy network); district academic officers (short course) / district education officers (policy network).

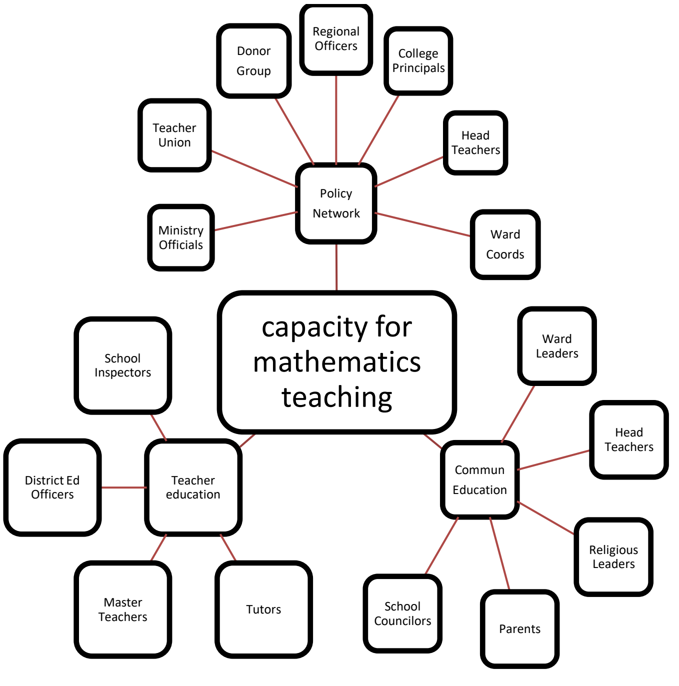


Figure 3. Network structure of capacity building project.

Although a project plan and general structure were in place, ongoing tinkering was essential to the success of the project. For example, when we found that the high level officials in the policy work were often called away from our sessions or sent representatives in their place, the relationships and redundancy that we attempted to build into the project was compromised. However, after inviting ward level officials and head teachers to the policy meeting to do a focus group, we learned how important their roles were in implementing policy. Given that they did not have the same demands on them as their higher ranking colleagues, they were able to participate more fully in the sessions. So, the policy network participation list was expanded to included ward coordinators and head teachers in the activities. This tinkering was critical to the success of the policy network.

EMERGENCE

In the development project, openness to tinkering and lessons from indigenous perspectives created a context in which emergence was fostered. Throughout the project we sought counsel from the people and participants with whom we were working. We asked, *What is working? What needs do members of the community identify? What traditional practices need to be incorporated? Who else should be involved?* The posing of these questions and the fostering of relationships among all members of the project (organizers and participants alike) provided the project with ongoing inspiration, effort, and coherence.

Upon reflection, it is clear that many of the decisions that contributed to the project's vitality and impact were those that were informed by the participants and triggered by emergent 'lessons', events and ideas. One of those critical decisions was to invite people in the short courses to engage in some work between short course sessions. To support their work, we offered access to very modest amounts of money. Offering micro-grants for participants' generated activities had not been built into the project design but with just small adjustments (a bit of tinkering) we could arrange for funds to be used for this purpose.

Another and very powerful example of emergence was the use of a project slogan. Early in the project the advisory board formulated a slogan, "*hisabati ni maisha*" [the phrase *hisabati ni maisha* roughly translates to mathematics is life / living]. One day when Joyce Mgombelo was trying to get the attention of the people working in their district groups she called out, "*hisabati*." Without hesitation someone in the group responded, "*ni maisha*" With a second call out, the whole group responded. The slogan and the chant became a touchstone for the people in the project.

As I reflect here and now I find a thread through all of my work. The thread is emergence—emergence of a fractal out of repeating a simple rule, emergence of order out of chaos, emergence of a learning system from the actions and interactions of youth in mathematics classrooms, teachers in professional development communities or stakeholders in mathematics education.

A NETWORK OF CANADIAN MATHEMATICS EDUCATION SCHOLARS

I want to close by pointing to the future. Over the course of my career I have had the great fortune to work with inspiring people. I have spoken at length about Tom Kieren and how the Pirie-Kieren students continue to work together (Figure 4). But there is another group of inspiring people. These are the graduate students who I have had the honour and pleasure of working with. They come us as students but leave as colleagues. Of course it is impossible to name a few of them not already mentioned. I encourage you to look for their work that uses enactivism and complexity in the context of mathematics education. Priscila Dias used complexity to frame designed based research where she studied the presence and fostering of mathematical proficiency by using modeling tasks in a high school mathematics classroom. Nat

Banting used enactivism and complexity to study teacher interactions in grade 9 mathematics where students worked collaboratively to pose and solve mathematics problems. Informed by complexity thinking, Lixin Luo is completing a hermeneutical study of a recursive mathematics curriculum. And Xiong Wang is using complexity thinking to understand an online mathematics education professional learning community. I am very aware that within the CMESG community there are many others using complexity in their work. I know Canadian mathematics education scholars will continue to contribute important insights and theories in mathematics education using complexity thinking.

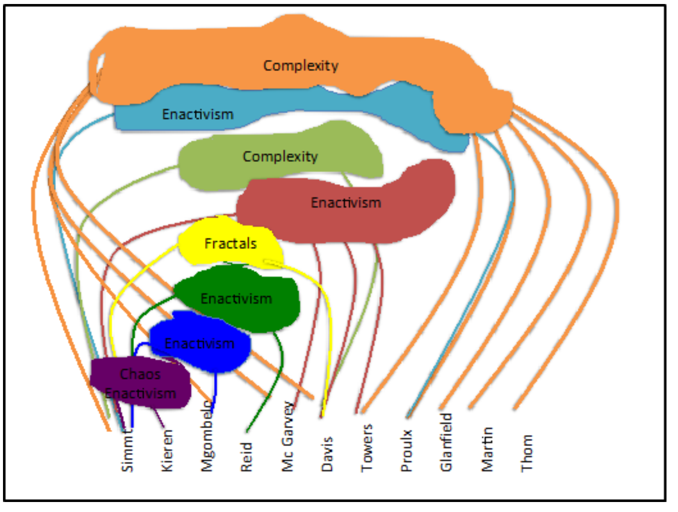


Figure 4. My Pirie-Kieren collaborators.

I want to end this paper by acknowledging my CMESG colleagues who have inspired and supported me in my work over the (almost 30) years I have been thinking about mathematics, mathematics teaching and mathematics learning.

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RETHINKING TEACHING FIRST-YEAR MATHEMATICS IN UNIVERSITY

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It is safe to claim that all universities in Canada offer some type of a calculus course, and that, for a fairly large number of students, calculus is the only mathematics they will experience in university. According to a recently compiled database of first-year mathematics and statistics courses taught at Canadian Universities (Jungic & Lovric, 2018) about 55% of mathematics courses offered are calculus courses (next on the list is linear algebra, which accounts for about 17%). If we consider the number of students taking these courses instead of counting the number of courses, then the percentage of students taking calculus is even higher. Clearly, calculus still dominates first-year mathematics curricula, and it seems that this situation will not change any time soon. However, the first-year database (Jungic & Lovric, 2018) reveals the existence of a number of non-calculus based first-year courses.

This paper aims to contribute to the discussions about rethinking teaching first year in mathematics by addressing the following questions: Should we continue to teach calculus as is, or should we teach calculus at all? If a student plans to take only one mathematics course in university, what are possible alternatives to calculus?

WHAT IS THE PROBLEM WITH TEACHING CALCULUS?

Tall and Schwarzenberger (1987) write, "Most of the mathematics met in secondary school consists of sophisticated ideas conceived by intelligent adults translated into suitable form to teach" (p. 44). We could undoubtedly replace "secondary school" with 'first year university'. The authors continue:

> This translation process contains two opposing dangers. On the one hand, taking a subtle high-level concept and talking it down can mean the loss of precision and an actual increase in conceptual difficulty. On the other, the informal language of the translation may contain unintended shades of colloquial meaning. (Tall & Schwarzenberger, 1987, p. 44)

It is not difficult to find support for their claim. In high school and university, the slope of the tangent to the graph of a function is introduced, formally, as the limit of the slopes of appropriately constructed secant lines. However, very often, this is followed by 'user-friendly' narratives, which discuss properties of tangents (Kajander & Lovric, 2009). Tangent lines are drawn (in textbooks and very likely in classrooms) with no reference to the limiting process that generated them. We find phrases such as "tangent is the line that touches the curve, but does not cross it" in both high school and university textbooks (see Kajander & Lovric, 2009, for a list of narratives that are commonly used to describe tangents; the 'unintended shades of colloquial meaning' are the misconceptions studied there).

The problems (Tall & Schwarzenberger's [1987] "increase in conceptual difficulty" [p. 44]) emerge when students are faced with having to figure out what the tangent to a corner or a cusp looks like, or when they have to work with discontinuous functions, commonly given in a piecewise form. By not going beyond the intuitive, students fail to build an abstract image of the tangent and find advancing to related concepts challenging (such as understanding what differentiability is, especially for functions of several variables).

Stewart (2012) uses the "zooming" argument—"by zooming in toward a point on the graph of a differentiable function, we noticed that the graph looks more and more like its tangent line" (p. 250)—to justify using the tangent line as the linear approximation. Consider the following example: the tangent to the graph of the function $y = x^2$ at the point (1,1) is $y = 2x - 1$. By using, for instance, graphing software, we can easily convince ourselves that $y = x^2$ 'looks more and more' like $y = 2x - 1$, *but also looks more and more like the lines* $y = 1.9x - 0.9$, $y = x$, $y = 2.05x - 1.05$, and many others. Which begs the question: If all these lines approximate the function (and they do), what is special about the tangent line that we take it as the linear approximation? This is a subtle, but important question, not addressed in most calculus books.

Further cases of perils of translation of subtle mathematical concepts into a "suitable form to teach" are presented in numerous studies, including Tall (1992, 2001), Edwards and Ward (2004), Kim, Sfard, and Ferrini-Mundy (2005), Kajander and Lovric (2018), and Burazin and Lovric (2018, 2019).

Another troubling aspect of calculus are applications, often referred to as 'real-life' applications even when it is absolutely clear that there is no real life in them. Calculus textbooks are full of artificial, 'ready-to-use' problems, such as most problems given in the related rates or optimization sections. These problems usually require very little in terms of reasoning and problem-solving, and the major effort is dedicated to applying known algorithms (for computing derivatives, finding extreme values, integration, and so on).

Calculus *does* offer opportunities to study true applications, but those require classroom time and conceptual proficiency and routines ('automation', as discussed by Peter Taylor, 2018) that are beyond what many first-year students possess. For instance, studying branching of a blood vessel from a main vessel (Stewart, 2012, p.336) requires knowing geometry, working with cotangent and cosecant functions, differentiating, and solving equations with trigonometric functions. (This indeed *is a real-life situation* as physicians can predict potential problems in blood flow if the branching angle is not close to the optimal.) Determining the shape of a honeycomb (Adler & Lovric, 2015, p. 372), combines geometry (trigonometry, tilings) with certain calculus techniques. Anecdotal evidence suggests that instructors shy away from discussing such topics, not just for the lack of time. Often students find themselves struggling with technical details which, unfortunately, take their focus away from the essential parts of the problem.

Ample evidence suggests that teaching theory and working on proofs in calculus are challenging (consult, for instance, Tall, 1992; Kim, Sfard, & Ferrini-Mundy, 2005; Zaskis, 2005; or papers on proof by Selden and Selden, such as Selden & Selden, 2003). Due to simplifications (thus superficiality) that must be made when presenting mathematical analysis in its 'simplest' form as calculus, students are unable to recognize the value and the depth of, say, the Intermediate Value Theorem ("Isn't it obvious?" is a common comment heard from students), or truly understand the concept of limit or convergence of infinite sums of numbers and functions (power series).

In summary, these arguments (which could easily be expanded) support the suggestion that we should not continue teaching calculus as we have been for quite some time but, instead, seriously rethink it. (If we have not managed to fix calculus in decades, what gives us confidence that we can do it now or in the future?) The fact that calculus still serves as gatekeeper for numerous university programs serves as further motivation to do so.

In terms of mathematics, university students fall into one of the three categories:

- those that take only one mathematics course, with a possibility of some further mathematics content offered through discipline-specific courses such as accounting or commerce;
- those who need mathematics for their discipline such as engineering and physical sciences students; and
- mathematics and statistics majors.

In this paper, we focus on the first group and suggest an alternative—a more relevant and useful course. Teaching mathematics to the second group could be rethought by creating courses which discuss genuine applications in the areas of student interest and guarantee that there is sufficient class time to cover the necessary background material for the more discipline-specific examples (see Taylor, 2018).

‘NUMERACY’ COURSES

We focus on the ideas and work done by the authors at McMaster University and the University of Toronto Mississauga.

There is no general agreement on what numeracy is. There are numerous understandings, definitions, and conceptualizations. Universities declare that their students will satisfy the ‘numeracy requirement’ by taking one of a wide range of courses, such as the history of mathematics, precalculus, calculus, computer science, statistics, or mathematical logic and foundations.

We view *numeracy* (or numeric literacy or quantitative literacy) as a combination of specific mathematical knowledge and skills that are needed to function in the modern world. (Many sources define numeracy as a set of skills. However, it is only when this set of skills is used with appropriate knowledge and applied to true-life, genuine contexts that numeracy becomes useful and meaningful.) Numeracy is about reasoning from and about quantitative information that is presented numerically, graphically, verbally, or in dynamic forms. Guided by the questions we routinely ask in mathematics (What? Why? How do we know?), we conceptualize numeracy to include critical and evidence-supported thinking, as well as logical reasoning in contexts that might not involve quantitative information. Most important, numeracy courses—having no outside constraints on their content, unlike calculus or linear algebra, which are prerequisites for upper-level courses—seem to be the right place to discuss true mathematical applications.

The *numeracy course* Math 2UU3 (‘Numbers for Life’) has been taught at McMaster University since Winter 2017. Among the themes discussed so far in the two iterations of the course, we found

- Language of mathematics and everyday language: contrasting definitions, theorems, proofs and algorithms in mathematics and in real life;
- ‘Quantitative’ in quantitative reasoning: numbers and related concepts; elementary relationships between numbers; proportional reasoning; currency conversion and units conversions as examples of proportional reasoning; fraction, ratio and percent;
- Scientific notation; visualizing numbers in narratives (developing a feel for numbers and their size); reasoning with numbers; analyzing narratives that involve quantitative information and/or logical reasoning;
- Patterns of change, in particular, linear, quadratic, cubic, and exponential;
- Models involving exponential functions, such as human population dynamics, steady growth in limited environments (demographic and economic ramifications) and exponential decay; logarithmic scale, Richter scale;
- Scatter plots and building models using linear regression; non-linear regression;
- Climate change parameters: modeling the amount of carbon dioxide in atmosphere, surface area and thickness of Arctic ice, average global temperature;
- Discussing appropriate and inappropriate use of visual representations of information/data; interpreting information presented visually; dynamic visualization;
- Personal and general finance: managing credit card debt, loans, mortgages; financial calculators; interest, inflation, CPI (consumer price index); financial market and economy indicators (Dow, S&P, TSX, NASDAQ);
- Economy and social indicators; Gross domestic product (GDP) and gross national product/income (GNP/GNI); human development index and Gini index;
- Case studies of randomness: lottery and games of chance; probability in contexts;
- Basic probability: law of large numbers; mean, spread, box-plot diagrams; normal distribution, and reasoning based on standard deviations;
- Intuition about total probability and Bayes’ theorem, applied to testing for a medical condition (false positives and false negatives); risk expressed as probability;
- Basics of statistical hypothesis testing, null hypothesis, *p*-value.

This course is open to any student in any faculty at McMaster University and has no prerequisites beyond what is covered in grade 10 mathematics in Canada. Consequently, it is a rare mathematics course where humanities and social sciences students sit alongside mathematics majors and business students. With such a wide range of students’ backgrounds, where does one start? Note that we did not specify *what* backgrounds. In this course where doing mathematics and writing narratives go hand in hand, we see students with poor mathematics skills who are good writers, as well as good mathematics students who struggle with putting a sentence together.

Anecdotal evidence, coming from observing students working on class activities, suggests that good students are willing to (and do) help their peers with mathematics; however, we are not sure if it goes the other way around (i.e., students helping each other in writing).

Course instruction is a mix of lecturing and class activities. Lecturing is used to introduce new themes and topics; whereas, class activities emphasize thinking and problem-solving. One particular aspect of problem-solving is the fact that, unlike in ‘real-life’ mathematics textbook problems, numeracy course problems do not give all data needed nor indicate what data is needed—students need to figure out what they need and use appropriate sources to find it.

How is it all done? As a guiding principle for class instruction and class activities, accompanying problem sets and assignments, we extended the notion of *mathematical habits of mind* (Cuoco, Goldenberg, & Mark, 1996). In particular, we want our students to engage in

- collecting, organizing, and analyzing quantitative data obtained from various sources; sampling, forming conclusions about a population based on samples taken from it, and building awareness of limitations of such approaches;
- 'pattern-sniffing' and use of patterns to extrapolate (as in regression);
- graphing and visualizing (i.e., creating, reading, and interpreting data in tables, charts, diagrams, graphs, and other dynamic forms);
- estimating and approximating quantities ('back of an envelope calculation');
- creating a sound quantitative argument, based on logical reasoning and backed by evidence (for instance, not mixing up causality and correlation);
- experimenting and reasoning with quantities, numbers (e.g., properties of exponential growth) and concepts (e.g., probability or statistics) to facilitate understanding of natural, physical, social, and behavioural phenomena; and
- communicating quantitative information verbally, in writing, and in graphic forms such as tables, charts, diagrams.

As an illustration of implementation of these principles, we present a group activity ("Wasted Water" problem) in which students had to estimate the amount of water wasted in Canada every day by people who do not turn the water tap off while brushing their teeth.

Considering approximately 200 students present in the class as a representative sample and using online clickers, we gathered information on how many times each day they brush their teeth and whether or not they turn the water tap off while doing so. (In future iterations, students will be given the problem only and will have to decide what data they need. That data will be collected on the spot by using online clickers.)

As no other information or data were given, students had to decide what else they needed and how to obtain, calculate, or estimate these quantities. This makes the "Wasted Water" problem different from problems we find in calculus (or other) textbooks, where the exact amount of data needed to solve the problem (i.e., no missing nor superfluous pieces of information) is offered.

As a conclusion of the activity, members from the two groups were called to present their solutions (Figure 1). Those presentations were a vivid evidence of students' engagement in a variety of mathematical habits of mind. They looked up the population of Canada (34 million in Estimate 1 and 35.16 million in Estimate 2) to scale up the sample (classroom) proportions to the entire Canadian population. Then, they estimated how long on average one person brushes their teeth (90 seconds in Estimate 1 and 2 minutes in Estimate 2; as reported, these were 'rough averages' based on the groups' personal experiences).

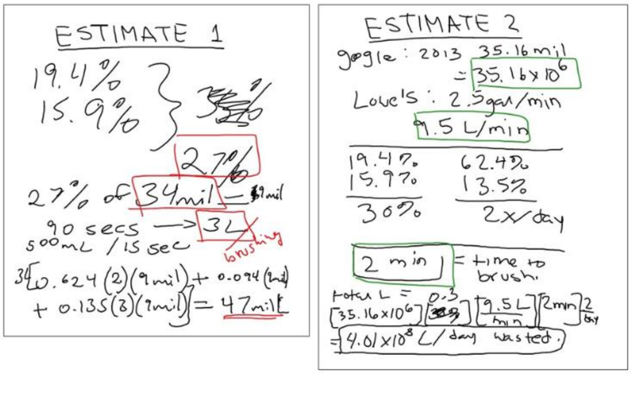

Figure 1. Students' presentations of solutions to the "Wasted Water" problem

Another piece of information was missing: water flow rates. Estimate 1 is based on scaling an estimate (which was a consensus of the group members, but not checked independently) that a 500 millilitres glass of water can be filled in 15 seconds, whereas Estimate 2 was informed by the water faucet flow rates provided by Lowe's (hardware store) website.

Following the presentations and prompting by the instructor, additional issues were identified. Students questioned whether the sample size (i.e., the number of students present in the classroom) is large enough to merit reliable statistics about the entire population. As well, an issue of bias was identified: Can the data taken from a sample of university students be viewed as representative of the entire Canadian population?

Related to the 10-fold difference in the two estimates (47 million litres in Estimate 1 and 401 million litres in Estimate 2), there was a suggestion to test their 'sensitivity' (the term was suggested by the course instructor). This amounts to answering the following question: If a certain numeric piece of information is increased or decreased by a small amount, say by 2 to 3 percent, how does that affect the final estimate?

When creating a story based on the obtained estimates, one has to find a way to help the reader visualize the large quantities of water wasted. In other words, 47 million litres in Estimate 1 and 401 million litres in Estimate 2 could easily be meaningless to a large segment of the population, unless related to something familiar. In searching for ideas, students recalled the previous lessons, when we discussed narratives that include numbers which are not close to everyday and intuitive experiences. Here is a sample of such narratives:

> […] specific diagnosis is rare, but being born with a blend of male and female characteristics is surprisingly common: worldwide, up to 1.7% of people have intersex traits, roughly the same proportion of the population who have red hair. (Kleeman, 2016, para. 4)

In this case, the possibly vague proportion given as 1.7% is visualized as a proportion of red-haired people in the general population. We know that we see red-haired people, but not often. Here is a second narrative:

> How many Canadians actually live up North? Approximately 118,000. That's one-third of one percent of the national population. To put it another way, about as many Canadians live in Australia as live in Nunavut. If the entire population of Northwest Territories decided to attend Edmonton Eskimos game, Commonwealth stadium would still have 10,000 empty seats. (Gilmore, 2016, para. 1)

The number (118 thousand) and the proportion ("one-third of one percent of the national population") are made more familiar by the comparisons that follow.

Based on these experiences, it was suggested that the amount of water should be expressed in the units of water needed to fill an Olympic-size swimming pool. Given the dimensions of such pool (50 by 25 by 2 metres), one can restate the volume of 47 million litres in Estimate 1 as "about 19 Olympic size pools" and 401 million litres in Estimate 2 as "about 160 Olympic size pools".

CONCLUSION

As it seems to be a more natural setting where *mathematical habits of mind* can be cultivated, a numeracy (for the lack of better word) course offers a productive alternative to teaching calculus to some university students, such as those planning to take only one mathematics course. The "Wasted Water" problem demonstrates the richness of ideas and concepts, as well as habits of the mind, that can be brought up and experienced in a single example. Through a carefully selected sequence of such examples, ideas and concepts that are planned to be covered in a course can be further strengthened. As well, the reality of the situations makes important issues (such as sample size and bias or sensitivity of an estimate) emerge naturally.

We do not wish to imply that the days of calculus are numbered. On the contrary, by offering alternatives (such as numeracy) and by no longer requiring calculus for every student who needs to take mathematics, we can free calculus from many constraints that a one-size-fits-all model brings and redesign mathematics courses to better serve the diverse student populations.

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REFLECTIONS AROUND THE NOTIONS KNOWLEDGE (-TO-BE)-TAUGHT AND KNOWLEDGE (-TO-BE)-LEARNED FROM THE ANTHROPOLOGICAL THEORY OF THE DIDACTIC—CHALLENGES FOR RESEARCHERS AND FOR TEACHERS

RÉFLEXIONS AUTOUR DES NOTIONS SAVOIR (-À-ÊTRE)-ENSEIGNÉ, SAVOIR(-À-ÊTRE)-APPRIS DE LA THÉORIE ANTHROPOLOGIQUE DU DIDACTIQUE—DÉFIS POUR LES CHERCHEU(RS)SES ET POUR LES ENSEIGNANT(E)S

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Université Concordia

SUMMARY

Over the last few years, we (graduate students under my supervision and I) have been inquiring into 'what undergraduate students (don't) learn in … [calculus / introductory analysis courses]'. In doing so,

- we are constantly reflecting on the different stages of the process of didactic transposition (quickly stated: the process by which knowledge is transposed from scholar knowledge to teachable/learnable knowledge), and
- we construct (praxeological) models of students' knowledge based on different pieces of data.

In this presentation, I share these reflections and some considerations regarding the (non)linearity of the didactic transposition process in the particular case of teaching and learning at the university level. I then delve into the challenges of building models of students' knowledge and gathering data to do so. I share examples of the models and methodologies we use. Finally, I address questions raised by university professors, who have been involved in our research (as researchers, reviewers, participants, observers), regarding the role and relevance of the models we build—what these mean to them.

RÉSUMÉ

Au cours des dernières années, nous (étudiants diplômés sous ma supervision et moimême) avons recherché « ce que les étudiants de premier cycle (n') apprennent (pas) dans … [leurs cours de calcul / cours d'analyse] ». Ce faisant,

- nous réfléchissons constamment aux différentes étapes du processus de transposition didactique (rapidement énoncé: le processus par lequel le savoir est transposé du savoir savant au savoir enseigné / appris), et
- nous construisons différents modèles (praxéologiques) des savoirs des élèves à partir de différentes données.

Dans cette présentation, je partage ces réflexions et des réflexions sur la (non) linéarité du processus de transposition didactique dans le cas particulier de l'enseignement et de l'apprentissage au niveau universitaire. Après, je partage les défis que nous rencontrons dans la construction des modèles des savoirs des élèves et dans la recollection des données pour le faire. Je partage des exemples de modèles et de méthodologies que nous utilisons. Finalement, je réfléchis aux questions soulevées par des professeurs, qui ont participé à nos recherches (en tant que chercheurs, évaluateurs, participants, observateurs), sur le rôle et la pertinence des modèles que nous construisons—ce que ces modèles signifient pour eux.

New PhD Reports

Présentations de thèses de doctorat

STUDENT ACTIONS AS A WINDOW INTO GOALS AND MOTIVES IN THE SECONDARY MATHEMATICS CLASSROOM

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Simon Fraser University

ABSTRACT

My dissertation concerns student actions in the secondary mathematics classroom, with respect to the nature of student actions, the particular activity settings in which certain actions are performed and how, over time and multiple viewings, an individual student's actions can provide sufficient information to substantiate a claim as to his or her goals and motives in the secondary mathematics classroom. Three different mathematics classes in three different British Columbian secondary schools participated in the research. Findings include the identification of five different student motives and their relative stability and discreteness, the number and settings of the necessary observances to arrive at these identifications, and the accuracy of predictions of student action based on a known student motive.

INTRODUCTION

Students' actions within the mathematics class often appear to occur with little reason. In any given setting, an individual student may perform one action one day and a different action the next, or exhibit two seemingly contradictory behaviours in two separate settings. "In many cases these behaviours are centred on proxies for learning and understanding, such as mimicking, that are not actually conducive to learning" (Liljedahl & Allan, 2013, p. 263). This seemingly random or unpredictable behaviour actually has a common source according to activity theory: motive. In fact, a main tenet of activity theory is that motive is the driver of all human activity. Thus, establishing the nature of this driving force plays a pivotal role in explaining the actions that students perform, and reconciling behaviours that appear to be irrational. According to Pintrich (2000), "students are not just 'motivated' or 'unmotivated' in terms of some general quantity…there are important qualitative differences in how students are motivated and these different qualities have a dramatic influence on learning and achievement" (p. 101). Pintrich's assessment supports an investigation of these motivations, the ultimate goal of the research described within this paper. A precursor to accomplishing this goal was to first establish what actions students actually perform within the mathematics classroom.

METHODS AND DATA SOURCES

As mentioned above, the force that drives student activity is motive and it is the key component of Leontiev's Activity Theory (1978). As a theoretical lens, activity theory permits a description of what students do and say without overlaying pre-existing assumptions or judgments. These observations, taken together, can then be used to develop a hypothesis for the student's motive, which may be something other than a desire to learn. For Leontiev (1978), "[a]ctivity does not exist without a motive; 'non-motivated' activity is not activity without a motive but activity with a subjectively and objectively hidden motive" (p. 99).

The fact that motive is often hidden from the subject suggests difficulty in determining the ultimate motive. This obstacle can be overcome by utilizing an 'actions first' strategy (Kaptelinin & Nardi, 2012). The strategy begins at the level of goals, which people are generally aware of and can express, and the analysis is subsequently expanded up to higher goals and ultimately the motive. In this regard activity theory serves as both a theoretical lens and an analytical tool.

Given that there are certain actions that students are not consciously aware of performing, even upon reflection (Leontiev, 1978), to obtain authentic results it was necessary to examine students actions 'in the moment' in the setting in which they occur. An ethnographic approach was then a natural choice and a complementary methodology for a theoretical framework utilizing activity theory because of its overall approach and methods. The distinctness of the ethnographic approach lies in a holistic analysis requiring "interpreting and applying the findings from a cultural perspective" (Wolcott, 1980, p. 59). In classroom ethnography this entails a holistic analysis that is sensitive to the contextual situation of the classroom and incorporates participants' perspectives of their own behaviour (Watson-Gegeo, 1997). This requires substantial fieldwork, observation, fieldnotes, and various forms of semi-structured and focus group interviewing.

An approach to data collection and analysis rooted in analytic induction[1] fits naturally within an ethnographic frame. In my interpretation, iterative analysis occurs throughout the process whereby data collected in one lesson provokes questions and shifts focus for subsequent observations and interviews. Ongoing analysis signals potential themes, which then frame the subsequent data collection and ultimately the eventual analysis (Wolcott, 1987). The sheer volume of information collected necessitates some form of structure for managing the data. Activity settings are a unit of analysis by which student actions can be organized and analysed. Defined as units of 'contextualized human activity' (O'Donnell & Tharp, 1990), activity settings are the specific settings that provide the context in which activities take place and that influence the types of activities subjects are likely to encounter. According to Tharp and Gallimore (1988), activity settings are the *who*, *what*, *when*, *where*, and *why* of everyday events. Within the classroom there are many different activity settings; identifying the particular activity setting in which a student participates provides a way to discuss the characteristic elements of the setting that influence the student's behaviour.

The data for my dissertation comes from research conducted in three secondary school mathematics classes in British Columbia during the 2013-2014 school year. Three classes were observed: two at the grade 11 level (PreCalculus 11 and Foundations 11) and one at the grade 10 level (PreCalculus 10). All teachers had at least ten years teaching experience. Throughout a semester each class was observed for twelve periods, each period ranging from 60 to 75 minutes. Classroom lessons and informal interviews were audio recorded and transcribed for later analysis and comparison with field notes taken during the class.

The process of collection, coding, and recoding data produced a veritable rainbow of student actions, wherein the smallest unit of data consisted of a single action coded with two pieces of information: the individual who performed the action and the activity setting in which the action occurred. I engaged in the sifting and sorting of these data in several different ways, all through the lens of Leontiev's activity theory, in order to determine the likely primary motive

[1] See Patton, 2002, p. 494 for more on analytic induction.

underlying the students' behaviour. Then actions were re-examined through the lens of the driving motive. Further analysis rooted in these key findings revealed interesting relationships between students' actions and motives within and across activity settings.

RESULTS

My findings derive from the results of four different analyses. The first two analyses are based on the initial raw data and discussion can be found in Parts II and III of my dissertation (Allan, 2017). These findings then drove the latter pair of analyses, both adding to and extending them, and comprise Part IV of my dissertation. This process is illustrated in Figure 1.

Due to space constraints, in this paper I will only address some of the findings of Parts III and IV.

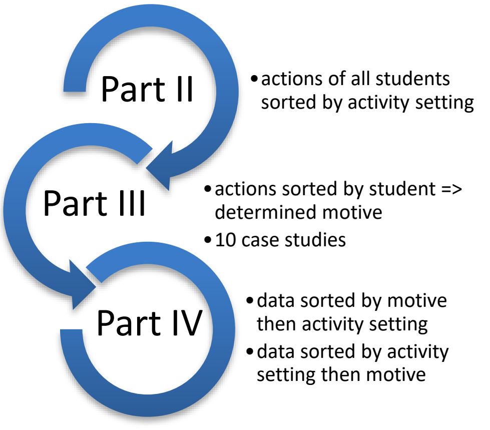

Figure 1. Illustration of analysis process and results.

IDENTIFYING MOTIVES

As mentioned above, the initial sorting of data by activity setting in Part II served to suggest potential themes for student actions in different settings. In Part III, I then took the same initial data and organized it by student. Each student's observed actions and interview data, over the entire semester, contributed to a holistic picture of that student within the mathematics classroom. This provided additional context for interpreting student actions and ultimately pointed to the primary motive driving each individual's actions. Consideration of the identified motivations of all individuals resulted in the identification of five different primary motives: understanding, get a good grade, get credit for the course, get through it, and avoid attention and/or work. I wrote ten case studies, comprising Part III of my dissertation, to illustrate and substantiate the identification of these motives.

Primary and secondary motives

As I engaged in the process of determining each student's motive I found myself struggling to reconcile certain aspects of the behaviour of students whom I had tentatively identified as having the same primary motive. For example, both Toby and Beth were established to have a motive of getting a good grade, but whereas Beth did every assignment and every assigned task to the best of her ability, Toby was less diligent about completing ungraded assignments or those that required more effort than what Toby was willing to devote. At this point I referred to activity theory and incorporated students' secondary, and sometimes tertiary, motives. I found the complex interplay between these hierarchical motives fascinating and very useful. I believe it contributes to a deeper and more accurate analysis. In the given example, it allowed me to understand that while Toby and Beth both held subsidiary motives of understanding, which supported their principal motive of getting a good grade, they also held other high ranking motives. Beth's tertiary motive of being seen as a good student supported her chief motive; whereas, Toby's motive of minimizing his effort mitigated his.

Incorporating secondary and even tertiary motives allowed me to explain why the actions of a given student might be different in the same setting on separate occasions and why two students with the same primary motive sometimes act differently.

COMPARING AND CONTRASTING MOTIVES

Identifying the types of actions in different activity settings and the motives driving student actions answered my initial questions, but also left me with more questions with respect to the similarities and differences in the actions I observed and the motives I deduced. I opted to probe further and was rewarded with the findings that follow.

Activity of students with same primary motive

Considering the five primary motives, I then compared and contrasted them by activity setting and actions to search for themes, as described in the example with understanding and getting a good grade below.

Figure 2 depicts the activity of students with a motive of understanding. The darker rectangles represent activity settings or specific actions (such as cheating). The ovoid bubbles represent actions that can be seen to comply with perceived expected actions in that activity settings—these are outside the larger 'circle'. The diamond shaped containers hold actions that would be seen as noncompliant. The diagram serves to highlight some particular features of the behaviour of a student who holds a primary motive of understanding mathematics. First, there is significant blank space in the inner region; there are very few activity settings, or aspects of classroom life in which these students choose not to participate. Some notable exceptions, for some students, are *doing homework in class*, *now you try one*, *cheating*, *asking for help*, and *review day*. A more detailed explanation of the reasons underlying these actions can be found in my dissertation (Allan, 2017).

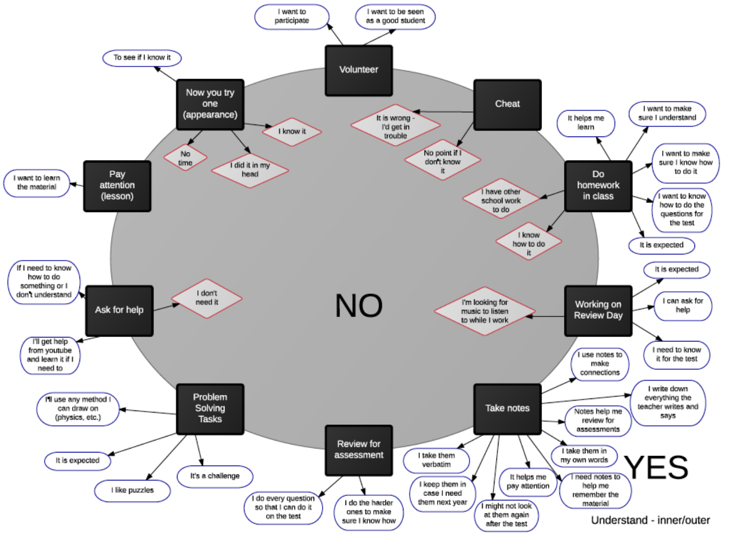

Figure 2. Activity of students with a motive of understanding.

Figure 3 shows the activity of students with a motive of getting a good grade. There is a glaring difference in the population of the inner region of the diagram as compared to the illustration above (Figure 2). There are two reasons for this: first, a much larger contingent of students were identified to a motive of getting a good grade than one of understanding (and thus there were a greater number of student actions); and second, students with a motive of getting a good grade were more varied in their actions both individually and across students. That is, a particular student might do homework in class on one day and not the next, and two different students with the same motive might act differently on the same day.

Figure 3. Activity of students with a motive of getting a good grade.

Three significant themes emerged from looking at the actions of students with the same primary motive. The first concerns the exchange of effort, which is not discussed here. The second and third are stability of the motive and its continuity (over time). Stability, as I use it, refers to the 'sticking power' of the motive within *localized* activity settings. Continuity is viewed across activity settings and over time and thus involves a more *global* perspective.

Considering the five primary motives through their relative stability, it is seen that understanding and avoidance are more stable than the other motives. They are not easily displaced as the primary motive in a given activity setting. In contrast, getting a good grade, getting credit and getting through the course are more susceptible to being displaced or influenced by factors in the social environment or other conditions (see Kaptelinin, 2005 for more on conflicting motives).

Looking across the same five motives, through the actions and explanations of the students, certain patterns emerged. I describe the motives as taking place along a continuum from continuous to discrete with respect to time. In this view, understanding and avoidance as primary motives are continuous as opposed to discrete. This means that actions that are consistent with these motives occur continuously as opposed to only at certain times or within particular settings. Further, these patterns of behaviour persist if we look across multiple days. 'Discrete' describes motives wherein the actions that are consistent with them occur more sporadically. There is more variability in activity and more tension or conflict between motives when actions are viewed across activity settings and over time. However, there is still some predictability in student behaviour despite these tensions. The following example portrays the difference between continuous and discrete.

Consider two composite[2] students: Jenna and Lisa. Jenna holds a primary motive of getting a good grade. This motive holds dominance when graded assignments are due and in the days or classes leading up an assessment, such as a quiz or test. At other times, Jenna may not prioritize participation in some activities, such as problem solving, or she may defer her learning to a more convenient time. When this occurs it is likely that a secondary motive of socializing or minimizing effort takes precedence. Her attention may not stay focussed on the lesson and she may not try examples perhaps because she feels she knows it or that she can try them later. In contrast, Lisa holds a motive of understanding. She sees opportunities for learning at all times, though she may not always do her homework in class or try every example. Regardless, understanding is always Lisa's primary motive and her actions almost always align with this motive.

Distinctions with respect to relative 'discreteness' of the three less stable motives exist and a discussion of these can be found in my dissertation.

Actions of students in particular activity settings

The determination of student primary motives provided an additional lens through which to view student actions in activity settings. I began with the student actions and organized them by activity setting, then examined the motives that linked to different groups of student actions. Figure 4 depicts the actions of students during problem-solving tasks.

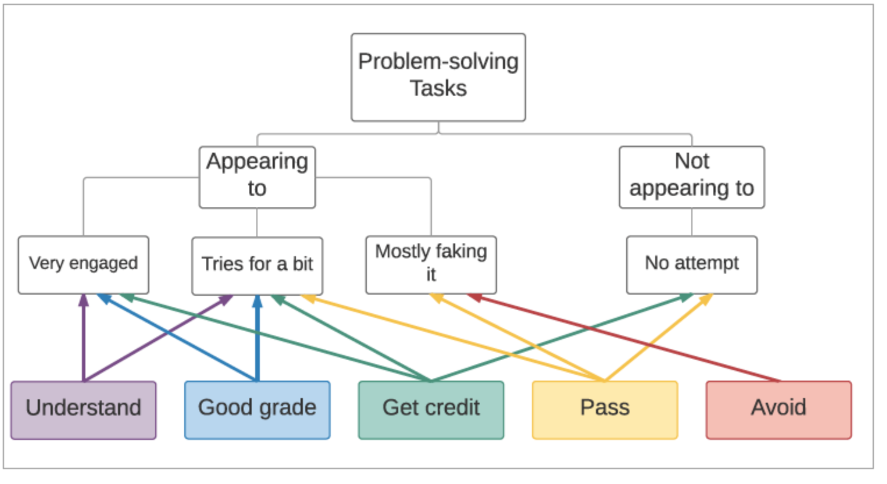

Figure 4. Motives linked to actions during problem solving tasks.

Student actions are grouped in categories to simplify the diagram and the analysis. In the above example, a student's actions in the activity setting of problem solving were grouped as being either very engaged; trying the task for a while; mostly faking trying the task; or making no attempt to try the problem. The arrows in the diagram link student motive to the associated action category but are not directional (i.e., an arrow from understanding to very engaged is not meant to imply that a student with this motive will be engaged, nor that an engaged student

[2] Though Jenna and Lisa themselves do not exist, they are composites of students whose actions are represented in the data.

implies a motive of understanding). What it depicts is that students who were observed to try a problem-solving task for a bit were identified to have a motives of either understanding, getting a good grade, getting credit, or passing the course. It should also be noted that an individual student with a motive of getting credit may have been observed to try for a short time during one problem-solving task and make no attempt another day (the student did not necessarily perform the same action in every observed incidence of the same type of activity setting). Another activity setting, doing homework in class, is represented in Figure 5.

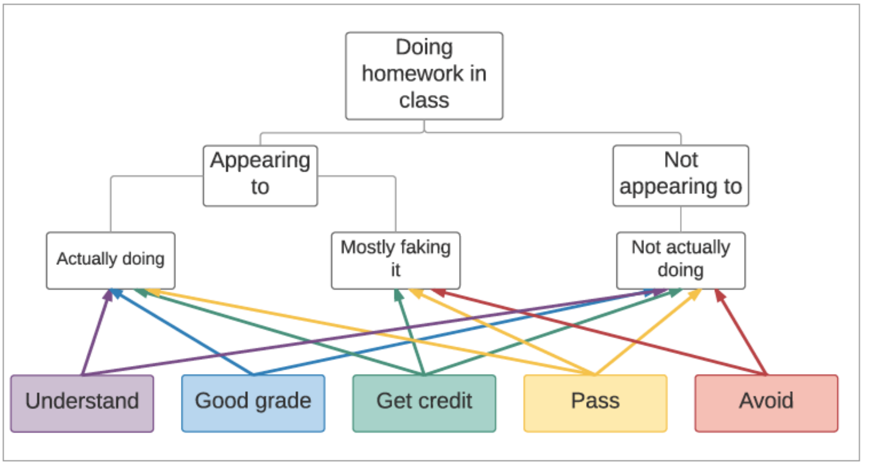

Figure 5. Motives linked to actions during homework in class.

One interesting finding that is evident in the diagram is that students who faked participation held motives of getting credit, passing, or avoiding; those with a motive of understanding or get a good grade did not bother to fake it, they just did homework in class or did not. What is most noteworthy in this diagram is that students who were observed to not do homework during class held any one of the five different motives. Thus, you cannot determine student motive by looking at their actions in only one activity setting. I discuss the implications of this finding and others in the section that follows.

DISCUSSION AND IMPLICATIONS

I began this study with the desire to document the forms of student behaviour exhibited by students within the secondary mathematics classroom and through this to better understand the motivations driving students' actions. I believe I have achieved the former and claim some success with respect to the latter. I identified five primary motives driving student actions and compared and contrasted actions, activity settings, and students' motives in several ways. The results of my study have implications for researchers and educators.

For researchers, there ismuch more work to be done in the area of student motive. Given the success I found with Leontiev's activity theory, I suggest it as a fruitful frame of approach to future work. Some potential avenues include further probing with the notions of continuous and discrete with respect to student motives and the stability of motives within activity settings with consideration of the hierarchical shifts between primary and secondary motives. Any such research will need to take place over time and multiple observances in light of the finding that student motives cannot be determined by looking at their actions in any particular activity setting.

For educators, I believe that the findings of this study suggest that we think deeply about what we value and how we demonstrate our values through our instructional practices and our assessment practices. How do our values and our practices influence or encourage certain student motives, and are these the motives that we want our students to hold?

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OBSERVING HIGH-SCHOOL STUDENTS' MATHEMATICAL UNDERSTANDING AND MATHEMATICAL PROFICIENCY IN THE CONTEXT OF MATHEMATICAL MODELING

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ABSTRACT

Teaching mathematics can be demanding for different reasons. One of these reasons relates to the challenges teachers may face due to the lack of students' mathematical understanding and proficiency. The present research (Dias Corrêa, 2017) intends to comprehend what forms of mathematical understanding and mathematical proficiency are observed and how they are expressed in the actions and interactions of high-school students when engaged in mathematical modeling tasks. The research study offers insight into the use of mathematics modeling by: 1) portraying how mathematical modeling tasks foster high-school students' mathematical understanding and proficiency; and 2) showing the feasibility of implementing this kind of task in mathematics classes with time and curriculum constraints. The study also revealed that students demonstrate mathematical understanding and proficiency during the course of the modeling tasks, even when they do not come to full resolutions of problems. Therefore, mathematical modeling is reinforced as a relevant and doable teaching approach in the teachers' challenging task of teaching for understanding.

INTRODUCTION

The use of mathematical modeling in education has been investigated for the last five decades. The benefits of bringing modeling into mathematics classes are well known and well accepted, and modeling is becoming more common and more appealing to mathematics teachers. However, there are still unanswered questions and conjectures to be explored so as to aid and encourage mathematics teaching through modeling. The present study (Dias Corrêa, 2017) uses classroom-based research to explore the use of modeling tasks within high-school mathematics classes in order to provide insight and motivation into the teaching of mathematics for understanding. In this study, participants were engaged in mathematical modeling tasks that required them to develop models for mathematical situations instead of using an already known mathematical model or a given one. The purpose of the research was to investigate students' mathematical understanding and proficiency in this teaching setting.

Teachers might argue that, although mathematical modeling has the potential to work on students' mathematical understanding and proficiency, it is not worth doing because it is incompatible with classroom constraints. Indeed, the difficulty to innovate in mathematics classes can be related to teachers' commitment with a tight curriculum—which is perceived as one they must cover—and to time constraints. As a consequence, teachers keep teaching in a more procedural and instrumental way that—apparently—is less time consuming and more successful in terms of covering the curriculum. However, procedural and instrumental ways of teaching mathematics are often unsuccessful in terms of promoting thoughtful inquiry and mathematical reasoning skills, which are fundamental features of mathematics. In accordance with that, Silver, Mesa, Morris, Star and Benken (2009) affirm that

> With amazingly little variation across teachers and over time, research has found that mathematics instruction and instructional tasks tend to emphasize low-level rather than high-level cognitive processes…, require students to work alone and in silence…, focus attention on a narrow band of mathematics content…, and do little to help students develop a deep understanding of mathematical ideas. (p.502)

It is hard to figure out new possibilities that challenge and change old entrenched ways of teaching and learning mathematics. This classroom-based research dealt with time and curriculum constraints and offers reasons and strategies for changing this scenario.

When referring to research published about ten years ago, Lesh, Galbraith, Haines and Hurford (2013) assert that "most of [modeling] studies investigate the development of ideas—not the success of treatments and interventions" (p.280). This research not only draws attention to examples of modeling tasks, it also presents an intervention to implement tasks in class, students' actual mathematical work on tasks, and a diagram-based approach to interpret and analyze students' mathematical work. Then, from students' work analysis, the current research dives into the accomplishment of the intervention in terms of students' mathematical understanding and proficiency. The research is fairly complete in portraying the implementation of mathematical modeling tasks in high-school classes, giving teachers a practical sense of what modeling is about, as well as reasons and encouragement to try it.

MATHEMATICAL MODELING: DEFINITION AND BENEFITS

The International Community of Teachers of Mathematical Modeling and Applications (ICTMA) gathers research on mathematical modeling in different areas of mathematics education. It is interesting to notice that different perspectives and views of modeling can be found within this community. Lesh and Fennewald (2013) agree that there is not an established definition of what modeling is. However, they assert that there is an agreement that model-development research analyzes relevant contextualized decision-making problems, tending to be more like an engineering approach in which "realistic solutions" are developed to "realistically complex problems" (p.6). For this research study, mathematical modeling is considered as the ingenious process of developing a mathematical model, which is a mathematical representation that expresses how a situation, an object or a process runs. It is relevant to emphasize that my understanding of mathematical modeling necessarily includes the *development* of a mathematical model. Cirillo, Pelesko, Felton-Koestler and Rubel (2016) draw attention to some relevant features of mathematical modeling, which are helpful in characterizing modeling processes, they are 1) the connection with ordinary situations; 2) the ill-defined nature; 3) the necessity of a creative modeler who is able to make assumptions, choices and decisions; 4) the recursive behaviour given that the modeler needs to constantly confront the model and the phenomena to validate the model; and 5) the non-unique or non-strict nature since the modeler can chose from multiple paths and get to different solutions.

As a learning resource—apart from drawing attention to mathematics usefulness—mathematical modeling presents considerable advantages for students. Blum and Ferri (2009) point out some of these advantages of mathematical modeling at the secondary level. They assert that mathematical modeling 1) helps students in understanding the world; 2) supports students' mathematical learning in that it improves motivation, conceptualization, comprehension, and retention; 3) improves mathematical skills and decision making; and 4) assists in proper framing of mathematics. As a high-order thinking activity, modeling prompts students to investigate, analyze, inquiry, reason, model, solve, conjecture, and validate. By working on these abilities, students potentially enhance their mathematical understanding and their mathematical proficiency.

As a teaching resource, according to Biembengut and Hein (2013), the use of mathematical modeling in class presents advantages to classroom teachers. The authors argue that teachers have the opportunity to 1) research and create their own classroom materials; 2) get a better grasp of the content knowledge involved in the modeling task; 3) implement tasks interacting with other disciplines; 4) be creative and critical in the model development and validation; 5) be more aware of students' work and struggles; and 6) review their assessment resources and criteria. Part of what makes the use of modeling in mathematics teaching a laborious process also helps in teachers' professional development.

CLASSROOM DESIGN AND RESEARCH METHODOLOGY

This research classroom design was based on complexity science underpinnings. As students' actions and interactions are believed to be non-linear, spontaneous and self-organized, a mathematics classroom is characterized as a complex system that allows mathematical understanding to emerge (Davis & Simmt, 2003). Davis and Simmt (2003) suggest that there are five necessary conditions to implement, develop and maintain a complex environment within mathematics classes. These conditions are *internal diversity*, *redundancy*, *decentralized control*, *organized randomness*, and *neighbour interactions*. This research endorses that these conditions are intrinsically present in the natural classroom setting (except when the schooling system imposes other circumstances) and can enable the emergence of thoughts that will allow for mathematical understanding, mathematical proficiency and mathematical knowledge production. Considering that these conditions can vary all the time in class, conditions were monitored during the intervention in order to ensure the appropriate setting.

According to The Design-Based Research Collective (2003), educational design-based research aims to combine theoretical research knowledge with every day practical experiences. This blend is expected to yield practical knowledge that can be helpful in developing teaching and learning processes in educational settings. Cobb, Confrey, diSessa, Lehrer and Schauble (2003) describe design-based experiments as "extended (iterative), interventionist (innovative and design-based), and theory-oriented enterprises whose 'theories' do real work in practical educational contexts" (p.13). Indeed, this study fits well in this description, given that it brings theoretical knowledge into mathematics daily practice by means of an iterative intervention, in order to analyze education practical issues related to students' understanding and proficiency. The research intervention was elaborated to allow students to engage in the modeling process and to work on their mathematical understanding and proficiency while data was being collected. The intervention cycle is illustrated in Figure 1.

The pentagon in Figure 1 represents the classroom in which Davis and Simmt's (2003) five conditions are sought and preserved. Within this classroom, a learning context is established. This context refers to the use of a mathematical modeling task as a resource for teaching and learning mathematics. Themes are chosen taking students input into consideration. Within this setting, students' mathematical understanding and proficiency are investigated. Students work in groups of three or four discussing and sharing their ideas. For this research purpose, audio and video recordings, students' mathematics journals and researcher field notes were collected. After class interventions, students were invited to participate in recall interviews. Research data was gathered during four of these interventions.

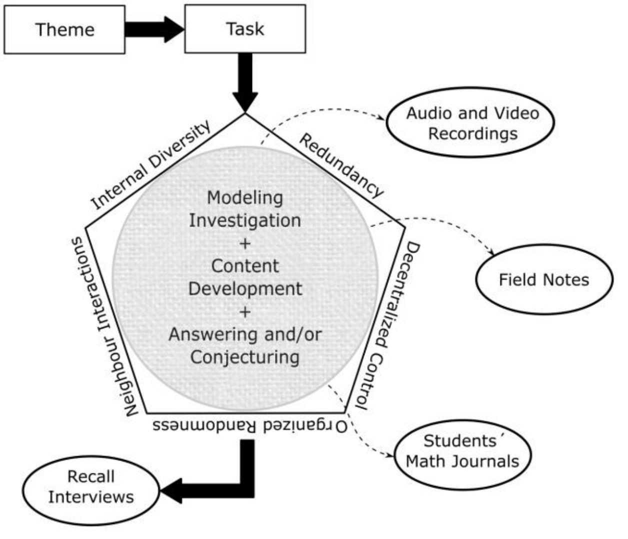

Figure 1. Intervention cycle.

DATA COLLECTION AND DATA ANALYSIS

In order to investigate students' mathematical understanding and proficiency while engaged in mathematical modeling tasks, four different tasks were proposed to a high-school class taking grade 11 mathematics. The class was composed of 27 students. Although all of them participated in the tasks, data was collected from the 12 students who provided consent. Tasks were applied during a four-month Alberta mathematics course.

Analyzing students' mathematical understanding is a challenging task; it depends on comprehending what ways students understand. Due to the difficulties in pointing out students' understanding, Kilpatrick, Swafford and Findell's (2001) mathematical proficiency model was chosen in order to operationalize this process through some indicators. Kilpatrick et al.'s model is composed of five strands of mathematical proficiency, namely *conceptual understanding*, *procedural fluency*, *strategic competence*, *adaptive reasoning* and *productive disposition*. The basis of this research data analysis consisted of identifying indicators of each of these five strands in students' work and then investigating how students demonstrate these strands during the modeling tasks. Table 1 shows the list of indicators used to detect each proficiency strand. This list was elaborated based on the aspects that Kilpatrick et al. understand as necessary to characterize each of the five strands of mathematical proficiency.

The use of this proficiency model presents some benefits for this research, which are 1) the inclusion of features of mathematical understanding that are present in the literature; 2) the description of students' mathematical performance in a way that relates to the daily practice of a mathematics teacher; 3) the possibility of teasing apart students' mathematical performance through a comprehensive model that includes aspects related to concepts, procedures, strategies, and reasoning; and 4) the inclusion of student-related aspects in the model.

To analyze the data from multiple sources as a whole, interpretive diagrams were designed to map student's journey through the task from introduction to completion. Diagrams present fragments of students' verbal communications and written solutions. Fragments were categorized based on the list of indicators presented in Table 1. A grey ellipse-shaped arrow shows the chronological order in which task fragments were gathered. The fragments that are not overlapped by the grey arrow were collected during recall interviews. Sixteen diagrams (four diagrams for each of the four tasks) were created to analyze the collected data. An example of an interpretive diagram can be seen in Figure 2.

Conceptual Understanding	Connect mathematical content.
	Retrieve mathematical content.
	Understand mathematical content.
Strategic Competence	Build strategy to understand problem.
	Build strategy to represent problem.
	Build strategy to solve problem.
Procedural Fluency	Choose right procedure.
	Choose right moment to apply procedure.
	Perform procedure correctly.
	Understand procedure.
Adaptive Reasoning	Logically relate contents.
	Logically relate situations.
	Logically relate content and situation.
	Transfer content between situations.
Productive Disposition	Perceive mathematics as worthwhile.
	Believe in his/her ability to learn mathematics.
	Believe in his/her ability to do mathematics.

Table 1. Strands of mathematical proficiency and their respective indicators.

A TASK AND A STUDENT'S INTERPRETIVE DIAGRAM

You have been hired to program a flight simulator for beginner pilots training. One of the features your team is supposed to develop is the feedback control towers send to aircrafts, which includes advice about approaching aircrafts and changes in route. Imagine that two aircrafts are approaching the same control tower from different directions and following a straight path.

a) How would you determine the distance between the two aircrafts considering that you have a GPS?

b) How would you determine the distance between the two aircrafts if the GPS were down and you needed a backup safety system?

The above task was intended to have students figure out the cosine law. Students had not been instructed in the cosine law prior to this lesson. Students were expected to go through all the necessary steps to model the cosine law. The idea was to aid students in realizing that the cosine law is a shortcut to a longer procedure that they were already able to implement. This task had no numerical values. It was anticipated that this could be an issue for some students. In case students were facing too much struggle, the teacher could provide them with some values. A couple possible extension questions were created in case students finished the task before the given time was over. Students had two consecutive 80 minute classes to work on the task.

Figure 2 illustrates a student's interpretive diagram based on the mathematical work he did when working on the flight simulator task. Fragment one portrays that the student suggests right angles to be forced into the problem so that right triangles can be created. In fragments two and three, he organizes his thoughts and uses a Cartesian plane to locate the two aircrafts and the control tower. The student explains how he uses his Cartesian plane knowledge to build on his strategy. The picture in fragment three speaks to the chosen procedure to determine the length of the sides of the right triangle that connects both aircrafts. Fragment four addresses his understanding of the overall process. In fragment five, based on previous knowledge, the student concludes he does not need angle measurements to solve the problem at this point. Then he explains how to calculate the length of the sides of the triangle. Fragment six presents the final step of his strategy in which he uses the Pythagoras theorem. Next, fragment seven is about the second question of the task. The student starts off by looking at extra available data given that the GPS is out of order. He uses the time the aircraft took off and its speed to figure out how far the aircraft has travelled. Then he subtracts this distance from the distance between the two original airports in order to figure out the remaining distance to the destination airport. In fragment eight, the student analyzes the triangle he has in hand. Fragment nine points out his strategy of splitting up the bigger triangle into two right triangles. The student needed some prompting to visualize he had different ways of splitting up the big triangle. He then realizes he does not need to split up a known angle. Finally in fragment ten, the student has a big triangle divided into two right triangles. He then plans to use trigonometric ratios to finish solving the task. Although—due to the lack of time—the student did not get to a final answer, his work confirms he went through all five strands of mathematical proficiency.

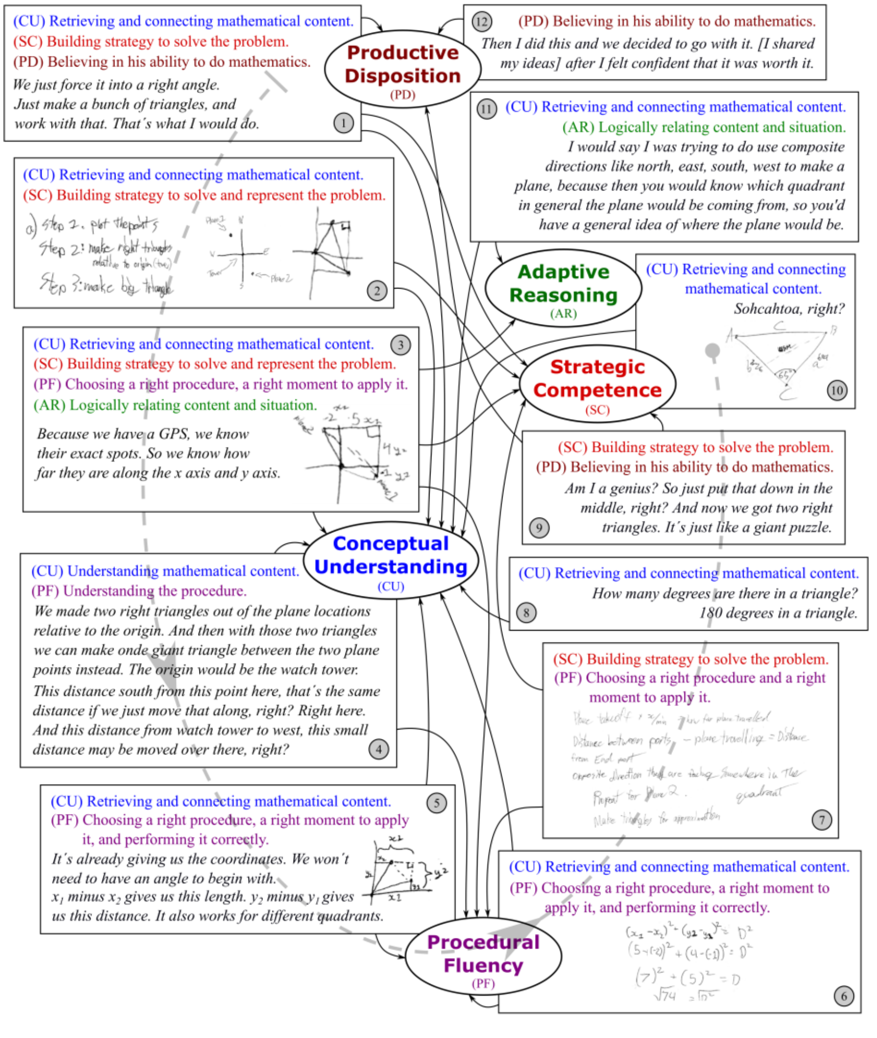

Figure 2. A student's diagram for the flight simulator task.

RESEARCH FINDINGS

To begin with, it is relevant to emphasize that this study reveals the feasibility of using modeling approaches in mathematics classrooms. The modeling tasks used in this research faced usual time and curriculum constraints. No special arrangements were made apart from working towards Davis and Simmt's (2003) five complexity conditions. In this sense, it can be said that the classroom environment was pretty close to what is commonly expected in a mathematics high-school classroom. This research reinforces the possibility of conducting mathematical modeling interventions along one regular mathematics course without hindering curriculum goals or wasting classroom time.

One of the main outcomes of this research speaks to the possibility of observing all five Kilpatrick et al.'s (2001) strands of mathematical proficiency when high-school students are engaged in mathematical modeling tasks. Indeed all 16 diagrams created for this research data analysis confirm that students worked on conceptual understanding, strategic competence, procedural fluency, adaptive reasoning, and productive disposition at least once during the mathematical modeling tasks. I am not claiming that the use of modeling tasks in class will necessarily promote all strands of mathematical proficiency; I am arguing about the richness and the possibilities of the use of mathematical modeling in class. This study confirms that mathematics classes based on modeling can play a relevant role in students' mathematical understanding and mathematical proficiency.

Another relevant outcome refers to the fact that students did not need to get to a final answer to work on all five strands of mathematical proficiency. In other words, the modeling process promotes the five strands of mathematical proficiency throughout the modeling task, which means mathematical understanding and proficiency are not dependent on tasks' final results. This is a significant advantage given that students do not need to be under pressure to reach a final formal solution: as if getting to a final solution was a condition for them to learn. Instead, students are learning and working on mathematical skills throughout the implementation of the task. Not needing to get to a final answer reinforces that processes should be given more attention than products.

Finally, this research also offers a doable approach to teachers and student teachers by offering a way of accessing and unpacking students' mathematical understanding and proficiency. This is possible by observing students' modeling processes during tasks and by means of implementing the diagram-based approach created for this research analysis. Carlson, Larsen and Lesh (2003) underline that the use of modeling practices within school is relevant for student teachers, because these practices allow teachers to have regular "access to the developing understandings and reasoning patterns of their students" (p.476). The use of modeling approaches in class is not only possible but also supportive of teachers' work.

FINAL CONSIDERATIONS

The research findings point to modeling as a potential way of teaching and learning mathematics as well as a potential way of working on students' mathematical understanding and proficiency. In agreement with that, English (2013) affirms that

> One means of preparing students for existing and future challenges is the inclusion of complex modelling problems within the curriculum. Such problems place students in multidimensional situations that require them to make reasoned and sophisticated choices about the knowledge they will apply and how, and ways in which they might communicate and share their products. (p.502)

One possibility to be implemented in future research is the use of the diagram-based approach created for this research purpose together with Kilpatrick et al.'s (2001) model of mathematical proficiency as a way of assessing students' understanding and proficiency in different performance tasks and not only when engaged in modeling tasks. This alternative rubric for assessment seems to have potential when compared with paper and pencil tests to observe not much more than some conceptual understanding and procedural fluency.

To conclude, it is worth noting that the research findings are of relevance for the mathematics education research community as they aim to draw teachers and researchers' attention to helpful aspects related to the challenging task of teaching for understanding through mathematical modeling approaches. By accessing students' mathematical understanding and proficiency, this study reveals an important aid for teachers to detect students' thoughts and misconceptions, plan their classes seeking for students' understanding, and enhance strategies for teaching for understanding and knowledge production. Lastly, the extension of Kilpatrick et al.'s (2001) work, by means of a diagram-based approach created to analyze student's work, is also an important contribution to the field: either due to its methodological nature or due to its assessment value.

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INCORPORATING MATHEMATICAL WRITING INTO A SECOND-YEAR DIFFERENTIAL EQUATIONS COURSE

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This is a summary of the paper "Student perceptions of the use of writing in a differential equations course" (DeDieu & Lovric, 2018).

INTRODUCTION

Incorporating mathematical writing into the curriculum can be an incredibly valuable tool. In addition to helping students better understand course content, precise mathematical writing can train students to become logical thinkers and construct rigorous arguments. It can also teach students how to effectively communicate complex ideas to non-experts.

The setting of this study was a second-year differential equations class containing assignments which emphasized writing quality over mathematical correctness. We explored whether students believed that the written assignments were an effective learning strategy and asked whether or not they believed that working on the assignments led to enhanced communication skills. We also analyzed the extent to which students' pre-existing beliefs may have contributed to these attitudes. Since psychological determinants can have a strong impact on student learning, the hope is that these insights can help instructors frame writing assignments in a way that will help achieve their desired learning outcomes.

METHODOLOGY

SETTING

The participants in this study were enrolled in a second-year differential equations course at a mid-sized research university in southern Ontario. The course was taught during a seven-week summer semester. There were 48 students enrolled in the course and the majority had no formal training in mathematical writing. The course emphasized both qualitative (theoretical, geometric) and quantitative (analytic, numerical) approaches to studying differential equations.

This course had weekly written assignments worth 10% of the students' final grade. The assignments emphasized writing quality over mathematical correctness and contained a variety of excogitative, expository, and reflective questions. The excogitative questions were theory-based questions which asked students to explain their mathematical thinking carefully and thoroughly. The expository questions asked students to use prose to explain complex mathematical ideas to non-experts. The reflective questions asked students to reflect on their thoughts, feelings, and experiences as they related to the content being learned. Full assignment prompts can be found in Appendix B of the full paper (DeDieu & Lovric, 2018).

DATA COLLECTED

The authors analyzed students' written assignments and administered a survey which consisted of 11 Likert scale questions and two free-response questions. The survey was inspired by Latulippe and Latulippe's (2014) Writing Project Survey. Fourteen students completed the online survey and 12 gave permission for their written assignments to be analyzed. We believe that the low response rate was due to the fact that we recruited students via email one month after the course had ended as it took longer than expected to gain ethics approval.

RESULTS

We found that students held a positive view towards the written assignments. Moreover, students viewed writing as an effective learning strategy in the sense that the written assignments helped to unify course concepts and pushed students to question their understanding, work harder, and seek help.

However, students did not tend to believe that these written assignments improved their general communication skills. Moreover, students tended to view writing and mathematics as distinct entities. This is encapsulated in the following student's response to the survey free-response question which asked the student to comment on how the written assignments impacted his/her learning:

> *At times, it felt like the written assignments were being given priority over the "doing mathematics" portion of the course. ... I agree that written assignments are an excellent way to develop a student's deeper understanding of the material—in any discipline, really—but deeper understanding must not come at the expense of the main goals of the course.*

We also analyzed students' narratives to understand more about the views they held about mathematics and their past mathematical experiences. We found that students were strongly predisposed to think in terms of symbols and were uncomfortable with analysis that involved other means, such as theory and geometry. Moreover, students had a tendency to view mathematics as a disjoint collection of subdisciplines. Students also shared that they refined their problem-solving skills throughout the course and many explained how challenging questions pushed them to refine their study skills.

DISCUSSION

The hope is that this study will provide ideas and insights to other instructors who are interested in using mathematical writing in their courses. In my talk at CMESG I outlined my personal takeaways from this study and the way it has influenced my teaching practice. I will briefly summarize those ideas here.

A personal goal of mine is that I want my students to cultivate their ability think logically and communicate complex ideas to non-experts. I use writing assignments in my courses to help achieve this goal. The students in our study did not tend to believe that the mathematical writing activities helped to strengthen their general communication skills. This inspired me to change the way I structured written assignments, the way I graded assignments, and the way I motivated mathematical writing.

In the differential equations course in this study, assignments contained a variety of excogitative, expository, and reflective writing questions. This made grading and motivating assignments challenging, as each of the three types of writing are very different and can have very different benefits. As such, when I used writing assignments in a subsequent course, I chose to separate these types of writing. I had students keep journals where they submitted weekly reflections which were graded based on completion. Students also completed four assignments which each contained one expository writing question. These expository assignments were worth approximately 15% of the final course grade and were graded using a rubric. Half of their assignment score was based on mathematical correctness and the other half was based on quality of writing. Moreover, students completed more traditional written assignments which contained excogitative writing questions. Many of these questions were also given separate scores for mathematical correctness and quality of writing.

Separating the different types of writing in this way made it easier to open a dialogue with students about why they were being asked to complete these writing assignments and what the potential benefits might be. I had students discuss these motivating questions in groups several times throughout the semester and submit reflections where they discussed how their future self might benefit from the skills they were developing through writing. I did not conduct a formal study to measure the effects, but end-of-term feedback suggests that students found value in the written assignments. In particular, in their final reflective journal prompt (which asked students to discuss whether or not writing enhanced their experiences in the course), many students specifically chose to talk about how mathematical writing helped to enhance their general communication skills. Even this student, who disliked the expository writing assignments, found value in them:

> *While I strongly disliked writing up the professional problems* [PP], *I feel they did the most to enhance my learning. Being able to do a homework problem is completely different from being able to communicate your findings in a concise, professional, and understandable manner. I feel like the PP write ups forced me to sit and communicate my findings in a simpler, less mathematical way which is similar to what math majors do in their careers.*

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DES DISCOURS RACIALISÉS PERSISTANTS POSENT UNE NOUVELLE DEMANDE D'ÉQUITÉ POUR LA FORMATION DES ENSEIGNANTS

PERSISTING RACIALIZED DISCOURSES POSE NEW EQUITY DEMAND FOR TEACHER EDUCATION

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RESUME

Cette étude s'appuie sur la théorie socioculturelle de l'apprentissage et la théorie de la race critique pour centrer et privilégier les récits racialisés de six élèves de sexe féminin de couleur inscrites dans un cours de statistiques avancées, caractérisé par la mise en œuvre d'une instruction de grande qualité ambitieuse et axée sur l'équité. Les données comprennent des entretiens avec des élèves, un entretien avec l'enseignant et six mois d'observations en classe. Le but de cette étude est d'examiner de près le contexte de cette classe de mathématiques afin de comprendre et de découvrir les moyens par lesquels un enseignant peut aider les élèves à négocier et à gérer leurs identités raciales tout en apprenant les mathématiques.

Les résultats indiquent que même dans un contexte de classe reflétant un enseignement axé sur l'équité et organisé pour soutenir l'apprentissage des mathématiques ainsi que l'identité académique des élèves, cette classe persiste à être un endroit dans lequel le discours racialisé persiste sur les capacités en mathématiques et la manière dont les élèves s'identifient, ou sont identifiés par d'autres, racialement. Ces résultats sont discutés, en accordant une attention particulière aux implications pour les enseignants et les formateurs d'enseignants. En particulier, je suggère qu'en tant qu'enseignants, il est nécessaire, mais pas suffisant que nous adoptions un grand nombre de pratiques que cet enseignant a adoptées pour aider les élèves. En même temps, en tant que formateurs d'enseignants, nous devons aider les enseignants à s'informer sur leur identité vis-à-vis leurs élèves et sur l'importance de la race et du racisme pour l'enseignement et l'apprentissage des mathématiques, afin d'engager les élèves dans des conversations sur le racisme et perturber les présuppositions concernant la capacité en mathématiques.

ABSTRACT

These proceedings draw on sociocultural theory of learning and critical race theory to centre and privilege the racialized narratives of six girls of colour who were enrolled in an AP Statistics classroom characterized by high-quality implementation of *ambitious and equity-oriented instruction. Data includes interviews with focal students, an interview with the teacher, and six months of classroom observations. The purpose of these proceedings is to look closely within this mathematics classroom context to understand and uncover some ways in which a classroom teacher can support students to negotiate and navigate their racial identities while learning mathematics.*

Findings indicate that even within a classroom context that reflects equity-oriented instruction and is organized to support students' academic identities and mathematics learning, this classroom persists to be a site in which racialized discourse persists regarding how students are positioned as doers of mathematics in relation to how they racially identify or are identified by others. These findings are discussed, paying close attention to the implications for teachers and teacher educators. Particularly, I suggest that as teachers it is necessary but not sufficient that we adopt many of the practices that this classroom teacher has adopted in support of students. At the same time, as teacher educators we need to support teachers to inquire into their own identities vis-à-vis their students', and the salience of race and racism for mathematics teaching and learning, to engage with students in conversations around race and racism, and disrupt assumptions regarding ability in mathematics.

> *Je pense qu'il est important que les enseignants associent l'actualité et les événements à leurs leçons pour que les élèves soient au courant de ce qui se passe dans le monde, car il est important qu'ils connaissent ces informations. Peut-être cette idée qu'ils ont entendue en classe, ce sujet réel encouragera autre chose. Peut-être qu'ils écriront un livre à ce sujet ou qu'ils commenceront un mouvement autour de lui. Vous ne savez jamais ce que vous pourriez inspirer en aidant simplement les enfants à savoir ce qui se passe.* (Leilani, entrevue d'élève)

Les enseignants doivent connecter leurs salles de classe à des événements réels et à la vie des élèves, car leurs effets peuvent encourager et inspirer les élèves à utiliser ce qu'ils apprennent pour engager et changer le monde qui les entoure. En tant que communauté en didactique des mathématiques, nous avons appris qu'un enseignement des mathématiques équitable doit être adapté aux élèves et à leurs identités académiques et sociales (par exemple, Gay, 2000; Gutstein, 2008; Ladson-Billings, 2014). L'instruction complexe (IC; Cohen, 1994) est un exemple d'une forme d'enseignement ambitieuse et axée sur l'équité, dont il a été démontré qu'elle soutenait l'avancement de la compréhension conceptuelle en mathématiques et aidait les étudiants à développer des identités mathématiques fructueuses (par exemple, Boaler & Staples, 2008; Jilk, 2011). Connaître les expériences racialisées d'après une diversité d'étudiants de couleurs peut être un avantage pour comprendre l'identité et le discours racial dans les milieux d'apprentissage des mathématiques, à travers les lignes raciales. Bien que cette enseignante, employant CI, ait aidé avec succès des élèves (généralement de couleurs) à participer et à réussir en mathématiques de niveau supérieur, ces élèves ont exprimé des récits racialisés autour de leurs apprentissages.

Ici, on s'appuie sur la thèse de l'auteur (Nazemi, 2017) qui a montré que, dans cette salle de classe ambitieuse et conduite équitablement avec une enseignante consciente de la race et de l'identité de ses élèves, des discours et des processus racialisés persistaient. Centrer la voix des élèves (à travers différentes identités « de couleur ») a montré que la complexité des identités raciales et académiques des élèves était liée à leurs expériences, intégrées dans le contexte d'apprentissage dans lequel ils se trouvaient (Nazemi, 2017). Les expériences racialisées des élèves, dans un cours de mathématiques axé sur l'équité, constituent une réalité à laquelle les enseignants doivent être conscients et prêts à réagir. Les suppositions sur les personnes qui peuvent ou ne peuvent pas faire et réussir en mathématiques affectent la façon dont les élèves interagissent les uns avec les autres, avec leur professeur et avec l'apprentissage des mathématiques avec lequel ils s'engagent. Ce travail a pour objectif de suggérer des implications pour la formation des enseignants, centrées sur l'identité et les expériences des élèves et adaptées à celles-ci.

> *I think it's important [for teachers to] involve current events that you have your students stay on top of what's going on in the world because it's important that they know this stuff. [M]aybe this idea that they heard in class, this real-life topic will encourage something else. Maybe they'll write a book about it or maybe start a movement around it. You never know what you could inspire by just helping kids know what's going on.* (Leilani, student interview)

Teachers need to connect their classrooms to real-life events and to students' lives, as the effects can be as far reaching, such that students might feel encouraged and inspired to use what they learn to engage with and change the world around them. As a mathematics education community, we have come to know that equitable mathematics teaching needs to be responsive to students and their academic and social identities (e.g., Gay, 2000; Gutstein, 2008; Ladson-Billings, 2014). Complex Instruction (CI; Cohen, 1994) is one example of an ambitious and equity-driven form of instruction which has been shown to support students' advancement of conceptual understanding in mathematics and support students to develop productive mathematical identities at the same time (e.g., Boaler & Staples, 2008; Jilk, 2011). Knowing about the racialized experiences of students, from a diversity of student of colour perspectives can be advantageous in understanding identity and racial discourse in mathematics learning, across racial lines. While this classroom teacher, employing CI, has successfully supported students (who are by and large of colour) to participate and succeed in upper level mathematics, these students expressed racialized narratives around learning.

These proceedings draw upon the author's dissertation work (Nazemi, 2017) which showed that, in this classroom context that is ambitious and equitably driven with a teacher who is conscious of race and her students' identities, racialized discourses and processes of racialization persisted. Centering students' voices (across various 'of colour' identities) showed the complexity of students' racial and academic identities through hearing from them about their experiences, as embedded in the contexts of learning in which they are situated (Nazemi, 2017). Students' experiences, in an equity-oriented mathematics classroom, as racialized in nature, is a reality that teachers need to be aware of and prepared to respond to. Prevalent assumptions around who can and cannot know, do and succeed in mathematics, affects how students interact with one another, their classroom teacher and the mathematics learning they engage with. This work intends to suggest implications for teacher education, centered around, and responsive to, students' identities and experiences.

LES CADRES THÉORIQUES

Ce travail s'appuie sur la théorie socioculturelle de l'apprentissage et la théorie de la race critique (CRT) pour centrer et privilégier les récits racialisés de six élèves de sexe féminin de couleur inscrites à une classe de Statistiques avancées, caractérisée par la mise en œuvre d'une instruction de grande qualité ambitieuse et axée sur l'équité.

Les premiers théoriciens socioculturels de l'éducation ont démontré que la pensée, le développement et l'apprentissage dépendaient beaucoup du contexte social et culturel dans lequel ils se déroulaient (par exemple, Lave & Wenger, 1991; Rogoff & Lave, 1984; Vygotsky, 1978). À travers des études ethnographiques, Lave et Wenger (1991) ont montré que l'apprentissage est « situé » ou a lieu dans un contexte social. Cela signifie que les contextes dans lesquels les étudiants se situent sont liés avec ce qu'ils apprennent, ce qu'ils sont en train de devenir et comment ils se voient. De cette manière, nous pouvons considérer l'identité comme un produit, jamais complet, des histoires socioculturelles (Hall, comme cité dans

Nazemi, 2017) ou « d'être reconnu comme un certain « genre de personne » dans un contexte donné » (Gee, 2000, p. 99). Wenger (1998) nous a montré comment l'apprentissage et l'identité sont liés, de sorte que participer au contexte social de l'apprentissage « façonne non seulement ce que nous faisons, mais aussi qui nous sommes et comment nous interprétons ce que nous faisons » (p. 4). Bien que la théorie socioculturelle aide à relier apprentissage et identité, cette approche ne reconnaît pas le rôle éminent de la race et du racisme dans tous les contextes, mais certains chercheurs ont relevé ce manque (Esmonde & Booker, 2016; Gutiérrez & Rogoff, 2003; Nasir & Hand, 2006).

En réponse aux lacunes des perspectives socioculturelles, de nombreux chercheurs se sont concentrés sur les expériences identitaires d'étudiants de couleur afin de reconnaître le rôle de la race et du racisme dans les contextes d'apprentissage (par exemple, Esmonde, Brodie, Dookie, & Takeuchi, 2009; Leyva, 2016; McGee, 2016; Nasir & Shah, 2013; Oppland-Cordell & Martin, 2015; Shah, 2017; Stinson, 2008; Zavala, 2014). Certains chercheurs—parmi lesquels certains des auteurs susmentionnés—expliquent la prévalence de la race et les processus de racialisation dans les contextes d'apprentissage, en utilisant un objectif de théorie de la race critique (CRT) dans l'éducation (Ladson-Billings & Tate, 1995; Martin, 2012). Le CRT aide les chercheurs à reconnaître, à centrer et à légitimer des contre-récits des élèves de couleur qui racontent leurs propres expériences eux-mêmes (Delgado & Stefancic, 1995; Solórzano & Yosso, 2001), tout en analysant la fonction de la race et le racisme dans la vie des élèves, dans les salles de classe et dans le système éducatif (Tate, 1997). Dans le contexte de l'apprentissage des mathématiques, les chercheurs ont utilisé le CRT pour examiner la complexité des expériences des étudiants en couleurs et pour fournir un cadre dans lequel les contre-récits de résilience et de réussite sont centrés et partagés (par exemple, Berry, 2008; Corey & Bower, 2005; Martin, 2012; Stinson, 2008; Zavala, 2014). Peu de ces études, comme celle-ci, ont utilisé à la fois la CRT et la théorie socioculturelle (par exemple, Esmonde et al., 2009; Leyva, 2016; Zavala, 2014). Alors que de nombreuses études se concentrent directement sur les étudiants afro-américains (Berry, 2008; Corey & Bower, 2005; Martin, 2012; Stinson, 2008), ces études, centrées sur les récits d'étudiants de couleur (par lentille de CRT), ont permis aux chercheurs de découvrir et de donner un sens aux expériences des élèves en mathématiques et à la manière dont ils ont interprété leurs identités et leur ont permis de naviguer, vis-à-vis de l'apprentissage des mathématiques. Ce travail a permis aux chercheurs de montrer comment les étudiants de couleur réussissaient en mathématiques en résistant, en persistant et en portant des identités positives, souvent en opposition avec les récits raciaux dominants sur leur identité (par exemple, Berry, 2008; Martin, 2012; Stinson, 2008). Ici, à travers le CRT, c'est par la découverte et le centrage des récits racialisés des élèves que nous comprenons les implications de ces contre-récits pour la formation des enseignants.

THEORETICAL UNDERPINNINGS

This work draws on sociocultural theory of learning and critical race theory to centre and privilege the racialized narratives of six girls of colour who were enrolled in an AP Statistics classroom characterized by high-quality implementation of equity-oriented instruction (CI).

Early sociocultural theorists in education, demonstrated that thinking, development and learning depend greatly on the social and cultural contexts in which they take place (e.g., Lave & Wenger, 1991; Rogoff & Lave, 1984; Vygotsky, 1978). Specifically, through ethnographic studies, Lave and Wenger (1991) showed that learning is 'situated', or takes place, within social context. This means that the contexts in which students are located have everything to do with what they are learning, who they are becoming and how they see themselves. In this way, we can think about identity as a never-to-be complete product of socio-cultural histories (Hall, as cited in Nazemi, 2017) or "being recognized as a certain 'kind of person' in a given context"

(Gee, 2000, p. 99). Wenger (1998) showed us ways in which learning and identity are connected, so that participating in the social context of learning "shapes not only what we do, but also who we are and how we interpret what we do" (p. 4). While sociocultural theory helps to connect learning and identity, this approach does not recognize the imminent role of race and racism in all contexts, a shortcoming some scholars have remarked (e.g., Esmonde & Booker, 2016; Gutiérrez & Rogoff, 2003; Nasir & Hand, 2006).

In response to the shortcomings of sociocultural perspectives, many scholars have focused on identity-related experiences of students of colour to recognize the role of race and racism in learning contexts (e.g., Esmonde, Brodie, Dookie, & Takeuchi, 2009; Leyva, 2016; Nasir et al, 2013; Oppland-Cordell & Martin, 2015; McGee, 2016; Shah, 2017; Stinson, 2008; Zavala, 2014). Some scholars—including some of the aforementioned—account for the prevalence of race and the processes of racialization in learning contexts, through employing a Critical Race Theory (CRT) lens in education (Ladson-Billings & Tate, 1995; Martin, 2012). CRT helps scholars recognize, centre, and legitimize counter-narratives that speak to the lived-experiences of students of colour in their own words (Delgado & Stefancic, 1995; Solórzano & Yosso, 2001), while calling out and analyzing the function of race and racism both in students' lives, classrooms, and the educational system (Tate, 1997). In the context of mathematics learning, CRT has been used by scholars to examine the complexities of students of colour's experiences and to provide a framework where counter-narratives of resilience and success are centered and shared (e.g., Berry, 2008; Corey & Bower, 2005; Martin, 2012; Stinson, 2008; Zavala, 2014). Few of these studies, like this one, employed both CRT and sociocultural theory together (e.g., Esmonde et al., 2009, Leyva, 2016; Zavala, 2014). While many of these studies focus squarely on African American male students (e.g., Berry, 2008; Corey & Bower, 2005; Martin, 2012; Stinson, 2008), these studies, centering the narratives of students of colour by way of CRT, allowed scholars to uncover and make sense of students' experiences in mathematics and how these students made sense of and navigated their identities, vis-à-vis learning mathematics. This work has allowed scholars to show ways in which students of colour have been successful in mathematics through resisting, persisting, and carrying positive identities, often in opposition to the dominant racial narratives about their identities (e.g., Berry, 2008; Martin, 2012; Stinson, 2008). Here, through CRT, it is through the uncovering and centering of students' racialized narratives, that we understand the implications these counter-narratives pose for teacher education.

MÉTHODES DE RECHERCHE ET CONTEXTE

Les données présentées ici proviennent d'une étude de six mois explorant la manière dont six élèves de couleur (Carlin, Gena, Jane, Leilani, Lia, Mya) ont navigué et négocié leur identité tout en apprenant les mathématiques dans une classe de statistiques avancées, avec une enseignante blanche qui attentive aux questions de race, et emploie une forme d'instruction (IC) axée sur l'équité. L'objectif principal de l'étude était de comprendre comment ces élèves de couleur estimaient que leurs identités raciales jouaient un rôle dans leur apprentissage des mathématiques, comment elles se percevaient par rapport aux autres élèves de leur classe, quelle sorte de récits racialisés persistaient malgré l'enseignement équitable, et comment ces récits racialisés ont reflété ou contredit les idéologies néolibérales dominantes. J'ai utilisé une méthodologie d'étude d'entrevue qualitative (Yin, 1994), puis une théorie de « standpoint » (Haraway, 1988), la CRT et des théories socioculturelles de l'apprentissage et de l'identité afin de centrer les récits de ces élèves de couleur, pour comprendre leurs expériences dans leurs propres mots. Les sources principales de données pour cette étude comprenaient huit heures de données d'entrevue, avec des élèves de couleur et leur enseignante, ainsi que des notes basées sur 23 heures d'observations en classe.

Les élèves ont été interrogées sur leurs identités raciales et ethniques et ont répondu par différentes identifications raciales. Elles se sont identifiées comme « *sud-asiatique, cambodgienne* » (Jane) et « *afro-américaine* » (Leilani), ou « *mélangée* » (Lia), « *multiraciale* » (Mya), « *biraciale* » (Carlin). L'identification de soi de Gena s'est distinguée par le fait que, même si elle s'est identifiée comme philippine, elle a tout d'abord énuméré toutes les manières dont elle ne s'identifie pas, mais était présumée par d'autres (Nazemi, 2017). En raison de la nature délicate à parler aux gens de leur identité raciale, j'ai accepté la langue fournie par les élèves et demandé respectueusement des éclaircissements lorsque des opportunités se présentaient. Par exemple, Lia s'est identifiée comme « mélangée » et elle a développé ce terme en disant : « *Ma mère est blanche et noire, puis mon père est noir* ». Pourtant, quand j'ai interrogé Mya sur son identité raciale, elle a dit : « *Un peu comme les multiraciaux* » et tout au long de notre conversation, il n'a jamais été approprié ni respectueux de sonder davantage pour connaître les spécificités de son identité raciale choisie.

Les analyses se sont déroulées en deux phases. Les données relatives aux élèves ont d'abord été examinées pour explorer et centrer les récits racialisés, puis les données d'observation en classe ont permis de décrire l'instruction en classe et la manière dont les élèves interagissaient, avec leur professeur et leur programme. La deuxième phase de l'analyse a également placé les étudiants dans leur salle de classe, leur école et dans un contexte social plus large, afin d'explorer la manière dont leurs récits se sont déroulés pour ou contre un discours plus large du racisme institutionnel et du néolibéralisme.

Cette étude découle de mon travail en tant qu'expert pédagogique en didactiques des mathématiques dans un programme de formation des enseignants du niveau secondaire dans la région du Pacifique nord-ouest. J'ai rencontré pour la première fois Mme Williams, l'enseignante, grâce à son partenariat avec mon université. J'ai vite remarqué et appris que, contrairement à tout autre enseignant que j'avais rencontré, l'attention de Mme Williams à ses élèves et à leur identité (scolaire et raciale) était exceptionnelle et méritait d'être mieux comprise. Ceci, en plus de la classe de statistiques avancées qui présente une diversité raciale différente de celle de nombreuses autres classes de mathématiques de niveau supérieur, a créé un phénomène que je souhaitais explorer davantage. Ainsi, à certains égards, cette salle de classe était typique des autres salles de classe urbaines du pays (par exemple, Grenfell & James, 1998; Howard, 1999; Nieto, 2005; Stiff &Harvey, 1988)—étant donné l'enseignante qui s'identifie comme de race blanche et les nombreux étudiants de couleur. Pourtant, cette classe était atypique pour une classe de mathématiques de niveau supérieur en termes de composition raciale des élèves, ce qui en faisait un site utile pour explorer les expériences des élèves en mathématiques (Bol & Berry, 2005; Viadero, 2002).

Mon identité de chercheur était au cœur de mes relations avec les élèves principaux, l'enseignant en classe et l'école en général. Je suis une femme de couleur qui, au moment de la collecte des données, était en train de terminer un doctorat en éducation. En raison de contraintes d'espace, je ne discuterai pas davantage de ma position en tant que chercheur ici. Pour plus d'informations sur mon positionnement et sur le plan de recherche de l'étude, voir Nazemi (2017).

RESEARCH METHODS AND CONTEXT

The data presented here is from a 6-month study exploring how six girls of colour (Carlin, Gena, Jane, Leilani, Lia, Mya) navigated and negotiated their identities while learning mathematics within an AP Statistics classroom, with a race-conscious white woman teacher who employs an equity-driven form of instruction (CI). The overarching goals of the study were to understand how these girls of colour felt their racial identities played a role in their learning mathematics, how they view themselves in relation to other students in their classroom, what sort of racialized narratives persisted despite the equitable forms of instruction taking place, and how these racialized narratives reflected or ran counter to dominant neoliberal ideologies. I employed qualitative interview study methodology (Yin, 2003) and drew upon standpoint theory (Haraway, 1988) and Critical Race Theory (CRT) along with Sociocultural Theories of Learning and Identity, to ensure that I was centering the stories of these girls of colour to understand their experiences in their own words. The major data sources for this study included eight hours of interview data, with focal girls of colour and their classroom teacher, and field notes based on and 23 hours of classroom observations.

Students were asked about their racial and ethnic identities, and responded with various racial self-identifications. They identified as "*South Asian, Cambodian*" (Jane) and "*African American*" (Leilani), or "*mixed*" (Lia), "*mixed race*" (Mya), "*multiracial*" (Carlin). Gena's self-identification stood out in that, while she identified as Filipino, she first listed all the ways in which she did not identify yet was assumed to by others (Nazemi, 2017). Because of the sensitive nature of asking people about their racial identity, I accepted the language that students provided and respectfully probed for further clarification when given the opportunity. For example, Lia identified as "*mixed*", and she expanded on this term to say, "*My umm my mom is White and Black, and then my dad is Black*." Yet, when I asked Mya about her racial identity, she said, "*Pretty much like mixed race*" and throughout our conversation it never felt appropriate or respectful to probe further to know the specifics behind her chosen racial identification.

Analyses took place in two phases: student level data was first examined to explore/centre racialized narratives, and then classroom observation data helped describe classroom instruction as well as how students interacted with one another, their classroom teacher and curriculum. The second phase of analysis also situated students within classrooms and within the larger social context to explore how their narratives ran in support of, or counter to, greater discourses of institutional racism and neoliberalism.

This study grew out of my work as a mathematics instructional coach in a secondary teacher education program in the Pacific Northwest. I first came to know the classroom teacher, Ms. Williams, through her and her school's partnership with my University. I observed and learned quickly that, unlike any other teacher I had met, Ms. Williams' attention to her students and their identities—both academically and racially—was something unique and worth understanding more closely. This, in addition to the AP Statistics classroom being racially diverse in ways unlike many other upper level mathematics classroom, made for a phenomenon I wished to further explore. Thus, in some ways this classroom was typical of other large urban classrooms in the country (e.g., Grenfell & James, 1998; Howard, 1999; Nieto, 2005; Stiff & Harvey, 1988)—given the white woman identifying teacher and the many students of colour. Yet, this classroom was atypical for an upper level mathematics classroom in terms of the racial composition of students, making it a useful site for exploring the experiences of students of colour in mathematics (Bol & Berry, 2005; Viadero, 2002).

My research identity was central to my relationship with the focal students, the classroom teacher and the school at large. I am a woman of colour, who at the time of data collection was completing a doctoral degree in education. Because of space restraints I will not further discuss my positionality as a researcher here. For more on my positionality and about the research design of the study, see Nazemi (2017).

QUELQUES CONCLUSIONS

Dans l'ensemble, les résultats indiquent que même dans un contexte de classe reflétant un enseignement ambitieux et axé sur l'équité et organisé pour soutenir l'identité académique des élèves et leur apprentissage des mathématiques, cette classe est un site où le discours racialisé persiste quant à la façon dont les élèves s'identifient racialement ou sont identifiées par d'autres élèves. Cette section s'appuie sur trois conclusions principales : la représentation ne suffit pas; il faut s'occuper à l'apprentissage et à l'identité des étudiants; et même avec l'instruction ambitieuse et équitable, on voit les récits racialisés. Ici, je ne discute que de la troisième de ces conclusions. Pour les autres, veuillez consulter Nazemi (à paraître).

MÊME AVEC L'INSTRUCTION AMBITIEUSE ET ÉQUITABLE, ON VOIT LES RÉCITS RACIALISÉS

Même avec une forme d'enseignement ambitieuse et axée sur l'équité par une enseignante soucieuse de la race, l'expérience des élèves montre que les discours racialisés sont profondément ancrés au sein de leurs pairs et de leurs groupes raciaux. Ici, je me concentre sur trois élèves de couleur qui ont identifié des récits racialisés qui façonnent comment elles sont perçues par les autres et comment elles se voient par rapport aux autres, dans la salle de classe de Mme Williams.

Gena, qui s'est identifiée avec la race asiatique, a été la seule élève à répondre « oui » quand j'ai demandé si elle pensait que son identité raciale avait une incidence sur son apprentissage des mathématiques. Dans sa réponse, elle a fait référence au stéréotype bien connu, que les Asiatiques sont bons en mathématiques (par exemple, Cvencek, Nasir, O'Connor, Wischnia, & Meltzoff, 2015; Nasir & Shah, 2011). Elle a estimé qu'en raison du stéréotype que tous les Asiatiques sont bons en maths, ses pairs s'attendent à ce qu'elle soit bonne en maths pendant les travaux en groupe. Plus précisément, elle a déclaré : « *Je pense que c'est parce que les gens sont comme 'Les Asiatiques peuvent faire des mathématiques', et je suis comme 'non, pas tous les Asiatiques peuvent faire des mathématiques, parce que je sais que beaucoup d'entre nous ne savent pas ce qui se passe !'* » En classe, j'ai vu Gena travailler avec son groupe, quand ils menaient une enquête auprès de leurs pairs à Champlain sur un sujet qu'ils étaient curieux d'approfondir en utilisant des méthodes statistiques. Le groupe de Gena était composé d'une élève musulmane (je me base sur son foulard), d'un étudiant musulman Afro-américain (je le sais parce que je l'ai interviewé) et de Carlin, qui est biraciale (élève principale dans mon étude). Le groupe de Gena a décidé d'explorer la relation entre le manque de sommeil et la race. Gena me parlait de sa petite amie (qui s'identifie comme blanche) qui ne comprenait pas pourquoi Gena—qui a un travail à temps partiel en plus de suivre ses études—ne semble pas pouvoir dormir suffisamment. Tandis que Gena me raconte ses frustrations de suivre son travail et son école et que sa petite amie « *ne comprend pas* », car elle n'a pas besoin de travailler pour aider sa famille[1], les pairs de Gena dans son groupe ont continué à poser des questions pour leur enquête. Gena semblait heureuse d'assumer un rôle de chef dans son groupe et d'utiliser ses expériences personnelles pour motiver l'exploration de l'enquête (mes notes, 24 mai 2016). La direction de Gena dans son groupe m'a suggéré que les autres étudiants la considèrent probablement comme intelligente et sachant, peut-être parce que, comme je l'ai observé, elle était la seule élève asiatique de son groupe.

Leilani, qui s'est identifiée comme afro-américaine, est l'autre élève à avoir évoqué le stéréotype « Les Asiatiques sont doués en maths », toujours dans le contexte du travail en

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[1] C'est un exemple où la classe semblait être un marqueur social important pour les étudiants. Gena et moi avons eu une conversation au cours de laquelle je disais qu'il pourrait y avoir des facteurs et des circonstances qui, avec la race, prédisent les habitudes de sommeil. Elle semblait disposée à écouter ma suggestion mais semblait toujours voir le contraste entre elle et sa petite amie comme strictement racialisé.

groupe. Dans son cas, cependant, cette supposition a été évoquée en termes de ce que ce stéréotype signifie pour les étudiants qui ne sont pas asiatiques, en particulier les étudiants qui s'identifient comme noirs ou afro-américains, comme elle.

> *Je pense que parfois les gens, vous savez, comme le stéréotype est que les Asiatiques sont vraiment bons en maths, alors, euh, quand vous êtes dans un groupe avec comme tous les Asiatiques et que vous êtes un gamin noir, parfois vous pouvez vous sentir d'accord, ils vont penser que je ne suis pas aussi bien qu'eux dans ce domaine. Ensuite, vous commencez à vous rendre compte que vous savez peut-être que je ne suis pas aussi bien qu'eux et que vous savez parfois qu'ils disent certaines choses. Vous serez d'accord avec cela même si vous savez que ce n'est pas correct parce que vous voulez dire « Oh, ils savent », mais vous avez parfois raison. Je pense donc qu'il est important que les gens soient simplement en sécurité avec ce qu'ils savent et qu'ils ne tentent pas d'alimenter un stéréotype ou quelque chose qui ressemble à marcher contre vous.*

La discussion de Leilani sur les stéréotypes entre les capacités des élèves asiatiques et noirs en mathématiques s'accorde avec ce que d'autres chercheurs ont découvert (par exemple, Martin, 2009; McGee, 2016; Nasir & Shah, 2011; Shah, 2017). Plus précisément, il est courant que les étudiants de couleur parlent—et se placent eux-mêmes et leurs pairs—dans ce que Martin (2009) appelle la « hiérarchie raciale » des mathématiques (p. 315). Dans cette hiérarchie raciale, « les étudiants identifiés comme étant asiatiques et blancs sont placés en haut de la hiérarchie et les étudiants identifiés comme afro-américains, amérindiens[2] et latino-américains sont classés dans la partie inférieure » (p. 315). S'appuyant sur les travaux de Martin qui ont examiné de près la prévalence du discours racialisé dans l'enseignement des mathématiques, Shah (2017) a interrogé divers étudiants de couleur dans le contexte de leur classe de mathématiques. Il a également constaté que, selon les élèves, les élèves asiatiques et blancs se situaient au sommet en termes de capacités et de performances en mathématiques, tandis que les élèves identifiés comme insulaires-Pacifique ou noirs se situaient en dernière position.

J'ai demandé à Leilani un exemple spécifique pour illustrer son affirmation sur la façon dont les étudiants noirs perçoivent leurs capacités en mathématiques comme inférieures par rapport aux étudiants asiatiques. En réponse à ma question, Leilani a de nouveau souligné les conséquences négatives du stéréotype « Les Asiatiques sont bons en maths » pour d'autres groupes d'élèves.

> *Je pense avoir vu des gens faire des suppositions sur eux-mêmes. C'est comme si parfois j'entendais des enfants noirs en classe et qu'ils seraient comme eux, ils obtiendraient de très bons rangs à un test et ensuite, peut-être que les enfants asiatiques auraient eu un rang plus bas et qu'ils seraient comme « Hou la la ! J'ai fait mieux que le gamin intelligent » et je suis comme « wow, vous pensez que vous n'êtes pas intelligents… comme vous devriez avoir le sentiment que vous êtes intelligents. Vous ne devriez pas penser cela, vous devriez savoir que vous n'êtes pas moins qu'eux dans tout ce que vous faites. » J'entends des gens dire : « Je dois faire le projet avec les pairs intelligents » ou « Je dois rester à côté des gamins intelligents » vous savez ? Donc, j'ai le sentiment que c'est vraiment dégradant de penser que vous n'êtes pas intelligents.*

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[2] Alors que la hiérarchie raciale des mathématiques de Martin (2009) fait référence aux étudiants amérindiens, l'omission des Amérindiens et des peuples autochtones constitue l'un des lacunes de la plupart des travaux importants sur la race et la racialisation dans l'enseignement des mathématiques (et la recherche pédagogique en général). Je reconnais que les Amérindiens et les peuples autochtones sont maltraité par le système éducatif. Je veux être explicite sur le fait que j'ai reconnu à quel point ces groupes ont ét autochtones sont maltraité par le système éducatif. Je veux être explicite sur le fait que j'ai reconnu à quel point ces groupes ont été ignorés et rendus invisibles, tant historiquement que présentement, et que les recherches futures tiendront davantage compte de cette lacune.

Cohérant aux conclusions de Shah (2017), Leilani, comme de nombreux élèves de couleur, considérait le stéréotype, qu'ils ne sont pas aussi intelligents que leurs pairs asiatiques, comme « dégradant » à leur identité. En même temps, son contre-récit en est un de résilience, de prise de conscience des suppositions qui peuvent avoir des conséquences pour elle tout en travaillant dur pour ne pas les laisser faire. Les sentiments de Leilani face au stéréotype selon lequel les élèves noirs ne sont pas aussi doués en mathématiques sont liés à ce que nous verrons ensuite avec l'expérience de Carlin.

Carlin, comme Gena et Leilani, a expliqué comment son identité raciale avait joué un rôle dans l'apprentissage en raison des suppositions sur l'intelligence formulées par ses pairs sur la base de son identité raciale perçue. Carlin s'est identifiée comme étant « biraciale », et spécifiquement noire et blanche. Contrairement à Gena et Leilani, Carlin a décrit la façon dont elle « passait comme [blanche] », soulignant à quel point l'expérience racialisée d'une personne dépend du contexte, en particulier lorsque celle-ci passait comme blanche. Ici, Carlin révèle à quel point il est difficile de naviguer son identité multiraciale, d'autant plus que les deux races avec lesquels elle identifie sont associées à des suppositions très contrastées concernant l'intelligence (emphase ajoutée).

> Nazemi *: Pensez-vous que votre identité raciale joue un rôle dans votre apprentissage ?*
> Carlin : *Pas tellement, mais* depuis que je suis Caucasien, les gens, les gens s'attendent à ce que je sois plus intelligent pour une raison quelconque. *Et je suis comme bien ça n'a rien à voir avec mon intelligence. C'est juste…un peu juste une statistique. Et j'étais comme…ils…ils aiment….* Je passe beaucoup de temps avec les Afro-Américains, les Noirs, et ils s'attendent à ce que je sois plus intelligente pour une raison quelconque. Mais je ne suis que moi. *J'apprends juste comment j'apprends, et c'est…c'est bizarre.*
> Nazemi : *Comment savez-vous qu'ils s'attendent à ce que vous soyez plus intelligent ?*
> Carlin : *Ils le disent*
> Nazemi : *Ils le disent ?*
> Carlin : *Ouais.*
> Nazemi : *Ils disent, comme tout droit. Comme on dit « tu es blanche, tu es plus intelligente ? »*
> Carlin : *Non, ils ne disent pas « tu es blanche, tu es plus intelligente », c'est comme* quand je me bats et que je leur demande de l'aide, ils diront : « tu es blanche…tu devrais le savoir. » J'aime bien c'est un peu bizarre. Je suis aussi noire aussi alors…devrais-je savoir…devrais-je ne pas le savoir parce que je suis noire *? C'est bizarre pour eux de dire ça, alors j'écarte un peu les épaules parce que je ne laisse pas ce genre de choses me déranger.*

Carlin explique ici que passer pour blanche signifie qu'on s'attend à ce qu'elle soit plus intelligente. Elle décrit ci-dessus que, lorsqu'elle se débat avec un concept de mathématiques, ses pairs lui diront qu'elle devrait le savoir, car elle est blanche, à quoi elle répond qu'elle est aussi noire. Cette conclusion est cohérente à celle de Hobbs (2014) qui a décrit les effets sur l'identité raciale et culturelle d'une personne lorsqu'elle passe au blanc. McGee (2016), qui s'est inspiré de ce travail, a montré que, même s'il peut sembler que les élèves de couleur tirent un bénéfice du passage au blanc, il y a aussi beaucoup de choses « perdues par le rejet partiel ou total de leur identité raciale et culturelle » (p. 1654). Elle décrit en outre que le sentiment de pression pour limiter certaines parties de l'identité raciale et culturelle d'une personne témoigne de la manifestation durable et continue du privilège des blancs par le racisme et l'hégémonie des blancs.

Les suppositions sur Carlin et son intelligence changent quand on la voit comparée à un camarade de classe blanc. Par ailleurs, Carlin décrit la manière dont elle est perçue par rapport aux élèves blancs de sa classe (Nazemi, 2017) et estime que cela est différent de ce que ressentent les élèves de couleur qui la voient comme blanche. Bien que je n'aie pas eu l'occasion d'observer Carlin interagir avec ses camarades de classe blancs (car elle n'était pas regroupée avec eux lorsque j'observais la classe), j'ai observé les deux élèves blancs comme dominant les discussions en classe entière, et même les discussions de groupe. Selon mes observations, les deux étudiants blancs ont souvent été les premiers à poser une question et/ou à proposer une réponse, et se disputant souvent avec les étudiants devant la classe. Plus généralement, alors que Carlin a raconté qu'elle est souvent présumée blanche et présumée intelligente en présence d'autres personnes, lorsqu'elle se trouve en présence de ces étudiants blancs, elle se positionne comme moins intelligent—probablement, car son identité multiraciale et de son identité de genre sont considérées comme inférieur à l'identité d'un type « *entièrement blanc* » (pour utiliser son langage). Vouloir gérer la manière dont on est perçu par les autres est aussi dégradant qu'inutile. Les récits de Carlin nous aident à comprendre que même si, parfois, nous ne pouvons pas verbaliser ni comprendre pleinement pourquoi ou comment nous sommes perçus par d'autres, les implications pour notre identité, si elles ne sont pas parlées ou comprises, sont profondément ressenties. Simultanément, tout comme ce que Gena a raconté avec moi à propos du fait qu'elle est supposée être intelligente en mathématiques parce qu'elle est asiatique, Carlin explique qu'elle est supposée être intelligente parce qu'elle est vue comme de race blanche. Et, tout comme l'explication de Leilani selon laquelle les étudiants noirs sont perçus comme intellectuellement inférieurs aux étudiants asiatiques, l'anecdote de Carlin décrit les élèves noirs comme étant intellectuellement inférieurs aux étudiants blancs. De plus, contrairement à la blancheur et à l'intelligence présumées de Carlin, lorsqu'elle interagit avec un élève de type « *entièrement blanc* », elle est impuissante en raison de la complexité de son identité, qui ne se limite pas aux blancs, mais inclut noire et du genre fille.

Plus généralement, les étudiants ont confirmé que les discours sur une « hiérarchie raciale des compétences en mathématiques », dans lesquels les étudiants asiatiques et blancs sont supposés être au sommet et les étudiants noirs au bas (Martin, 2009, p. 315), étaient bien vivants dans le cours de Mme Williams. Indépendamment de la position de Mme Williams à l'égard des identités raciales des élèves et de sa mise en œuvre de pratiques équitables favorisant l'apprentissage des élèves, des récits raciaux hiérarchiques circulent et ont des conséquences profondes sur le sentiment de soi des élèves. En outre, bien que les récits de Carlin et de Leilani nous aident à comprendre le décalage complexe entre notre perception de nous-mêmes et la façon dont nous sommes perçus par les autres, ils rappellent également la manière dont nos identités sont complexes et varient d'un contexte à l'autre. La fluidité de l'identité est particulièrement claire dans les récits de Carlin sur la manière dont elle est perçue par les autres, mais aussi sur son évolution face à un élève de race blanche qui exerce un pouvoir sur son identité raciale et son genre. Dans une perspective socioculturelle, nous avons vu ici que l'identité est une notion complexe en évolution, qui dépend du contexte, et qui est composée de la façon dont nous nous voyons, ainsi que de la façon dont nous sommes perçus par les autres.

Si dans la salle de classe ambitieuse et équitable de Mme Williams qui soutient l'apprentissage des élèves tout en tenant compte de leur identité et de leurs expériences, les récits racialisés persistent, qu'est-ce que ça veut dire pour les enseignants et leur formation ?

KEY FINDING(S)

Overall, findings indicate that even within a classroom context that reflects ambitious and equity-oriented instruction and is organized to support students' academic identities and mathematics learning, this classroom is a site where racialized discourse persists regarding how students are positioned as doers of mathematics in relation to how they racially identify or are identified by others. This section draws from three main findings: Representation is not enough; It is Important to Attend to Student Learning and Identities; and Even with Equity-Minded and

Ambitious Instruction, Racialized Narratives Persist. Here, in these proceedings, I only discuss the third of these findings. For the others, please see Nazemi (forthcoming).

EVEN WITH EQUITY-MINDED AMBITIOUS INSTRUCTION, RACIALIZED NARRATIVES PERSIST

Even with the affordances of an equity-driven and ambitious form of instruction, and a race conscious teacher, students' experiences showed that racialized discourses run deep within and amongst their peer and racial groups. Here I focus on three focal girls of colour who identified racialized narratives that shape how they are perceived by others and how they see themselves with respect to others in Ms. Williams' classroom.

Gena, who identified racially as Asian, was the only student who answered "*yes*" to my question regarding whether she felt her racial identity impacted her learning of mathematics in Ms. Williams' class. In her response, she referenced the well-known stereotype about Asians being good at math (e.g., Cvencek, Nasir, O'Connor, Wischnia, & Meltzoff, 2015; Nasir & Shah, 2011). She felt that because of the assumption that all Asians are good at math during group-work she is assumed by her peers to be good at math. Specifically, she said, "*I think it's cuz people are just like 'Asians can do math', and I'm like no, not all Asians can do math, cuz I know a lot of us don't know what the heck is going on!*" In class, I observed Gena working with her group on their project where they were administering a survey (to their classmates at Champlain) based on a topic they were curious to explore further using statistical methods. Gena's group was made up of one Muslim student (I am basing this on her headscarf), one Black Muslim student (I know this because I interviewed him), and Carlin who is mixed race (focal student in my study). Gena's group decided to explore the relationship between lack of sleep and race. Gena was telling me about her girlfriend (who is White identifying) that does not seem to understand why Gena—who has a part-time job on top of keeping up for her studies—cannot seem to get enough sleep. As Gena tells me about her frustrations of keeping up with her job and school and that her girlfriend "*doesn't get it*" because she does not need to work to help her family[3], Gena's peers in her group kept asking questions about considerations they need to make for their survey. Gena seemed happy to take on a leading role in her group and use her personal experiences as a motivation behind exploring lack of sleep and race together (Field Notes, May 24, 2016), yet her positioning as a leader in the group and as knowledgeable about the topic—perhaps more experientially rather than mathematically—was noteworthy. Gena's leadership in her group suggested to me that other students likely see her as intelligent and knowing, possibly because, as I observed, she was the only Asian student in her group at that time.

Leilani, who identified herself as African American, is the only other student that brought up the 'Asians are good at math' stereotype, again in the context of group-work. In her case, however, this assumption was brought up in terms of what this stereotype means for students who are not Asian, specifically students that identify as Black or African American, like her.

> *I think sometimes people, you know like uh the stereotype is that Asians are really good at math, so umm when you're in a group with like all Asians and you're a Black kid sometimes you might feel like okay they're gonna think I'm not as well as them in this. Then it starts to get to your head, that you know maybe I'm not as well as them, and you know sometimes they say certain things. You'll agree with it even if you know it's not right because you're like, "Oh, they know," but sometimes you're right. So, I*

[3] This is one instance where class appeared to be a salient social marker for students. Gena and I had a conversation where I was suggesting that there could be factors and circumstances that, along with race, predict sleeping patterns. She seemed willing to listen to my suggestion but still seemed to see the contrast between her and her girlfriend as strictly racialized.

> *think it's important that people just be secure with what they know and not try to feed into a stereotype or umm something that's like working against you.*

Leilani's discussion of the stereotypes between Asian and Black students' contrasting mathematics abilities fits with what other scholars have found (Martin, 2009; McGee, 2016; Nasir & Shah, 2011; Shah, 2017). Specifically, it is common for students of colour to talk about—and place themselves and their peers within—what Martin (2009) calls the "racial hierarchy" of mathematics. In this racial hierarchy, "students who are identified as Asian and White are placed at the top, and students identified as African American, Native American[4], and Latino are assigned to the bottom" (p. 315). Building on Martin's work, which looked closely at the prevalence of racialized discourse in mathematics education, Shah (2017) interviewed various students of colour in the context of their mathematics classroom. He also found that, according to students, Asian and White students were positioned at the top in terms of ability and performance in mathematics, while students who identified as Pacific Islander or Black were positioned near the bottom.

I asked Leilani for a specific example to illustrate her claim about how poorly Black students see themselves and their mathematics abilities, as compared to Asian students. In response to my query, Leilani further highlighted the negative consequences of the 'Asians are good at math' stereotype for other groups of students.

> *I think that I've seen people make assumptions about themselves. It's like, sometimes I'll hear, Black kids in class and they'll be like umm, they'll get really good scores on a test and then they, maybe the Asian kids got a lower score and they'll be like "Wow! I did better than the smart kid" and I'm just like wow you, you think you're not smart… like you should feel that you're smart. You shouldn't think that, that you know that you're less than them in whatever you're doing." I hear people say like "I gotta get to do the project with the smart kids" or "I gotta sit by the smart kids" you know? So I feel like that's really degrading to, to think that uh you're not smart.*

Consistent with Shah's (2017) findings, Leilani, like many students of colour, experienced the stereotype that they are not as smart as their Asian peers as 'degrading' to their identities as Black learners of mathematics. At the same time, her counter-narrative is one of resilience, of being aware of assumptions that can have consequences for her yet working hard not to let them. Leilani's feelings around coping with the stereotype that Black students are not as good at mathematics relates to what we next see with Carlin's experience.

Carlin, like Gena and Leilani, spoke about how her racial identity played a role in learning because of assumptions about intelligence that peers made based on her perceived racial identity. Carlin identified herself as "*multiracial*", which she later elaborated as Black and White. Different from Gena and Leilani, Carlin described how she "*pass[ed] as White*", pointing to how context-dependent one's racialized experience can be, especially when one is White passing. Here, Carlin reveals how complicated it is to navigate her multiracial identity, especially given that both racial markers that she identifies are associated with very contrasting assumptions around intelligence (emphasis added).

> Nazemi: *Do you feel like your racial identity plays a role in your learning?*

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[4] While Martin's (2009) racial hierarchy of mathematics refers to Native American students, an unfortunate shortcoming of much of the important work around race and racialization in mathematics education (and educational research in general) is the omission of Native American and Aboriginal Peoples. I recognize that Native American and Aboriginal Peoples have been the most underserved populations educational system. I want to be explicit that I recognized how ignored and made invisible these groups have been both historically and presently and that future research will tend to this shortcoming more carefully.

> Carlin: *Not so much,* but since I'm since I'm Caucasian, people, people kinda expect that I'm smarter for some reason. *And I'm like well that has nothing to do with my intelligence. It's just…kinda just a statistic. And I was like…they…they like…* I hang around African American, Black people a lot, and they expect me to be smarter for some reason. But I'm just me. *I'm just learning how I learn, and it's…it's weird.*
> Nazemi: *How do you know they expect you to be smarter? Like…*
> Carlin: *They say it*
> Nazemi: *They say it?*
> Carlin: *Yeah.*
> Nazemi: *They say it, just like straight up. Like they say "you're White, you're smarter"?*
> Carlin: *No, they don't say "you're White you're smarter", it's like* when I struggle and I ask them for help, they'll be like "you're White…you should know this". I'm like well that's kinda weird. I'm also Black as well so…should I know…should I not know it because I'm Black? *Like that's weird for them to say like, so I kinda just shrug it off 'cuz I don't let that stuff bother me.*

Carlin explains here that passing as White for her means that she is expected to be smarter. She describes above that when she is struggling with a mathematics concept, her peers will tell her that she should know it because she's White, to which she responds that she's also Black. This finding is consistent with Hobbs (2014) who outlined the effects on one's racial and cultural identity when white-passing. McGee (2016), who drew upon this work, showed that while it might appear that students of colour benefit from passing as White, there is also much "lost by partial or full rejection of one's racial and cultural identity" (p. 1654). She further describes that feeling pressured to limit parts of one's racial and cultural identity is an attestation to the enduring and continued manifestation of white privilege through racism and white hegemony.

Assumptions about Carlin and her intelligence shift when she is seen in comparison to a white (boy) classmate. Elsewhere, Carlin describes the way she is viewed in relation to white students in her class (Nazemi, 2017) and that this feels different than when students of colour are seeing her as white. While I did not have the opportunity to observe Carlin interact with her white classmates (as she was not grouped with them when I was observing the classroom), I did observe the two white (boy) students as dominating whole class discussions and even small group discussions. In my observations, both white students were often the first to ask a question and/or offer an answer, often arguing with students in front of the class. More generally, while Carlin relayed that she is often presumed White and presumed smart in the presence of others, when she is in the presence of these white students she is positioned as less knowing—likely because both her multiracial identity as well as her gendered identity is seen as inferior to a "*full white guy*" (to use her language) identity. Attempting to manage how one is seen by others is as degrading as it is futile. Carlin's stories help us to see that while sometimes we cannot verbalize or fully make sense of why or how we are seen by others, the implications for our identities, if not spoken or understood, are deeply felt. At the same time, much like what Gena shared with me about being assumed to be smart in mathematics because she's Asian, Carlin shares that she is assumed to be smart because when she is read as white. And, much like Leilani's explanation of Black students seen as intellectually inferior to Asian students, Carlin's anecdote describes that Black students are seen as intellectually inferior to White students. Additionally, in contrast to Carlin's presumed whiteness and smartness, when interacting with a "*full white guy*" student she is powerless due to the complexity of her identity, which is not limited to white but includes Black and girl.

More generally, students confirmed that discourses regarding a "racial hierarchy of mathematical ability", where Asian and White students are assumed to be at the top and Black students are assumed to be at the bottom (Martin, 2009, p. 315), were alive and well in Ms. Williams' class. Regardless of Ms. Williams' stance towards students' racial identities along with her enactment of equity-oriented practices in support of student learning, hierarchical racial narratives circulate and thrive having deeply felt consequences for students' sense of selves. Furthermore, while Carlin and Leilani's narratives help us to understand the complicated disconnect between how we see ourselves and how we are seen by others, they also remind of the ways in which our identities are complicated and vary/shift from context to context. The complexity and fluidity of identity is particularly clear in Carlin's stories about how she is seen by others but how that shifts when faced with a white boy student who exerts power over her gendered and racial identities. Consistent with a sociocultural perspective, we have seen here that identity is an ever-changing complex, and context-dependent, notion that is made up of how we see ourselves, as well as how we are seen by others.

If Ms. Williams' ambitious and equity-driven classroom attends to supporting student learning while being responsive to students' identities and experiences, yet racialized narratives persist, what does this mean for teachers and teacher education?

QUELQUES IMPLICATIONS POUR LA FORMATION DES ENSEIGNANTS

Divers récits d'élèves ont révélé que leurs pairs formulaient des hypothèses racialisées à leur sujet et de leurs capacités en mathématiques. Les récits concernant le positionnement des élèves en termes d'intelligence en mathématiques et en tant que membre d'un groupe racial reflètent ce que Martin (2009) a appelé la hiérarchie raciale des capacités en mathématiques. En particulier, les étudiants ont raconté comment leurs pairs ont présenté les étudiants noirs comme étant moins capables en mathématiques que les étudiants asiatiques et blancs. Cependant, les élèves ont également présenté des contre-récits et ont résisté à la hiérarchie raciale. Par exemple, Leilani, une élève noire, a expliqué que, tout en restant consciente de la hiérarchie raciale, elle s'efforçait de s'opposer activement aux récits raciaux dominants selon lesquels les Asiatiques étaient bons en mathématiques à cause de ce que cela signifiait pour elle et les autres étudiants afro-américain(e)s (Nazemi, 2017). Son contre-récit lui a permis de ne pas laisser le stéréotype lui nuire. Plus généralement, cette étude suggère que les innovations pédagogiques telles que l'IC, bien qu'axées sur l'équité, doivent encore prendre en compte les récits racialisés qui circulent dans la classe et parmi ses membres, et trouver des moyens de les perturber. Après tout, même avec l'adoption de l'IC par Mme Williams, et en particulier la manière dont elle attribuerait la compétence pour améliorer le statut d'un élève, les étudiants continuaient de fonctionner dans la « hiérarchie des compétences mathématiques », ce qui leur permettait de s'épanouir malgré un traitement équitable.

Reconnaître la persistance de récits racialisés, même dans une salle de classe équitable, avec une enseignante qui enseigne en relation et en réponse à l'identité et aux expériences de ses élèves, constitue une nouvelle demande d'équité. Ces résultats suggèrent qu'en tant qu'enseignant(e)s, nous devons continuer à faire ce que fait Mme Williams : prendre le temps d'explorer nos propres identités et de reconnaître leurs effets sur la façon dont nous enseignons et interagissons avec nos élèves, apprendre davantage sur les élèves et leurs identités et expériences afin que nous puissions construire notre programme et notre pédagogie d'une manière qui est centrée sur la vie des élèves; et à répondre aux suppositions de nos classes qui placent les élèves dans la « hiérarchie raciale des compétences en mathématiques » en attribuant des compétences et en améliorant le statut des élèves. En même temps, nous devons faire plus. La persistance des récits racialisés des élèves dans la classe de Mme Williams suggère que des formes d'enseignement équitable telles que l'IC, qui soutiennent en partie l'apprentissage et l'identité des élèves, sont nécessaires, mais pas suffisantes.

À travers les voix des élèves au sujet de leurs propres récits, les enseignants peuvent mieux comprendre les expériences des élèves de couleurs, qui apprennent les mathématiques. Une lettre anonyme sur leurs identités et leurs expériences raciales est un moyen par lequel les enseignants peuvent commencer à mieux comprendre leurs élèves. Sans chercher à connaître les élèves, un enseignant a un accès limité à l'identité des élèves. Cet accès est restreint à la manière dont les élèves sont identifiés par les données au niveau de l'école et à ce que l'élève a partagé avec l'enseignant. En notant et en privilégiant les voix et les expériences racialisées des élèves de couleur dans la classe de mathématiques, les enseignants peuvent mieux communiquer avec les élèves, en apprendre davantage sur eux et soutenir en même temps l'apprentissage et l'identité des élèves. C'est une tâche complexe. Cela demandera un travail très difficile de la part des enseignants—dont une grande partie sont des femmes blanches—qui doivent prendre le temps d'examiner leur propre identité raciale, pour eux-mêmes et en relation avec leurs élèves. En plus de collecter des informations et de prendre conscience des expériences et des identités racialisées des élèves, les enseignants doivent trouver des moyens de dialoguer avec eux au moyen de conversations sur la race et le racisme, dans le contexte de tâches mathématiques et dans le contexte plus large de l'apprentissage et des suppositions autour les capacités en mathématiques. Comme Shah (2013) l'a suggéré dans sa thèse (qui examinait de près le discours racialisé parmi les étudiants de couleur en mathématiques), les enseignants peuvent immédiatement et systématiquement aborder les suppositions de race et d'aptitude en traitant « les invocations explicites de récits raciaux-mathématiques avec le même niveau gravité comme une déclaration raciste flagrante » (p. 120). Comme le fait Mme Williams, une façon de faire consiste à attribuer des compétences pour rehausser le statut d'un élève qui pourrait autrement être considéré comme occupant un niveau inférieur dans la « hiérarchie raciale des capacités en mathématiques ». L'apprentissage et le sens de soi des élèves sont en jeu lorsque des récits racialisés circulent dans la classe. Il appartient aux enseignants de noter et de perturber ces discours racialisés qui ont un impact néfaste sur l'identité des élèves, qu'ils soient mathématiques ou autres.

Le travail identitaire des enseignants, ainsi que l'apprentissage de la reconnaissance et de la perturbation des suppositions racialisées en classe, peut commencer avant d'entrer en classe, dans le contexte de la formation des enseignants. Prendre le temps d'explorer, d'articuler et de comprendre nos propres identités peut contribuer à une meilleure compréhension de l'identité de nos étudiants et peut nous aider à réfléchir sur les avantages et les contraintes de nos relations à l'identité et à la vie de nos étudiants. Lorsque nous en venons à reconnaître nos étudiants (et les enseignants en formation), tout comme nous-mêmes, en tant que détenteurs d'identités changeantes, complexes et de nature intersectionnelle, nous pouvons mieux nuancer tous les éléments constitutifs de notre identité et comment cela peut être respecté dans les contextes dans lesquels nous enseignons et apprenons les mathématiques.

SELECT IMPLICATIONS FOR TEACHER EDUCATION

Various focal students' narratives revealed that racialized assumptions were routinely made by their peers regarding these students and their abilities in mathematics. The narratives regarding students' positioning in terms of intelligence in mathematics and racial group membership reflected what Martin (2009) called the racial hierarchy of mathematics ability. Particularly, students recounted ways in which peers positioned Black students as less capable in mathematics than Asian and White students. However, focal students also exhibited counter-narratives and resisted the racial hierarchy. For example, Leilani, a Black student, explained that while she remained aware of the racial hierarchy, she worked to actively resist dominant racial narratives around Asians being good at math because of what this meant for her and other African American students (Nazemi, 2017). Her counter-narrative allowed her not to let the stereotype work against her. More generally, this study suggests that pedagogical innovations like CI, while equity-oriented, still need to consider the racialized narratives that circulate within the classroom and amongst its members and find ways to disrupt them. After all, even with Ms. Williams' enactment of CI, and particularly the way in which she would assign competence to uplift a student's status, students continued to function within the 'racial hierarchy of mathematical ability' allowing it to thrive despite an equity-minded classroom.

Recognizing the persistence of racialized narratives even in an equity minded classroom taught by a teacher who works to teach in relation and in response to her students' identities and experiences, poses a new equity demand. These findings suggest that as teachers we need to continue to do the things that Ms. Williams does—take the time to explore our own identities and recognize how these affect the way we teach and relate to our students, learn about students and their identities and experiences so that we can build our curriculum and pedagogies to be centered around students' lives, and respond to assumptions in our classrooms that position students in the 'racial hierarchy of mathematical ability' by assigning competence and uplifting students' status. At the same time as doing these things, we must do more. The persistence of students' racialized narratives in Ms. Williams' classroom suggests that equitable forms of instruction such as CI, which partly support students' learning and identities, are necessary but not sufficient.

Through students' own voices about their own narratives, teachers can help surface what students of colour learning mathematics might be experiencing in their mathematics classrooms. Ms. Williams' students' anonymous letter about their racial identities and experiences is one way teachers can begin to do this. Without reaching out to know students, a teacher has limited access to students' identities, usually only how they are identified by school level data and what the student has shared with the teacher. By noting and privileging the voices and racialized experiences of students of colour in the mathematics classroom, teachers can better connect with students, learn more about them, and support student learning and students' identities at the same time. This is a tall task. It will require seriously difficult work on the part of teachers—many of whom are white women—who must take the time to examine their own racial identities for themselves and in relation to their students. In addition to gathering information and making oneself aware of students' racialized experiences and identities, teachers need to find ways to engage with students in conversations around race and racism within the context of mathematics tasks as well as the broader context of mathematics learning and assumptions around ability. As Shah (2013) suggested in his dissertation (that looked closely at racialized discourse amongst students of colour in mathematics), teachers can immediately and consistently address assumptions around race and ability by treating "explicit invocations of racial-mathematical narratives with the same level of gravity as they would blatantly racist statement" (p. 120). One way of doing this, as Ms. Williams does, is through assigning competence to uplift the status of a student that might otherwise be seen as occupying a lower tier on the 'racial hierarchy of mathematical ability'. Student learning and students' sense of selves are at stake when racialized narratives circulate in the classroom. It is up to teachers to make note of and disrupt these racialized discourses that have a harmful impact on students' identities, mathematical and otherwise.

The challenging identity-work of teachers, along with learning to recognize and disrupt racialized assumptions in the classroom, can start prior entering the classroom in the context of teacher education. Taking time to explore, articulate and understand our own identities can contribute to our fuller understanding of our students' identities and help us to think about how the affordances and constraints of how we relate to the identities and lives of our students. When we come to recognize our students (and teacher candidates), just like ourselves, as holding shifting identities that are complex and intersectional in nature, we can better nuance all the parts that make up who we are, and how these parts are navigated, negotiated and can be upheld in the contexts in which we are teaching and learning mathematics.

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MATHEMATICAL TOOL FLUENCY: LEARNING MATHEMATICS VIA TOUCH-BASED TECHNOLOGY

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ABSTRACT

Recent advances in the study of mathematics embodiment have given rise to renewed interest in how mathematical learning relates to our bodily actions and the sensorimotor system. In this dissertation, I explore the embodiment of mathematics learning with a particular focus on the relationship among gestures, hand and finger movements, and the use of mathematical tools. The theoretical lens of perceptuomotor integration enabled me to articulate mathematics learning through the development of tool fluency within a non-dualistic view of mathematical tools. The dissertation is structured as three stand-alone descriptive case studies that adopt Husserl's phenomenological attitude in analysing participants' lived experience while using mathematical tools. The results provided evidence for a high degree of gestural and bodily engagement while learning, communicating, and playing with mathematical tools. In the third paper, I have proposed a new methodological approach to systematically analyse video data using StudioCode. This methodology enabled me to catalogue interactions in order to monitor and assess the emergence of mathematics expertise while the learner interacted with the mathematical tool.

INTRODUCTION

My dissertation focuses on different aspects of the role of bodily interactions on 'mathematical tools' and 'learning mathematics'. In particular, it concerns the development of a better understanding of the role of gestures and bodily interactions with the mathematical tools in learning mathematics and the embodiment of cognition. The dissertation consists of three stand-alone chapters written in paper format. Each paper has its own goals, framework, and participants; however, they all have some common themes. The first and last studies focused on learning mathematics via a touchscreen-based device, while the second study investigated the role of tactile graphs in teaching a pre-calculus course to a blind learner. I initially concentrated on the role of mathematical embodiment in learning mathematics. Having a non-dualistic view of mathematical instruments and beginning with an assumption of intertwined perceptual and motor aspects of tool use as perceptuomotor integration, I attended to the bodily gestures, activities and interactions with the mathematical instruments. The common themes arose out of the adopted perceptuomotor integration theoretical framework (Nemirovsky, Kelton, & Rhodehamel, 2013). That is to say, embodied mathematical tool (instrument) fluency is intertwined with learning mathematics, and touch interactions play a vital role in tool fluency and mathematics learning.

FIRST STUDY: EXPLORING CARDINALITY IN THE ERA OF TOUCHSCREEN-BASED TECHNOLOGY

It has become increasingly apparent that the use of fingers fosters flexible calculation strategies, such as composing and decomposing numbers, especially with respect to five and ten, which are essential. Number representation with finger symbols is related to the nonverbal-symbolical form of representation (Domahs, Moeller, Huber, Willmes, & Nuerk, 2010). This is rooted in sensory and bodily experience and enhances embodied cognition, which is termed 'embodied numerosity' by Moeller et al. (2012). Butterworth (2000) notes that developmental and cross-cultural studies have shown that children use their fingers early in life while learning basic arithmetic operations and the conventional sequence of counting words. In addition, Crollen, Mahe, Collignon, and Seron (2011) summarize the fingers' contribution to number sense, specifically regarding the iconic representation of numbers; keeping track of number words uttered; base ten and sub-base five numerical systems; and developing the one-to-one correspondence and stable-order principles by tagging fingers and countable objects with the saying of number words in sequential culturally-specific ways.

The study explored how a fifty-six-month-old boy named Alex learned cardinality through using *TouchCounts*. The most prominent focus of the study was the role of hands in the development of number sense, as well as the role of *TouchCounts* in preserving this development in a digital world.

TouchCounts (Jackiw & Sinclair, 2014) is an iPad application that supports the substantial role of body engagement and senses in education. Having indicated the important role of the fingers for numerosity, we postulate that using fingers to create numbers, when it is supported by tactile, auditory and visual modes of perception, will support and augment cardinal and ordinal understanding of numbers. *TouchCounts* enables young learners to summon numbers into existence and manipulate them in a digital space: it also offers visual and audible provisions in two sub-applications, namely the Enumerating and Operating Worlds. Multi-touch affordance of the iPad, empowered through *TouchCounts*, enables one-to-one correspondence among tap, count and object (de Freitas & Sinclair, 2013). *TouchCounts* also preserves various modalities of verbal counting, numeral notation and finger-counting to represent an ordinal or cardinal number.

We utilized Nemirovsky et al.'s (2013) perceptuomotor integration, Vergnaud's (2009) definition of cardinality, and a Husserlian descriptive phenomenological attitude (Husserl, 1991) to conduct an in-depth case study when Alex was developing mathematical expertise and tool fluency using *TouchCounts*. We reported on Alex's emerging perception and motor integration: from not being able to make a six out of necessity of sequentially, to the development a successful six by using his twin's fingers and *colliding* two herds of threes. The discussion progressed through analysis of three different episodes when Alex was playing with cards and *TouchCounts* in the classroom setting. We found Alex's gradual increase of perceptuomotor skills and an improved understanding of cardinality, accompanied by a unified and holistic continuity between his temporal flows in experiential time. Our study highlighted that playing with *TouchCounts* preserves what Vergnaud (2009) called 'effectiveness of counting strategies' as the one-to-one-to-one correspondences amongst the movement of fingers, eyes and words represents "three different repertoires of gestures" (p. 85).

In this study, we distinguished between showing a cardinal number (1-10) on young learners' fingers or finger-showing (known as finger-montring ["montring" is the original spelling]); obtaining a number via finger-counting, which links an ordinal process; and the development of both in creating numbers on *TouchCounts* via *Finger-touching*. The role of *TouchCounts* in transforming children's finger-showing and finger counting to finger-touching through card play also was addressed. This could be seen as the transition from Deleuze and Bacon's (2003) 'digital' relationship to the 'tactile'. Finger-touching in TouchCounts permitted Alex to coordinate his sense organs—eye, hand, ear—and gestures in different modes: e.g., number of fingers touches the screen in form of taps, number on the created herd on the screen, number name such as three and spoken "three". So the results of study highlighted the integration of a perceptual understanding of cardinality and motor activities that manifested in terms of successful *finger-touching*. In other words, the perceptuomotor aspects of operating on numbers on *TouchCounts* are tangled up with the concepts of ordinality and cardinality. (Please read the full paper at Sedaghatjou & Campbell, 2017.)

SECOND STUDY: ADVANCED MATHEMATICS COMMUNICATION BEYOND MODALITY OF SIGHT

The second study explored "Advanced mathematics communication beyond modality of sight". Being one of the few studies at the university level, in this research I explored how assistive technology and an innovative method of tactile graphing could enable a blind undergraduate student, named Anthony, to learn pre-calculus concepts. In this study, firstly, I discussed some of the problems that Anthony encountered during the lecture while being tutored or while accessing the course's written and pictorial materials (graphs). So the first research question was to find out how do mathematical tools and resources (such as tactile graphs, screen readers, etc.) make mathematical communication and learning possible for the blind learner in pre-calculus courses? Also, to find how tactile mathematics tools support the process of learning that coordinates the body in mathematical activity, I explored, how do the emergence of the blind student's bodily activities and gestures embody and express mathematical learning?

To answer the first research question, I discussed how using Braille, as the tactile writing system, was not the optimal choice for the pre-calculus written materials. Then I introduced alternatives of JAWS and VoiceOver to read the written digital materials and computer screen as well as Nemeth Coding and LaTex to communicate mathematically when needed. In this research, some of the very challenging problems facing the visually impaired student in teaching and learning mathematics are uncovered. For instance,

- reading and comprehending printed/drawn mathematical graphs in absence of a sighted assistant;
- strong reliance of mathematical communication on gestures, body language, pointing, etc.;
- enormous use of deixis by sighted mathematics community counterparts when referring to a mathematical graph or its parts.

Anthony faced these difficulties in various forms: in reading a textbook's pictorial information and graphs, on sketched figures on the board during the lecture time or tutoring sessions, when he was doing exercises at home, and tests or quizzes at school. To tackle this challenge, we invented two methods for graphing: Sketch graphing and Permanent graphing. The Sketch graphing enabled Anthony and his assistants/teachers to quickly and efficiently draw a tactile graph during the lecture, tutoring time, or test. Permanent graphing empowered Anthony to comprehend and read a drawn graph with all given details, presented just like the original resource (textbook, class notes, etc.) in absence of a sighted assistant.

The second part of this chapter explores how using the tactile graphs as a mathematical tool supported Anthony's learning. In this section, I detailed and analysed Anthony's lived experience when he was verbally and gesturally describing the given rational function graph's behavior. The emergence of Anthony's coordination of perceptual and motor activities was illustrated and his temporal flows of perceptuomotor activities—inhabited bodily and interpersonally in experiential time—were analysed (Husserl, 1991; Nemirovsky & Ferrara, 2009; Nemirovsky et al., 2013). The results showed Anthony's tool fluency as the enactment of his body orientation and appropriate use of the tactile mathematical instrument. I found changes in the scale and modality of mathematical activity as an historical disruption of relatively stable, culturally-understandable demonstrations of a rational function's behaviour with great precision. That is to say, the tool fluency phenomenon was fulfilled through a fluent demonstration of a function's behaviour that was a "culturally recognized creation" by the members of "mathematical communities" (Nemirovsky et al., 2013, p. 373).

My direct contribution on this study consisted of identifying some obstacles that a blind undergraduate student encountered in learning pre-calculus concepts and investigating possible aids to assist his mathematics learning at the university level. I also invented tactile graphs (sketched and permanent graphs) and examined their functionality in different contexts to provide readable graphs for the blind learner. Theoretically, the study suggested the transition from *active sensations* to *tactile perception* as a sign of tool fluency. While *active sensing* refers to controlling the finger movements (Gibson, 1962) while contacting a stimulus, *tactile perceptions* were evidenced when Anthony's acquired information formed the tactile graphs, and his learning guided his gestures and body coordination (in the environment) when demonstrating a rational function's behaviour (Lepora, 2016). Also, Anthony's tactile perception revealed his understanding of a rational function as a dynamic entity that comes from $-\infty$ and moves toward $+\infty$, despite the static nature of static tactile graphs, per se. (Please read the full paper at Sedaghatjou, 2018.)

THIRD STUDY: TOUCHSCREEN-BASED TECHNOLOGY IN EXPLORING GEOMETRIC TRANSFORMATION: USE OF TIMELINE AS AN ANALYTICAL TOOL

The third study, entitled "Touchscreen-based technology in exploring geometric transformation: Use of timeline as an analytical tool" addressed the following questions:

- What are the types of interactions in a specific touchscreen dynamic geometry environments (DGE) geometrical context named BlackBox?
- How do customized expansions on Arzarello, Bairral, and Danè's (2014) codes of touchscreen-based DGE interactions clarify emergence of tool fluency on a mathematical instrument?
- How does a prospective teacher learn geometric transformation via interacting with the touchscreen-based GSP?

In this study, I discussed how a prospective teacher named Anna learned geometric transformation via direct interactions with a touchscreen-based dynamic geometry environment (*The Sketchpad Explorer*, Jackiw, 2014). *Sketchpad Explorer* is an iPad application that enables users to manipulate sketches created using *The Geometer's Sketchpad* (Key Curriculum Press, 2001). Anna, who had no experience playing with a touchscreen-based device, was asked to identify the type of geometric transformation in the given task, named BlackBox. I adapted Arzarello et al.'s (2014) theory of touchscreen interaction to identify different modes of actions: basic actions, and active actions. The basic actions were defined as the mode of interaction with the touch interface, while the combination of basic actions and performed finger interactions were categorized as active actions. I extended Arzarello's modes of interactions and defined *new codes* in terms of perceptual and motor integration, rather than cognitive domain of mathematical thinking.

The results showed touchscreen-based DGE maintains geometrical relationships between components of shapes by offering continuous and real-time transformations. It also allowed direct interactions with the geometric objects with the hand. Thus, it prompted an environment for conjecturing, reasoning, developing explicit descriptions of geometric relationships and shapes by drawing different sketches, and tracing their effects even with no explicit instruction (Battista, 2008; Leung, 2008; Vrahimis, 2016). Also, the analysed data indicated Anna's tool fluency, exhibited in the form of accelerated active actions (namely, drag-touches and rotations) combined and co-joined with her verbal explanations. The result suggested Anna's active actions transformation from discordance between perceptual and motor aspects of learning, to a holistic sense of unity and flow in terms of mathematical tool fluency (Nemirovsky et al., 2013).

My contribution in this study also was to suggest a new methodology for analysing video data systematically. Appealing to Nemirovsky et al.'s (2013) perceptuomotor integration theory allowed me to trace the emergence of tool fluency in terms of integration of temporal streams of perceptual and motor activities using a detailed analysis of the video's timeline as an analytical tool. I used the video timeline to trace the Arzarello's adapted codes in video data, which enabled me to trace the paths of interactions and discuss the emergence of tool fluency in terms of developing active actions over the stretch of time. With the suggested methodology, I analysed a four-minute episode concentrated on the touchscreen-based communications for Anna, as she thought aloud to solve a geometric rotation task.

CONCLUDING REMARKS

This dissertation followed the statement that the interactions exhibited thinking not as a process that takes place 'behind' or 'underneath' bodily activity but is the bodily activity itself (Nemirovsky & Ferrara, 2009; Nemirovsky et al., 2013). Also, informed by my studies, I have broadened the notion of what constitutes a mathematical tool to include the body and extensions of embodiment in interacting with the world. Taking the three studies together, I found a high degree of embodied mathematics and temporal coordination of the capacities of sense organs (hands, ears and eyes) in response to the mathematical instrument. Bodily interactions with mathematical instruments navigate the edge of the actual and the potential: the potentiality of the body's contribution versus the actuality of the designed instrument. Thus, in different touchscreen-based DGEs, inventive instrumental gestures tap into the potentiality of the body's engagement and reconfigure various relationships between different sensations, respectively (Sinclair & de Freitas, 2014). For example, *TouchCounts* leaves visual and aural traces on the fingers' path on the screen. It also engages the potentiality of small physical gestures: the actuality of a pinch, for instance, is a metaphor for addition. The touch gestures make concrete quantities for young children, for whom those quantities are still 'abstracts' (Sinclair & de Freitas, 2014).

Alex's bodily orientations were brought from a *finger-counting* and *showing* to a *finger-touch*; for Anthony, they developed from an *active sensation* to a *tactile perception*; and for Anna from rotation and push actions to *drag-touch-to-approach* and *drag-touch-free* active actions. This is also where the participants harmonized words and involuntary bodily activities (motoric), providing evidence of their ability to anticipate the next step of the task (perceptual). Thus, their motor activities were involuntary and enacted as a part of perceiving. This is where the perceptuomotor integration met Husserl's phenomenological attitude. Also, direct manipulation via touchscreen-based device allowed back and forth interactions between hands, eyes, and ears for gathering meaning, forming undergoing retentions and anticipations. These shifts in thinking could be the result of attention to the relationships between hand, eye and ears in producing gestures in DGEs. They also could be, theoretically, the result of an 'unmet protention', the unanticipated result of a miscalculated 'retention' in ongoing sequential time (i.e., learning from mistakes). Therefore, the continuous growth of embodied skills, and integration of perceptual and motor activities through emerging paths of lived experiences within the social and environmental contexts forms ideas and learning, without being or end (Ingold, 2014; Nemirovsky, 2017). Such interactions highlighted the roles of mathematical instruments in the evolution of our ways of sensing and reflecting, as well as the fundamental role of instruments in the ways we come to know and arrive at fluency of use. In the first and last studies involving DGEs, I found DGE prompts to be an environment for reasoning, conjecturing, and explicit description of geometric or arithmetic relationships. The tactile graphs played the same role for Anthony.

The findings presented in this dissertation are far from providing a complete explanation about the contribution of the body—especially fingers, eyes, and ears—in learning mathematics. They rather contribute to a new path for the studies in the embodiment of mathematics with a focus on the coordination between finger, eyes, ears and touchscreen-based DGEs. While the case studies described here were meant to investigate mathematics learning in the form of tool fluency and body coordination, each alternative way focused on a different aspect. The chosen small sample size in my thesis helped me to focus on qualitative case study research exploring for possible phenomena of interest of emerging mathematical tool fluency in different context, not a quantitative consideration of the probability of occurrences of such phenomena. Considering larger groups of participants may result in complementary findings. So future studies with larger sample sizes, in actual classroom settings, should further investigate the possible benefit of using touchscreen-based DGEs on teaching and learning mathematics.

Also, the first study took place in kindergarten, while the researcher who conducted the interview was not a kindergarten teacher. So further studies where *TouchCounts* is utilized by an elementary or kindergarten teacher are suggested. In addition, using multiple iPads, or even an iPad Pro instead of an iPad (with a larger screen size than an iPad) may bring a greater level of collaborative engagements among the students. Also, further investigations on two-digit number combinations and creation using *TouchCounts* are suggested.

For the second study, I invited the participant by snowball sampling. I also was unable to find further participants to investigate the functionality of the suggested methods on their learning. It remains for further research to shed some lights on the presented tactile inventions for blind students and adopt and extend them in other undergraduate coursework.

In the third study, I examined mathematics learning in terms of tracing adapted codes based on Arzarello at al.'s active actions (2014) in a very small excerpt of videos in which temporal flows of perceptual and motor activities were observed. I offer the methodology and analysis as a starting point for further studies. Adapting my suggested methodology in a large data set would enable a team of researchers to collaboratively analyse and draw evidenced-based conclusions.

At the theoretical level, some explanations regarding what exactly learners' 'body' entails and how mathematics partakes the body would be helpful. Also more clarity in terms of the implications of the 'tool fluency' is needed. For example, if this perspective tool fluency could mean that a master carpenter is by definition an expert geometer by nature of the fact that they can adeptly use tools that pertain to the geometric properties of objects.

Theoretically speaking, I have extended the perceptuomotor integration approach to trace mathematics learning in terms of tool fluency. The chosen theoretical lens was helpful in speaking about the potential and the active role of the multitouch screen technology (*TouchCounts* and *Geometer's Sketchpad*), as well as describing active and dynamic sensation transformation toward tactile perception in learning mathematics for blind learners. The rationale was rooted in the idea that learning entails an interpenetration of the perceptual and motor aspects of activity with a tool, and this interpenetration is part of developing fluency with this tool. Husserl's phenomenological descriptive attitude (1991) provided a rich framework to analyse temporal flows of perceptual and motor activities. That is where I identified and adapted perceptuomotor integration in the form of unity, coordination, and harmony of bodily activities while using mathematical instruments by introducing different notations. In addition, the expanded Arzerallo et al.'s (2014) modalities (modes of active actions) and the systematic video coding could be adapted in the first study to categorize types of finger-showing, finger-counting, and finger-touching (see Sedaghatjou and Rodney, 2018, for more details).

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EXAMINING CHANGES IN SPATIALIZED GEOMETRY KNOWLEDGE FOR TEACHING AS EARLY YEARS TEACHERS PARTICIPATE IN ADAPTED LESSON STUDY

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ABSTRACT

Research suggests there is a crucial need for improved teaching of geometry and spatial reasoning in early years classrooms; however, limited teacher knowledge of geometry and spatial reasoning are significant challenges to this goal. This case study investigated five teachers' learning about spatialized geometry when participating in an adapted lesson study which included clinical interviews, exploratory lessons, and resource creation. The teachers demonstrated substantial expansion of their spatialized geometry knowledge for teaching (SGKT)—a new concept that encompasses learning about students and teaching as well as about specific geometry concepts. The substantial learning about SGKT evidenced in this case study suggests that the adapted lesson study model has potential to address the limited geometry knowledge typically found in early years teachers.

INTRODUCTION

Very little mathematics is taught in early years classrooms despite increasing evidence of its importance (Bruce, Flynn, & Moss, 2016). Of special concern in early mathematics education is geometry. Despite calls for geometry and spatial thinking to be forefront of early math curricula (National Association for the Education of Young Children & National Council of Teachers of Mathematics, 2010), geometry receives the least attention (Sarama & Clements 2009; Sinclair & Bruce, 2015). Of further concern is the narrow focus on names and describing shapes when geometry is taught (Clements & Sarama, 2011).

Examining the work of geometers provides a broader view of the nature of geometry. Geometer's regularly work with invariance, transformations, and symmetry, visualization, diagrams, dimensional reasoning, modelling and solving problems in real world contexts (Royal Society & Joint Mathematical Council, 2001; Venema, Barker, Farris, & Greenwald, 2014). Spatial reasoning is integral to this rich view of geometry. Unfortunately, most early years teachers do not have this rich dynamic/spatial understanding of geometry (Clements & Sarama, 2011; Lee, 2010). Because teacher mathematics knowledge is highly predictive of student learning (Darling-Hammond, Chung Wei, Andree, Richardson, & Orphanos, 2009; Hill, Rowan, & Ball, 2005), it important to develop early years teachers' understanding of spatialized geometry.

FRAMEWORKS FOR THIS STUDY

This study examines changes in teachers' spatialized geometry knowledge. The term spatialized geometry is used to emphasize the spatial aspects of geometry that are part geometers' work and contrast the typical early years focus on names and classifications. Spatialized geometry is highlighted in the National Council of Teachers of Mathematics (NCTM; 2000) geometry and spatial reasoning standards: 1) characteristics of shapes; 2) location and spatial relationships; 3) transformations and symmetry; and spatial reasoning, visualization, and modeling.

Research in mathematics education (Hill, Ball, & Schilling, 2008) highlights the complexity of teacher knowledge; Hill and Ball developed the construct Mathematics Knowledge for Teaching (MKT) to research this complexity. This research introduces Spatialized Geometry Knowledge for Teaching (SGKT) to capture the complexities of Spatialized Geometry. Like MKT, SGKT incorporates knowledge of content (KSG-SG), of students (KSG-Students) and knowledge of teaching (KSG-Teaching). SGKT provides a framework for examining potential changes in teacher knowledge about spatialized geometry.

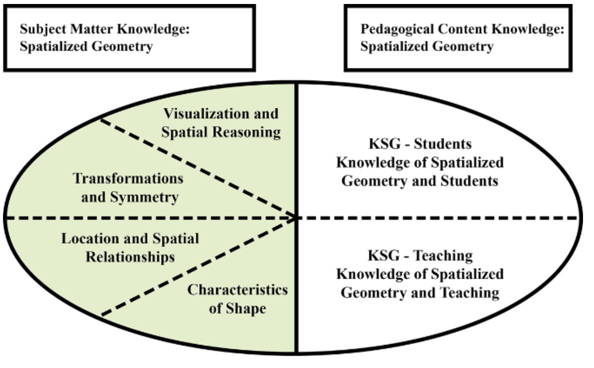

Figure 1. Spatialized Geometry Knowledge for Teaching (SGKT).

This study used adapted lesson study as a strategy for developing teacher SGKT. While there is little research on effective professional development for early years geometry, lesson study is considered the gold standard of professional development for teachers of older students (Moss, Bruce, & Bobis, 2016). Lesson study traditionally includes a four-step cycle: 1) setting the area of investigation, 2) planning a research lesson, 3) implementing the research lesson, and 4) debriefing the research lesson.

However, some research on lesson study in the North American context suggests some challenges. Because research suggests that North American teachers' have difficulty noticing student thinking and evidence of student learning (Fernandez, Cannon, & Chokshi, 2003; Tepylo & Moss, 2011), clinical interviews were added to focus attention on children's thinking. Clinical interviews in mathematics professional development have been shown to develop teacher knowledge about students (Clarke, Clarke, & Roche, 2011). Exploratory lessons were also added to provide multiple opportunities for teachers to extend and deepen their knowledge (Bruce & Ladky, 2011). With exploratory lessons, the lesson study team co-designs activities to test teachers' hypotheses about effective teaching of the mathematics topic under study. The teachers then implemented the lesson in their own classroom and returned to the next meeting with samples of student thinking to discuss student learning in response to their pedagogy. In this iterative process, teachers collected meaningful data about their students' thinking in response to their own teacher 'moves'. Finally, with the goal of adding to the field's knowledge of effective instructional models for spatialized geometry in the early years, the lesson study group created resources from their lessons. The complete adapted lesson study model is illustrated in Figure 2.

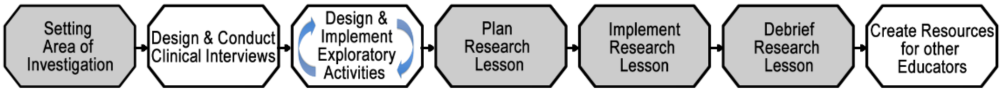

Figure 2. Adapted lesson study model.

RESEARCH QUESTIONS

The calls for increased spatialized geometry instruction in the early years are impeded by many factors, central to which is teacher knowledge. This study investigates whether an adapted lesson study model for teaching spatialized geometry improves teacher knowledge of SGKT. The research questions guiding this study were

1. How does teachers' content knowledge of spatialized geometry (KSG-mathematics) change throughout this adapted lesson study?
2. How does teachers' spatialized geometry knowledge for teaching (SGKT) change through participation in this experimental lesson study?

METHODOLOGY

Data for this study were collected in the first year of the Math for Young Children (M4YC) project which aimed to broaden the teaching and learning of early years mathematics. The project investigated professional learning teams as they explored an area of early mathematics using case studies. Case study methodology was well suited for this study given the complexity of teacher knowledge and professional development and the exploratory nature of the first year of the M4YC project.

This research specifically examined 'what is possible in early geometry teaching and learning' with a group of educators and researchers experienced in collaborative action research. The study closely examined five teachers from rural schools (Table 1). The team also included a school board mathematics consultant, educational assistants, and researchers who were not studied in this research.

Teachers	Teaching Context
Cherie	Grade 2-3 split class; school in small industrial town
Donna	Grade 2-3 split class; same school as Cherie
Barb	Kindergarten-grade 1 split class; rural school
Jessica	Full day kindergarten class; small agricultural town
Heather	Full day kindergarten class; rural school

Table 1. Participating teachers.

To capture changes in the teachers' SGKT, the following questions were asked before and after the adapted lesson study: 1) Write down everything you know about geometry (KSG-SG), 2) What are the important concepts you might focus on when teaching geometry? (KSG-Teaching), 3) Where might your students struggle with geometry? (KSG-Students). Additional data was collected during all planning and debriefing sessions and included recordings of meetings and clinical interviews, as well as artifacts prepared for teaching and for other teachers. A focus group interview was also collected at the end of the study.

To analyze the data, all recordings were transcribed verbatim and artefacts were added where they extended the verbal data. The transcripts were coded for evidence of changes in teachers'

SGKT and the focus within spatialized geometry: characteristics of shapes, location and spatial relationships, transformations and symmetry; and spatial reasoning, visualization, and modeling.

FINDINGS

KNOWLEDGE OF SPATIALIZED GEOMETRY (KSG-SG)

The first research question examined changes in teachers' content knowledge of spatialized geometry. Analysis of teacher artefacts and utterances demonstrated notable changes in the breadth and depth of teachers' content knowledge about spatialized geometry. Initially, Heather accompanied her scant concept map with a note, "*I hope this will be much fuller after our work together*".

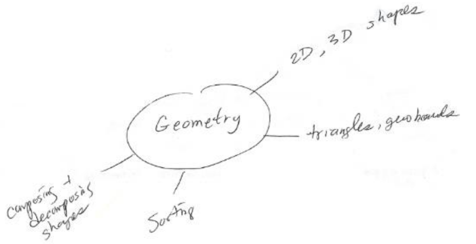

Figure 3. Initial concept map.

Indeed, her final concept map included far more detail as illustrated in Figure 4. Additionally, the content analysis of the teachers' entries in their concept maps indicated a broadening of their understanding of geometry. As demonstrated in Table 2, most of the teachers' entries related to the first strand: Properties of Shape. Only one teacher initially mentioned anything that related to transformations and symmetry.

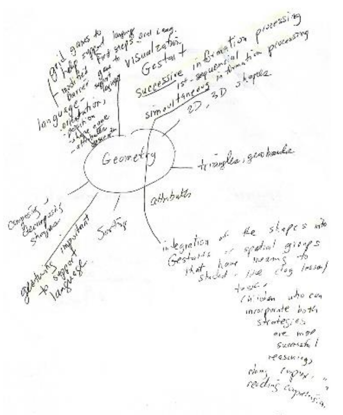

Figure 4. Final concept map.

After the adapted lesson study, the entries were well distributed within NCTM's four geometry strands with the most entries within spatial reasoning and visualization strand. Additionally, each of the five teachers mentioned content in each category, indicating a broadening of their understanding of geometry.

NCTM Category	Entries in Inventory Before Lesson study	Entries Added After Lesson Study
Properties of Shape	37	25
Locations and Spatial Relationships	2	10
Transformations and Symmetry	5	20
Spatial Reasoning and Visualization	0	82

Table 2. Changing foci within spatialized geometry.

CHANGES IN SPATIALIZED GEOMETRY KNOWLEDGE FOR TEACHING

The second research question examined how teachers' Spatialized Geometry Knowledge for Teaching (SGKT) changed though the adapted lesson study. Here I examined the substantial growth in teachers' SGKT during the clinical interviews and from the exploratory lessons and the consolidation role of creating resources for other teachers.

<u>Developing SGKT through Clinical Interviews</u>

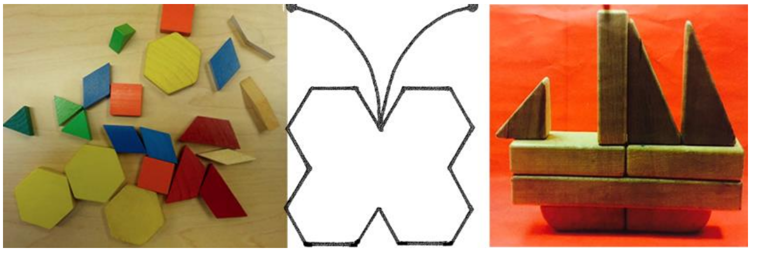

Figure 5. Composition clinical interview tasks.

The adapted lesson study began with the teachers reading about visualization in mathematics and creating clinical interview tasks in collaboration with the research team. The teachers created two composition tasks (Figure 5). For the butterfly composition task, students were asked to fill the butterfly completely in two different ways. For the boat composition task, students were asked to construct the boat from the picture. Each teacher then interviewed two boys and two girls from their own class with the author observing and making detailed notes. From these clinical interview tasks, the teachers noticed that students generally knew shapes: even the youngest children "*generally selected the correct shapes and sizes*". However, some students "*used trial and error where others knew which shape to use*". Barb noticed how some students tried to use a square to compose a butterfly and never realized that it would not work. The teachers also noticed that students could solve tasks without correct geometric language. "*We realized that they didn't have a lot of language to describe what they knew. They described the size and color* [...] *rather than the attributes of the shapes themselves*." One student "*called a hexagon a circle, but placed it in correct spot in the butterfly*".

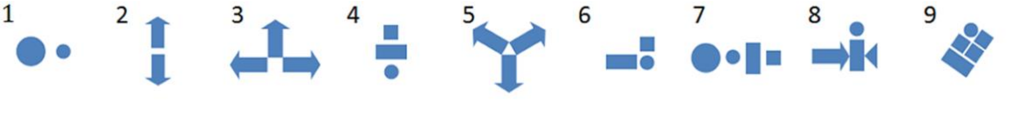

Figure 6. Visualization tasks for clinical interviews.

The teachers also created a series of Look-Make-Fix arrangements (Figure 6). Here students viewed one arrangement for five seconds before the arrangement was covered, and students made the arrangement from memory. Once the arrangement was uncovered, students were to fix their arrangement if necessary. Students continued until they struggled with two successive arrangements.

The teachers were surprised by the variety of visualization strategies used by students, including negative space ("*I saw a white triangle in the center*"—arrangement 5) or a gestalt ("*a man with funny arms*"—arrangement 8). The teachers commented "*I would never have seen that!*" The teachers also realized that students who rehearsed verbally (the strategy that they would have used), often missed spatial features (Figure 7). The teachers noted that students who used multiple strategies flexibly were the most successful.

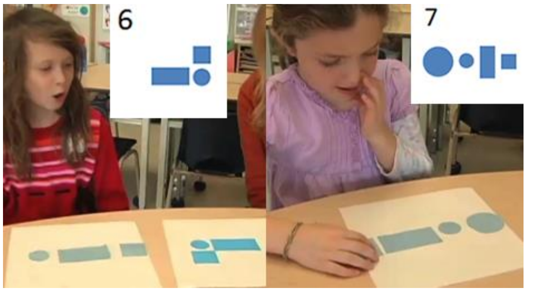

Figure 7. Students who verbally rehearsed arrangements.

During the fix portion of the Make-Look-Fix tasks, most students easily saw the difference between their arrangement and the stimulus (the surprise on the faces in Figure 7). Some students easily moved and rotated shapes to match the stimulus while others could see the difference but could not see how to move shapes. Even more surprising were students that indicated that two arrangements side-by-side were the same (Figure 8) when it was obvious to the teachers that the arrangements were different. Figure 8 illustrates arrangements that students indicated were the same, despite distinct differences in orientation.

Figure 8. Arrangements that students identified as the same.

Summary

Clinical interviews were added to the traditional lesson study model to help teachers notice student thinking: KSG-Students. As predicted, teachers in this stage noticed both student strengths ("*students knew shapes*") and weaknesses ("*students did not use geometric language*", "*some students did not see orientations or transformations*"). It also appears that teachers developed their understanding of spatialized geometry, KSG-SG, during the clinical interviews. In preparing for the exploratory lessons, the teachers set broad geometry learning goals for their students: 1) to recognize shapes from different perspectives, 2) to attend to position and orientation (spatial relationships), 3) to see transformations, and 4) to visualize using a variety of strategies.

<u>Developing SGKT through Exploratory Lessons</u>

The teachers refined their developing SGKT through the exploratory lessons. Their first lesson was a Look-Make-Fix activity intentionally designed to elicit the gestalt visualization strategy: a dog with squares in different orientations. The teachers purposefully chose one shape and one colour in an attempt to focus attention on orientation. The teachers noted that a little instruction on visualization had a large impact: after a brief discussion of the gestalt strategy, students recreated most features and used more geometric language. However, the teachers also recognized limitations of the gestalt including a lack of position vocabulary: "*we didn't get positional language; there was no use of 'on top of' or 'below'*".

The teachers then refined the visualization activities to focus attention on position, creating Look-Make-Fix activities with a variety of shapes on grids. They were surprised that students found it easier to work with familiar such as stars, hearts and arrows in comparison to geometric shapes, musing "*maybe it is harder to visualize shapes where they have to figure out what it is*" and "*it shocked me that for grade two and three students, familiarity of shapes is key*". Realizing that "*students need to play with the shapes*", the teachers introduced new shapes with playtime such a picture creation before starting new activities.

While the teachers noted student improvement in visualization after playing with geometric shapes, they realized that students still were not using positional language. While the teachers no longer equated geometry with vocabulary, they still debated the role of vocabulary. Through readings and discussions, they decided that knowing geometric language would support student in more complex geometric tasks by lessoning the cognitive load, allowing students to perform more complicated geometric activities. To build geometric language, the teachers designed Follow-the-Directions activities on grids to build receptive understanding of the geometric terminology before expecting students to produce the vocabulary independently.

Evidence of Refinement of SGKT from the Research Lesson

The development of the teachers SKGT was obvious in the task selected for the research lesson and the lessons created to prepare students. For the research lesson, teachers decided a barrier game—an activity where one student would describe an arrangement to another student who would then build it without looking. The teachers chose the activity to promote visualization of shapes, position and transformations as well as providing an authentic reason for geometric language. The teachers scaffolded the barrier game by first playing with ordinary items—dressing a clown with different patterned clothing. Students then built pictures with carefully selected shapes such as trapezoids, irregular quadrilaterals and isosceles triangles to highlight geometric properties.

After completing these explorations, Grade 1 students successfully completed several barrier games in front of an audience for the research lesson. The students were able to distinguish and discuss advanced geometric properties such as parallel sides and right-angle triangles. Students often used gestures and informal language to distinguish between shapes and asked questions about geometric properties: "*Is it the triangle with the square corner?*" Grade 1 students were seeing and describing shapes and their positions well beyond the Grade 1 expectations.

<u>Consolidating SGKT during Resource Creation</u>

The teachers also identified creating resources as an important part of their learning. Unanimously the teachers indicated that creating resources was important in consolidating their understanding. As the teachers created visualization activities, they realized how their understanding of visualization had changed:

> *Visualizing in my mind was like reading. Now I know that visualizing isn't just the whole gestalt, but the need to know what the parts are and that to me is where the geometry all tied in, the orientation, the position.*

The teachers also noted how creating the resources was a reflection on the process as a whole. Cherie argued "*you don't see the whole* [while working. It is] *getting the pieces together*". Jessica added that she "*needed the gestalt*" provided by creating resources for other teachers. The teachers also reflected on the entire adapted lesson study process: "*Geometry isn't going* *to be a short unit squeezed in before report card writing*", "*It's easier to teach because I know why I am teaching it*".

DISCUSSION

This paper argued that there is a crucial need for early years teachers to dedicate more time to learning and teaching spatialized geometry. This case study utilized an adapted version of lesson study to address the challenge that limited teacher knowledge poses for incorporating spatialized geometry in the early years classrooms. Within the small scale of this study, this intervention successfully tackled the challenge of limited teacher knowledge about spatialized geometry.

This study provided evidence of substantial increases in teachers' content knowledge of spatialized geometry (KSG-SG). Teacher entries in the pre-survey indicated they saw geometry mostly as naming and categorization, comparable to the research. During the research lesson, resource creation, and post survey, all the teachers demonstrated a broad understanding of spatialized geometry. The teachers indicated that visualization was central, that knowing transformations involves seeing before naming, and that location and spatial relationships are important mathematical concepts and need to be learned. The teachers also realized that knowing shape included recognizing shapes from multiple perspectives and understanding how shapes work together to form other shapes.

Indeed, the entire construct of SGKT showed continued development throughout the adapted lesson study. The teachers developed their KSG-Students, learning that children can know shapes even when they do not know shape vocabulary. They also learned that young students often do not 'see' location, spatial relationships, and transformations and that they find it easier to work with familiar shapes than unfamiliar shapes.

These new understandings of KSG-Content and KSG-Students seemed instrumental in pushing teachers to become more nuanced in their teaching. They carefully selected shapes to focus attention on specific geometric properties and scaffolded their introduction to help children see spatial and geometrical attributes. They learned to build geometric vocabulary first with receptive language activities and then by tasks that authentically required students to use geometric terminology.

In this research, the new construct—Spatialized Geometry Knowledge for Teaching—helped analyze the complexity of teacher knowledge about spatialized geometry much as MKT provided a framework to analyze the complexities of mathematics teaching in general in previous work. While the spatial nature of geometry is not new to geometers, it is to many early years teachers. Using the term spatialized geometry in this research helped highlight this important aspect of geometry as did the NCTM framework.

Given the complexity of Mathematics Knowledge for Teaching in general and the identified weaknesses in early years teachers' SGKT, the adaptations to traditional lesson study seemed important to the teachers' learning demonstrated in this study. The clinical interviews focused teacher attention on student understandings. The teachers' new learning about student strengths and weaknesses appeared to propel changes in the teachers' understanding of geometry. The exploratory lessons provided the opportunity for teachers to refine their understanding of the mathematics, of students, and of teaching geometry. The resource creation stage seemed important in helping teachers consolidate their considerable learning from the lesson study. The adaptations to lesson study implemented in this research provided the time and collaboration needed to continually revisit and develop SGKT.

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MATHEMATICAL REASONING AMONG ADULTS ON THE AUTISM SPECTRUM: CASE STUDIES WITH MATHEMATICALLY EXPERIENCED PARTICIPANTS

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I investigate the unique or unusual characteristics of mathematical problem-solving among adults on the autism spectrum by conducting and analyzing three case studies. The case studies involve providing individuals with a variety of mathematical problems divided into four main groups: paradoxes of infinity, problems emphasizing algebraic or geometric solution, probability, and logic and proof. Participants are given individual interviews intended to facilitate the communication of their thought processes when solving these problems. Results are analyzed with a variety of constructs from a perspective that is rooted in Vygotskian ideas and supportive of neurodiversity.

OVERVIEW OF AUTISM AND EDUCATION

The Autistic Self Advocacy Network (ASAN, 2014) states that autism is a neurological difference with certain characteristics, which are not necessarily present in any given individual on the autism spectrum. These include differences in sensory sensitivity and experience, atypical movement, a need for particular routines, and difficulties in typical language use and social interaction. They also list 'different ways of learning' and particular focused interests (often referred to as 'special interests'), which are especially relevant for research in education. Typical other characteristics include unusually high or low sensory sensitivities and particularly focused interests (often referred to as 'special interests'). More formally, the primary characteristics of autism from the fifth edition of the Diagnostic and Statistical Manual of Mental Disorders (DSM-5) are "persistent impairment in reciprocal social communication and social interaction" and "restricted, repetitive patterns of behavior, interests, or activities" (p. 50). This now includes other classifications such as Asperger syndrome (a change from previous editions), and the DSM-5 criteria parallel the last two of the ASAN's main criteria. Generally, I find that the descriptions largely agree, though the ASAN's description is more positive in tone, using the phrase 'neurological variation' and avoiding words such as 'disorder' or 'impairment'. The ASAN works from a perspective in support of neurodiversity, a term coined by Judy Singer in the 1990s, and generally referring to a positive and inclusive perspective on not only autism but also Attention-Deficit Hyperactive Disorder (ADHD), dyslexia, dyscalculia, and other neurological differences (Silberman, 2015). It is this more positive perspective that I intend to work from in my own research and analysis.

There are a variety of suggestions regarding types of learning differences among people on the autism spectrum in broader education research. For instance, a study by Klinger & Dawson (2001) suggested that people on the autism spectrum did not form prototypes of objects (that is, form an idea based on one or a few typical examples) when given tasks asking about group membership, instead taking an approach based on lists of rules. A second is the systemizing or empathizing/systemizing theory particularly associated with Simon Baron-Cohen (e.g., Baron-Cohen, Wheelwright, Burtenshaw & Hobson, 2007). This links systemizing, defined as “the drive to analyze and/or build a system (of any kind) based on identifying input-operation-output rules” (p. 125), and autism. Some versions also claim that this systemizing is in a sense mentally opposed to empathizing, or competes with it. A third is the idea of ‘theory of mind’ (Baron-Cohen, Leslie, & Frith, 1985), which typically focuses on problems encountered by people on the autism spectrum predicting the behavior and beliefs of others.

Much of the research currently done specifically on mathematics learning in people on the autism spectrum is focused on young children (e.g., Iuculano et al., 2014; Simpson, Gaus, Biggs & Williams, 2010) or looks at mostly arithmetic. There is also a notable strain of work done on the population of research mathematicians (e.g., Baron-Cohen et al., 2007; James, 2003), which outweighs the number of studies done on groups in the middle (mainly high school and college students, or adults other than career mathematicians). In the mathematics-focused work that has been done, some authors report strong mathematical interest or ability (e.g., Iuculano et al., 2014; James, 2010). However, the existing research is generally sparse, particularly for high school students and adults. Much of the previous work both in education generally and mathematics education specifically is from some type of deficit-based model, although this has changed somewhat with more recent studies. I seek to help close the gap in research concerning adults on the autism spectrum in general and mathematical reasoning with my work.

THEORETICAL FRAMEWORK

The theoretical framework that guides my research starts with the work of Vygotsky. In Vygotsky's writing, there is some work that directly addresses the study of ‘defectology’. At the time, this was used to refer to studies involving children with certain disabilities (of a narrower scope than we might consider today; Gindis, 2003). One of the main characteristics of Vygotsky’s (1929/1993) conception of ‘defectology’ was the idea of overcompensation. Vygotsky explained this initially in a framework of physical overcompensation, such as a kidney or lung necessarily strengthening when the other one is missing or by analogy to vaccination. He argued that overcompensation also occurred in psychological development, both in its general course and in particular in the presence of various disabilities. Based on this, he criticized the education of children with disabilities of the time as inappropriately focusing on only the weaknesses, not the strengths, of their students. Vygotsky’s emphasis on the social reasons for psychological differences among people with disabilities also has much in common with modern social constructionist views of disability. The Vygotskian model is thus highly compatible with a neurodiversity-based perspective on autism and education.

While my views are informed by the Vygotskian framework, there are some issues with using it directly. Some parts that are particularly relevant in autistic people, such as the ideas about atypical development and concept formation, particularly concern things that have already occurred far before starting university coursework, and thus cannot be observed in my interview subjects. Thus, while those ideas from Vygotsky inform my views, additional constructs were required for the data analysis.

The first of these consists of ideas regarding intuition. In Fischbein’s (1979) use of the idea, intuition is separated into different categories, particularly ‘primary intuition’ (developed outside of a systematic instructional setting) as opposed to ‘secondary intuition’ (developed in a systematic instructional setting). The division of categories here has similarities to Vygotsky’s distinction between everyday and scientific concepts, and I find it reasonable to consider the primary and secondary intuition used by Fischbein as identifying intuitive reasoning related to everyday or scientific concepts, respectively. It is the primary definition that is closest to what is typically meant when 'intuition' is named but not explicitly defined, which is useful for situating other work that mentions intuition but does not focus on it.

The second of the main theoretical constructs added is the case study approach (Yin, 2009). Given the overall structure of focusing in-depth on interviews with a small number of people, an approach of multiple case studies was a natural fit for my work. A case study in this sense focuses on in-depth understanding of the case in question, and only secondarily on generalizations from that understanding. Additionally, while generalization is possible, it is not of the same nature as generalization in other types of research (Stake, 1995). The focus can be on a single case or multiple cases, but the structure is still primarily that of a single case where the multiple cases are mostly examined individually. In accordance with this, the multiple case studies can be viewed in the sense of replication rather than the sense of sampling (Yin, 2009) with an element of comparison added in after the initial analysis.

PARTICIPANTS AND DESIGN

The data for my study comes from sets of clinical interviews with three participants that I conducted, focusing on a variety of problems (with most given to at least two participants as appropriate). My original plan was to recruit more and follow them through mathematics courses they were taking, but ultimately only one participant was recruited through those methods prompting an expansion and change of focus to include additional participants who had finished their undergraduate work. For each of the participants, we agreed on pseudonyms which carry similar cultural connotations to their given names.

Joshua was my first participant recruited from my university's centre for students with disabilities. He received an Autism Spectrum Disorder diagnosis at age 18 (changed from a previous diagnosis of Obsessive-Compulsive Disorder) and was in his early twenties at the time of interview. He reported a strong interest in chemistry (which he was majoring in) as well as a particularly low level of interest in subjects unrelated to the sciences and a strong inclination to work alone. He was taking integral calculus and linear algebra courses during the time he participated in interviews.

Cyrus was recruited in the community outside of the university, received an ASD diagnosis at the age of 13, and was in his thirties at the time of interview. His mathematical background included a bachelor's degree in mathematics, and he was working in computer programming at the time of interview. In contrast to Joshua, none of his special interests were strongly apparent in the interviews (although mathematics or computing in general may be an exception).

Mark was also recruited in the broader community, received a diagnosis of Asperger syndrome at age 21 (this occurred before the release of the DSM-V) and was in his mid-twenties at the time of interview. Like Cyrus, no strong special interests appeared in Mark's interviews. His mathematical background included a bachelor's and master's degree in mathematics.

The tasks used in their interviews were chosen for several reasons with two main strands: one focusing on what is considered paradoxical or counterintuitive, and one focusing on what methods of solution are expected in a problem. These strands are based primarily on some of the prior findings in education research of autism-related learning differences, such as prototype formation and systemizing, which I expected might relate to findings about those tasks in the mathematics education literature. The tasks chosen have been separated into four categories: Paradoxes of Infinity, Algebraic/Geometric Divide, Probability, and Logic/Proof.

Overall, I had eleven hours of interviews with Joshua, seven with Cyrus, and four with Mark using a variety of these problems. I will present two main strands of results from these interviews with selected excerpts.

INTUITION AND FORMAL REASONING

From Joshua's interviews, one notable feature occurred when using the Painter's Paradox problem (for more information on the problems, see Truman, 2017). This concerns the surface area and volume of the object created by rotating the graph of the function $1/x$ around the x-axis (starting from $x = 1$), particularly the fact that the surface area is infinite but the volume is finite. Joshua's response to this problem was unusually accepting of this mathematical result, as in this excerpt:

> Interviewer: *Of the two things that are in conflict, which one of them feels more correct?*
> Joshua: *I wouldn't really say that any of them are correct or incorrect, because-because mathematically they're, you know, that's… the law, I don't know if you want to call it the law, but the equations show what they show, and if they show what they show, you know, if they show that the surface area is infinite but the volume is defined, and not infinite, then that's what they show, you know, there's nothing we can really do about it, I mean, I'm sorry.*

From Cyrus' interviews, a particular highlight occurred with the Problem of Three Prisoners, which is essentially a less-recognizable rephrasing of the Monty Hall problem. Three prisoners (A, B, C) are awaiting execution, and one is selected at random to be pardoned. Each prisoner cannot be informed of their own fate, but A bargains to be told the name of one prisoner who will be executed, arguing that it is already known that at least one will be. This was done with several variants, where the prior probabilities of each prisoner being pardoned were changed in different ways. The first quote comes from discussion of the biased variant, when Cyrus is asked how the informed prisoner should feel (before calculation):

> Cyrus: *So, I think he should be sad, as well, if that's the case. That's what I'm going to say. This is a little bit of intuition here.* [Interviewer: *Okay, ah, how come?*] *My guess, based on the previous one, is it's going to be a similar calculation, that's going to make the probability lower, when you multiply some things.*

Here, we see that intuition for Cyrus in this context is not an intuition based on physical or other real-world considerations, but instead, intuition based on the mathematical structure witnessed in previous problems. The majority of the time when Cyrus uses something he calls intuitive, it is something like this. Additionally, his focus on intuition in any form is somewhat lower than usual, as seen in a second excerpt:

> Cyrus: *Because I was kind of stumped in terms of, arriving at an answer just thinking in terms of intuition. It was like, I got to the point where I really just had absolutely no idea what the outcome would be. So I felt I needed to work it out in order to come to any kind of conclusion.*

This is illustrative of Cyrus' general tendency toward using known methods of problem-solving over intuitive considerations. Overall, Cyrus is both less likely to use intuition in any form over working out a solution, and less likely to use primary intuition in particular when a consideration of intuition is made. This ties in somewhat to Mark's example where I highlight an exchange that came from the coin box problem. This is a probability problem with some similarities to the Monty Hall and Three Prisoners problems, described as follows:

> *You are given three boxes. You know that each box contains two coins: one has two gold coins, one has two silver coins, and the third has one gold and one silver coin. You choose a box, and take one of the coins, finding that it is gold. What is the probability that the other coin in your box is also gold?*

When asked about his answer to this problem and confidence in it, Mark's response shows what he considers to be important in being certain of an answer:

> Interviewer: *This is the answer that you are confident in?* [Mark: *Yeah.*] *Okay. Let's say that, on the scale from 1 to 10, to which extent are you confident in your answer?*
> Mark: *Maybe nine point nine?*
> Interviewer: *Okay, because, when you ask such a question and get something else but not 10, the next natural question is, what can happen in order to increase your confidence?*
> Mark: *Well, I think that, if I wrote out the Bayes' formula stuff, I'd put my confidence at maybe, nine point nine nine nine or so.*

This demonstrates a general pattern in Mark's interviews, where confidence in mathematical results is primarily rooted in scientific concepts (in the Vygotskian sense) and formal mathematical work, and thus an increase in that confidence is brought by more mathematical rigor, rather than any link to real-world considerations or the problem context.

The excerpts seen here relate to a general trend among the participants of putting higher importance on scientific concepts and formal mathematics, and trusting those more, while putting less emphasis on real-world considerations and intuition. The participants tend to have a high level of trust in the truth and consistency of mathematics, not displaying many of the typical objections to paradoxical tasks outlined in prior research. When faced with apparent contradictions, they are more likely to question an intuitive response rather than a formal mathematical result. Additionally, contextual considerations of problems phrased in a 'real-world' setting appear to be given less relevance. This may be related to the observation with Cyrus in another problem that having a real-world context did not help him to solve the problem more easily (while it had with others in other research). This also appears to be a logical result of an orientation toward structure, or in Vygotskian terms, systematic over intuitive reasoning.

TENDENCIES IN MODES OF REASONING

A second trend is that all three participants had a strong tendency toward particular modes of reasoning over others, which was particularly brought out by tasks in the second category. For Joshua, this was geometric reasoning and visual interpretation, as demonstrated in tasks related to integral calculus and linear algebra. This geometric/visual focus is something also demonstrated by Temple Grandin, one of the most famous individuals on the autism spectrum, in her work (Grandin, Peterson, & Shaw, 1998). However, Cyrus and Mark displayed a strong tendency toward algebraic over geometric reasoning. Each of the participants had different patterns, but each participant's pattern was consistent across their interviews.

As an example, I present an excerpt from one of Joshua's interviews. The problem used here was the first of several Magic Carpet tasks, introduced by Wawro, Rasmussen, Zandieh, Sweeney, & Larson (2012). The task is formulated as follows:

> *You have two modes of transportation: a hoverboard and a magic carpet. The hoverboard moves along the vector* $(3,1)$ *and the magic carpet moves along the vector* (1,2). *Can you get to a cabin at* $(107,64)$ *using these modes of transportation? If so, how? If not, why is that the case?*

This problem is typically used in an inquiry-oriented introduction to linear algebra working toward the use of linear combinations. While Joshua had been exposed to the algebraic ideas in his linear algebra course already, Joshua's response was mainly based on geometric reasoning. Joshua starts off by drawing the destination point, using a ruler to measure precisely where the point should be to get a drawing that is properly scaled. For the first step in the sketch, he draws the given vectors for the magic carpet and the hoverboard, scaled up by a factor of 10 (given the rest of the scaling, they would be barely visible otherwise). After doing this, he moves the vector $(20, 10)$ so that its tip touches the destination point, and then extends the two vectors so that they intersect, concluding that the intersection point is where you should change from one mode to the other (see Figure 1).

This solution is unlike those of any of the students observed by Wawro et al. (2012) in their use of this task. The drawings used by Joshua were produced using a ruler and were very precise, enough to give a correct solution (note that this was done on lined paper, not a square grid). However, it is notable that Joshua terms this (Figure 1) a "*sketch*" (possibly ignoring some of that word's connotations) and seems to not regard this solution as 'mathematical'.

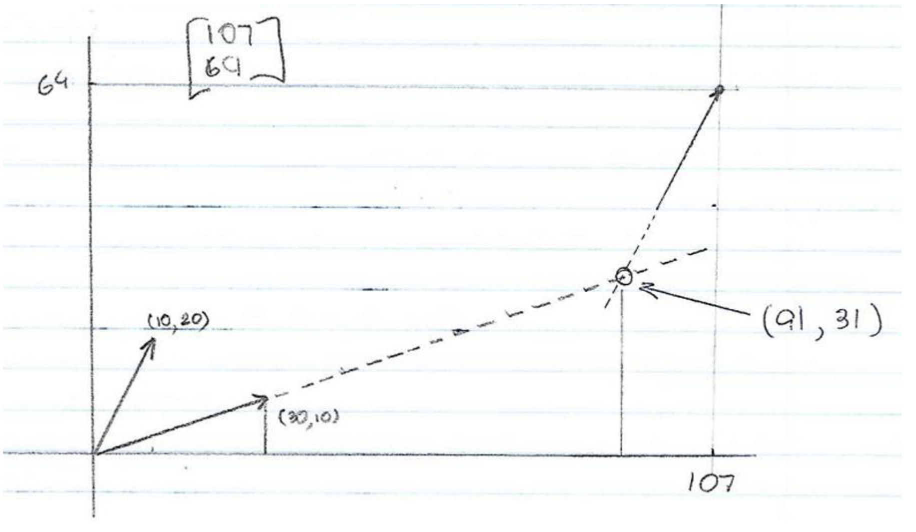

Figure 1. Joshua's "sketch".

The drawings were measured after they were produced, and the point of intersection found by the drawings was the correct point (though the coordinates written above are very slightly off). The interpretation as the location where the person in the problem changes from one mode to the other is also correct. Thus, this solution accomplishes the stated goal of the problem (to find a way to get to the cabin) perfectly well, although by approaching the problem this way, Joshua avoids the intent to push the student toward a standard linear algebra solution. Joshua's solution is less directly related to linear combinations of vectors, though a geometric version of the idea can be brought out from the drawing used for the solution. In particular, vector scaling is used to arrive at the solution, as well as vector addition (which is geometrically accomplished by placing the start of one vector at the end of another).

CONCLUSIONS

The results of my research show a variety of different ways that adults on the autism spectrum can approach and solve mathematical problems. Those results and the differences between them show that there is not necessarily a single approach that students on the autism spectrum can be expected to use and that a variety of forms, which may be considered unusual, can produce successful results. This highlights the importance of being able to see validity in unusual student work and interacting with students without deficit-based preconceptions, something which holds importance across a variety of forms of disability-related education research and beyond. In a classroom context, it is important to be able to strike a balance between encouraging students who present such solutions, using them to broaden the class discussion, and maintaining the original pedagogical intent and goals of the given tasks.

In addition to particular findings regarding mathematical problem-solving, the experiences in the case studies presented here can be viewed in comparison to more general results in autism-related research. For instance, my research findings are consistent with the theory that people on the autism spectrum learn in a manner that relies less on prototypes (Klinger & Dawson, 2001). I believe that the examples of problem-solving in the interviews show that this can produce positive results, and does not need to be viewed as a deficiency. They are also consistent with the systemizing theory (e.g., Baron-Cohen et al., 2007), where systemizing is viewed as an inclination to create or analyze a system based on the formulation of rules. However, this could also come from the mathematical inclinations already known about and sought in the participants. This should not be taken as support for the suggestion by many proponents of the systemizing theory of its opposition with empathizing (viewed here as the recognition of what someone else is feeling), since the nature of the interviews shows very little about any skills in that category, either positively or negatively. However, some speculations about the intent of the question and its designer do push back against some more extreme interpretations of theory of mind deficiency claims (which suppose that people on the autism spectrum are to various degrees unaware of the minds of others), while more complex uses of theory of mind again do not occur in the context of the interviews.

FUTURE RESEARCH

Due to the overall problems with finding a wide range of participants, as well as the gender difference in diagnosis, I only found male (or male-presenting; I did not ask them for their own identification) participants for my study, which I acknowledge as a shortcoming that I plan to rectify in future research. There were also difficulties in finding participants overall, which led to the small number of interviewees with varying mathematical backgrounds. I was unable to follow multiple undergraduates through their mathematics courses, which was my original intent. However, this did allow for a greater depth of study of the interviews from each individual participant and led to a focus on certain types of problems that has been fruitful.

In extending the research that I present here, I believe that a helpful comparison group for these problems would be adults who have taken some mathematics courses, but who are focusing their studies in a field which is not heavily mathematics-based. This could help to separate the effects of mathematical training from more fundamental inclinations. Additionally, it would be helpful to find participants of different genders and cultural or geographic backgrounds, as well as increasing the number of participants overall.

I have found some evidence that a significant portion of people on the autism spectrum, while unwilling to participate in verbal interviews, might be open to answering similar questions given on paper or electronically; this is one possibility for increasing the number of participants. I think that this may mitigate the discomfort with social interaction and might provide some (though less) help in dealing with prior negative experiences with researchers. There may also be more complex results of neurological differences and past experiences that contribute to a preference for non-verbal interviews. I adapted some of my questions to a written format for this reason but did not succeed in getting responses to this version during the time for this study.

Other notable traits which are not common to all participants, such as Mark's strong inclination to confirm calculations, can be viewed as possibilities for further inquiry. It could be that other situations were less conducive to that effect appearing with the other participants, so it would be something to consider when designing further studies.

Another possibility is to extend similar questions across the broader range of neurodiversity to see if there are distinctive traits in the problem-solving of people with other neurological differences. However, this may be better examined by researchers with a more in-depth familiarity with those neurological differences.

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KEY IDEAS IN PROOF IN UNDERGRADUATE MATHEMATICS CLASSROOMS

LES IDÉE CLÉS D'UNE PREUVE DANS LES CLASSES DE MATHÉMATIQUES UNIVERSITAIRES

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ABSTRACT

The mathematics education literature reveals an ongoing interest in fostering students' ability to construct and reconstruct proofs. One promising tool is the concept of 'key idea'. This study investigated how undergraduate mathematics students identify the key ideas in a proof and use them in reconstructing it. The findings show that while most of the students reported that they consciously identified key ideas in proofs, they varied widely in their understanding of the concept itself. Very few students were able to use precise language and point to an idea that helped them both understand the proof and reconstruct it. The findings suggest that mathematics educators, in their desire to see students enhance their understanding of proof and proving by the use of key ideas, will need to extend considerable support to students by actively intervening to draw their attention to features of proofs that are candidates for key ideas.

La littérature en didactique des mathématiques témoigne d'un intérêt soutenu porté à l'amélioration de la capacité des étudiants à construire et reconstruire des preuves. Un outil prometteur pour ce faire est le concept « d'idée clé ». Cette étude porte sur la manière dont les étudiants identifient les idées clés d'une preuve et les utilisent pour la reconstruire. Les résultats montrent que bien que la majorité des étudiants indiquent qu'ils ont délibérément identifié les idées clés dans les preuves, leur compréhension du concept même d'idée clé varie considérablement. Très peu d'étudiants étaient capables d'utiliser un langage précis et d'indiquer une idée clé qui les avait aidés à comprendre une preuve et à la reconstruire. Ces résultats suggèrent que les didacticiens ou enseignants de mathématiques qui souhaitent améliorer la compréhension et la rédaction de preuves par l'utilisation d'idées clés devront appuyer significativement les étudiants en intervenant de manière à attirer leur attention aux aspects des preuves qui pourraient être considérés comme des idées clés.

INTRODUCTION

Since Hanna (1990) highlighted the distinction between proofs that prove and proofs that explain, it has been widely accepted that the purpose of using proof in the classroom is not merely to convince students that a mathematical statement is true but also, and more importantly, to provide them with mathematical insights (e.g., Hanna, 1995; Hersh, 1993; Mejia-Ramos, Fuller, Weber, Rhoads, & Samkoff, 2012; Thurston, 1994). Closely associated with the notion of 'proofs that explain' is that of a *key idea* in a proof. For some scholars this term refers to a central mathematical idea, method, or strategy used in a proof; whereas, for others it connotes the outline, overview or architecture of a proof. Despite these different interpretations of *key idea*, this notion has had pedagogical implications that are consistently aligned with the promotion of students' ability to undertake proof construction and reconstruction on the basis of richer understanding (Detlefsen, 2008; Duval, 2007; Gowers, 2007; Hanna & Mason, 2014; Leron, 1983; Raman, 2003; Robinson, 2000).

From a mathematician's perspective, Gowers (2007) ascribes great importance to the main ideas (or key ideas) in a proof and relates the ability to reconstruct a proof to the number of main ideas it contains. He defines the 'width' of a proof as the number of pieces of information or main ideas (key ideas) one must carry in one's head. In his opinion, the fewer main ideas needed for a proof, the more memorable that proof will be; he then maintains that memorability fosters proof reconstruction (Hanna & Mason, 2014). In the interest of reconstructability, Gowers suggests, then, that one must first make an effort to identify the key idea(s) in a proof.

Despite the increasing interest among mathematics educators in fostering the ability of students to construct and reconstruct proofs, and in proof reconstructability in particular, my review of the literature does not reveal any studies that investigated how university mathematics students go about identifying the key idea(s) of a proof, consciously or unconsciously, or how key idea(s) are actually used to help students grasp the overall structure of a proof or the mathematical insights embedded in it. Of those empirical studies that have investigated students' comprehension of proof, to my knowledge, none have even probed the identification of key ideas as a factor that promotes the ability of undergraduate university mathematics students to construct and reconstruct proofs. This study sought to close some of the gaps in past research by addressing questions related to the effective teaching of mathematical proof at the undergraduate level with specific reference to the use of *key ideas*.

THEORETICAL FRAMEWORK

Practicing mathematicians have paid considerable attention to the concepts of understanding, remembering, and reconstructing proofs (Byers, 2010; Manin, 1998; Thurston, 1995), but there is limited literature on the application of these concepts in mathematics education. Gowers (2007) proposes a critical dimension of proof, which he terms the width of a proof and underscores the relationship between width of a proof and its memorability and reconstructability.

WIDTH OF A PROOF

Once a proof is constructed, it can clearly be seen as having length and depth, where length refers to the total number of words, symbols and numerals, and depth refers to the richness of its connections to other mathematical ideas and in particular to other mathematical domains (Hanna & Mason, 2014). Gowers (2007) suggests a third dimension: the 'width' of a proof, referring to the number of distinct main ideas one has to keep in mind (or memorize) in order to remember a proof.

The notion of width introduced by Gowers (2007) is borrowed from theoretical computer science where it represents the amount of storage space needed to run an algorithm on a computer. Applying it to proof, Gowers defines the width of a proof as the number of steps or step-generating thoughts that one has to hold in one's head at any one time. When a proof is communicated, in either written or oral form, the proof could have a high or low width. Analogically speaking, as artificial intelligence would benefit greatly from a new paradigm of searching for ultra-low-width computations (Gowers, 2007), a proof of low width, with fewer pieces of information to carry in one's head, would be more useful than one of high width (Hanna, 2013). The advantages of a low-width proof lie at the heart of proof presentation and reconstruction by making the proof more transparent and more easily remembered (Gowers, 2007).

KEY IDEA

The notion of *key idea* in proof construction and reconstruction stems from both mathematical and pedagogical considerations. Lai and Weber (2014) reveal that mathematicians value and emphasize the presentation of the main ideas of a proof over that of a completely rigorous proof. Similarly, mathematics educators have discovered that to help students understand and remember proofs it is more beneficial to put an emphasis on the main ideas contained in the proofs than to teach the students how to build valid sequences of logical steps (Durand-Guerrier, Boero, Douek, Epp, & Tanguay, 2012; Hanna & Mason, 2014; Hemmi, 2008; Knipping, 2008; Malek & Movshovitz-Hadar, 2011). Furthermore, Duval (2007) suggests using main ideas to overcome students' mental blocks in proof construction. Such ideas described above are 'key ideas', which carry the flow of information in mathematical proof (Detlefsen, 2008) and capture the gist of a proof (Robinson, 2000). Though Gowers and many researchers use 'main ideas', 'big ideas', or 'critical ideas' to describe 'the hints' in a proof, they are simply using different terms, and for the sake of consistency and simplicity this thesis uses the term *key ideas*.

The term *key ideas* may mean different things to different people. Some scholars refer by that term to the most important mathematical ideas, methods, or strategies used in a proof, whereas others have in mind an outline, overview or architecture of a proof. The former maintain that a proof can foster understanding more successfully if it is constructed on the basis of well understood and internalized key mathematical ideas (Gowers, 2007; Hanna & Mason, 2014; Mason, Burton, & Stacey, 1985; Yang & Lin, 2008). The latter, while also focusing on the central constructive idea, propose a general approach in which a proof task is broken into chunks to highlight its overall structure (Leron, 1983, 1985; Mejia-Ramos et al., 2012; Robinson, 2000; Selden & Selden, 2015). Specifically, for the purpose of obtaining high-level understanding, Robinson (2000) suggests that it is essential to ignore low-level details while highlighting the overall structure of the proof. Selden and Selden (2015) suggest the construction of a proof framework or outline specifically to reduce the burden on the working memory of the mathematician or student.

Raman (2003) understands the term *key idea* differently. For her, it refers to a heuristic idea that one can map to a formal proof with an appropriate degree of rigor. This concept of key idea is related to that of explanatory proofs and brings together a private aspect (engendering understanding) and a public aspect (containing sufficient rigor; Hanna & Mason, 2014), and therefore offers a way to both understanding and conviction.

METHODS

This study is driven primarily by qualitative methods while incorporating a complementary quantitative component. The participants were first-year (n=17) and mixed-year (2^{nd} to 4^{th}) (n=42) undergraduate mathematics students at an urban university in Ontario. The majority of the students were in the program of honours mathematics or mathematics education.

A quantitative survey with a sample size of 59 participants was administered first and then classroom observations and student interviews were conducted throughout the semester. At the end of the semester, a follow-up question sheet was also administered. As student work was a major source for investigating how students might go about identifying key ideas in a proof, I collected student work including worksheets, individual assignments, group work, mid-term tests, final exams, as well as snapshots of students' proving products on a board or on paper. Based on the teaching materials of each course, five complete and correct proofs were used for students to identify and formulate key ideas. In this study, I focused on two proofs for investigation and discussion: (1) the carpet proof of the irrationality of $\sqrt{2}$, and (2) the proof of the irrationality of $\sqrt{k}$ for non-square k.

The survey data, consisting of 26 Likert-scale items in total (including an item on *The Geometer's Sketchpad* {Key Curriculum Press, 2001} for class C only) and six demographical questions, was analyzed on www.surveymonkey.com. Responses between first-year and mixed-year (2^{nd} to 4^{th}) mathematics students on the use of key ideas (five items) and on their proving practices (five items) were compared. Students' responses to the interviews were analyzed using NVivo 10 software for qualitative analysis to explore themes and patterns of responses. The analysis of student work on the two proofs started separately yet ended simultaneously. The unit of analysis was a proof statement. Each participant's work was divided into proof statements and grouped in categories using the constant-comparative method (Kolb, 2012). This process continued until a set of categories was formed.

KEY FINDINGS AND DISCUSSION

WHAT ARE STUDENTS' PERCEPTIONS OF THE ROLE OF KEY IDEAS IN A PROOF?

The online survey aimed to investigate how undergraduate mathematics students perceive the role of key ideas in a proof from three aspects: 1) the role of key ideas in reading a proof, 2) the role of key ideas in constructing a proof, and 3) the role of key ideas in reconstructing a proof. The findings show that the majority of the mathematics students, from year I to year IV of their undergraduate studies, acknowledged the importance of key ideas in a proof when reading it and claimed that they attempted to identify key ideas of a proof. About half of the students recognized the value of key ideas in the two proofs and reported that they reflected back on the key ideas when constructing and reconstructing these proofs.

Concerning the role of key ideas in reconstructing a proof, the students agreed that key ideas in a proof could play an important role in helping proof reconstruction. Nevertheless, a large number of students also stated that they needed to have taken notes on the details of each step of a proof in order to be able to reconstruct it. In fact, more than half of the students claimed that both the key ideas and details in a proof were equally important for reconstruction. Interestingly enough, the more senior mathematics students seemed to favour paying attention to the details of each proof step, while the more junior students seemed to prefer knowing the key ideas used in a proof.

WHICH FEATURES OF A PROOF DO STUDENTS IDENTIFY AS KEY IDEAS?

Three foci on key ideas were identified through analysis of students' work: 1) focusing on the mathematical ideas used in the proof, 2) focusing on the methods used in the proof, and 3) focusing on the details of the proof. The findings show that most students seemed to focus on mathematical ideas when they attempted to identify a key idea in a proof; a few students focused on the methods, while very few students focused on details of a proof, such as algebraic manipulation (see Figures 1-3).

(e) What is the idea of this proof?

The proof proceeds as if $\sqrt{2}$ is a rational number until a contradiction appears which shows that it is not. The contradiction is that the number a, though specified to be the smallest integer for which $a^2=2b^2$ is ~~not~~ found not to be, ~~and~~ proving that there is no rational representation of $\sqrt{2}$

Figure 1. A key idea that focuses on the mathematical idea used in the proof.

I believe what the proof is trying to show that proofs that don't look geometric really can be. Using squares an irrational ~~it~~ it is shown that literal squares can sub algebraic ones like $a^2=2b^2$. Looking at a problem a different or breaking it down to things you like geometry, helps see the problem better

Figure 2. A key idea that focuses on the method used in the proof.

There is a square of side length a, which has area a^2. There are two squares of side length b and area b^2. Then $a^2 = b^2+b^2 \Rightarrow a^2 = 2b^2 \Rightarrow a=\sqrt{2}b$. The two squres of side length b overlap within a square of side length a. The overlap is equal to the empty space within the square of side length a.

Figure 3. A key idea that focuses on the details of the proof.

The findings show that a key idea may have been judged as such by students for a variety of reasons, such as its value as a conceptual explanation, its procedural usefulness, its usefulness for self-clarification, and its association with previous learning. In other words, the status and content of a key idea of a proof may depend upon why it is judged to be a key idea. In this sense, the choice of key idea may simply be based on pragmatic reasons and depend on a variety of contexts, and therefore the idea of key idea may best be seen as a multi-faceted concept.

Although the identification of key idea may be subjective and learner dependent, as discussed above, the findings indeed indicate that the key ideas identified by the students did overlap to some extent. The overlaps do not appear to be coincidental but rather suggest that certain key ideas such as the methods used are seen as salient.

This study's findings in showing students' success in the proof reconstructions of the irrationality of $\sqrt{2}$ for a non-square k were somewhat in line with Raman et al. (2009) in that a proof should not be thought as having only one particular key idea, "We refer to 'a' key idea rather than 'the' key idea, because it appears that some proofs have more than one key idea" (p. 156), and did support Gowers' view that a key idea provides a clue how to remember a proof and to write it. In fact, the findings of this study show that the students' proof reconstructions were closely aligned with the key ideas they identified—varied in type as those key ideas may have been.

Perhaps due to the nature of this particular proof, its key ideas seemed to provide students with something striking to remember and to rely upon later in rewriting the proof. Different proofs could be considered in future studies to seek other possibilities of what a key idea could lend to a proof that meets both explanatory persuasion (Gowers, 2014) and the standard of mathematical rigor.

PEDAGOGICAL IMPLICATIONS

The key ideas of a proof identified by students reflected not only their understanding of the concept of *key idea* as it had been presented to them, but also, and perhaps even to a greater degree, which feature of the proof most attracted their attention. This may be one reason for the lack of clarity in some students' work, but other reasons could range from an insufficient understanding of the proof and its mathematical subject matter to a simple struggle with poor language proficiency. To avoid erroneous judgments of students' work, it is important for mathematics educators to be aware of these possibilities.

Students' work on proof reconstructions demonstrated some variations. These variations stemmed in part from the different interpretations students gave to the concept of key idea, as discussed above. Though some students struggled with identifying key ideas, the majority of the students, in one way or another, came very close to capturing what clearly were the key ideas of the two proofs used in the study. However, some students focused on the methods used in a proof, while others focused on the details and major steps. Indeed, when asked, "What is the key idea of the proof?" the students gave answers that were based on how they had interpreted the notion of key idea.

These results suggest that, in order to help students in their struggle to succeed in proof and proving, instructors would do well to create more opportunities for students to reconstruct a proof and, just as importantly, be attentive to students' inevitably differing identifications of the aspects of a proof and their differing approaches to proof reconstruction. In other words, despite the considerable overlap in what students see as key ideas, there is room for more pedagogical attention to the variations in key-idea identification.

The findings also imply that mathematics educators, in their desire to see students enhance their understanding of proof and proving by the use of key ideas, will need to extend considerable support to students by very actively intervening to draw their attention to features of proofs that are candidates for key ideas.

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Ad Hoc Sessions

Ad hoc sessions

WHAT IS AN AUTHENTIC ASSESSMENT TASK?

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Reform perspectives of mathematics education include reforms in how students' learning and performances are assessed in the mathematics classroom with emphasis on formative assessment. Thus, there is a need to support mathematics teachers' development of their assessment literacy in designing and implementing classroom assessments, such as authentic assessment tasks, that are well aligned with the objectives of reform-oriented mathematics curriculum.

Mathematical tasks are central to learning and doing mathematics. They "provide the stimulus for students to think about particular concepts and procedures, their connections with other mathematical ideas, and their applications to real-world contexts" (National Council of Teachers of Mathematics, 1991, p. 24). Tasks have received extensive research attention from a variety of perspectives dealing with their nature/attributes, design, and uses in the teaching and learning of mathematics and in teacher education and professional development. Mathematical tasks used in educational settings have been categorized in a variety of ways including worthwhile, authentic, rich, complex, traditional/routine/closed/algorithmic, non-routine/non-algorithmic/open, inquiry, and cognitive level/demand. Except for the traditional/routine tasks, the other categories are similar in terms of offering experiences to develop students' mathematical understandings and mathematical thinking.

In this ad hoc session, the focus was on authentic assessment tasks, that is, tasks that replicate real-world problems that require students to demonstrate their mathematical understanding through the application of essential mathematical knowledge and skills in problem solving (Koh, 2014). In addition to participants' perspectives of authentic tasks, we discussed Koh's (2011) *criteria for authentic intellectual quality*, which include the following five criteria and their respective elements: (1) depth of knowledge (factual knowledge; procedural knowledge; advanced concepts), (2) knowledge criticism (presentation of knowledge as a given; comparing and contrasting information; critiquing information), (3) knowledge manipulation (reproduction; organization, interpretation, analysis, evaluation, synthesis of information; application/problem solving; generation/ construction of new knowledge), (4) extended communication, and (5) making connections to the real world beyond the classroom. The mathematics indicators for each of the elements are detailed in Koh (2011). They are the basis of a Social Science and Humanities Research Council (SSHRC) funded project (Koh and Chapman as co-principal investigator) aimed at developing prospective elementary school mathematics teachers' expertise in the selection, adaptation, and design of authentic assessment mathematics tasks.

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CMESG 2018 AD HOC

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Version française à suivre (traduction : Ariadna Gula et Sofia Berezovskyj).

The idea for the ad hoc session came out of an earlier discussion about numeracy and the challenge of making it well-defined, in particular in relation to mathematics. Some (academics, instructors, teachers, professionals) see numeracy simply as a subset of mathematics; whereas, others see it as something larger than mathematics, some see the two as mutually exclusive. Perhaps the best illustration of this variety of views are the courses that universities in North America declare as falling into 'numeracy', or 'quantitative reasoning' domains: these courses range from remedial precalculus, functions and algebra courses, to symbolic logic and computer literacy, to modelling, history of mathematics and 'math for liberal arts' courses.

Two questions guided our discussion:

- *Can one participate in numerate behaviour without invoking mathematics?*
- *What would be the characteristic of a problem/scenario that would evoke numerate behaviour without mathematical behaviour in students and/or citizens?*

The challenge was met head on, but, of course, the tension between mathematics and numeracy was not resolved in the short time we had. A variety of real world situations (such as setting up a satellite dish) and designed school tasks (unresolved wager problem, sharing of purchase of buy two get one free, grandpa birthday sharing costs among children with unequal incomes) were discussed—energetically of course. Due to the large audience many voices were not heard.

How to think about numeracy? The ad hoc participants suggested the following:

- perhaps numeracy for many should be renamed numberacy to highlight its focus on numbers and proficiency in working with numbers, such as mental arithmetic;
- the understanding of the difference between mathematics and numeracy can be enhanced by taking the perspective that what distinguishes mathematics is that it is the abstract study of structure; whereas, numeracy is rooted in true life contexts;
- one characteristic that can distinguish numeracy from mathematics is that it provides room to broaden the discussion of problems being examined, bringing it into the realm of ethics in decision making (e.g., splitting equally versus fairly and what that can mean); perhaps in this case numeracy can be taught within social sciences courses?
- Perhaps one difference lies in behaviours and intentions: typical mathematicians would be interested in building models that attempt to provide precise responses to the presented situations or problems; whereas, a numerate response would beless interested in precision ('back of an envelope calculations', as in Fermi problems).

This topic is clearly one that is of interest to a large part of the CMESG community.

L'idée de la session ad hoc est née d'une discussion antérieure sur la numératie et le défi de définir le concept avec plus de précision, en particulier en relation aux mathématiques. Certains (universitaires, instructeurs, enseignants, professionnels) considèrent la numératie comme un sous-ensemble des mathématiques, tandis que d'autres la considèrent comme un concept qui est plus grand que celui des mathématiques; certains voient les deux concepts comme étant mutuellement exclusifs. La meilleure illustration de cette diversité d'opinions se trouve dans les cours que les universités en Amérique du Nord considèrent comme reliés aux domaines de la « numératie » ou du « raisonnement quantitatif ». Ces cours varient : cours de rattrapage pour accéder aux cours de calcul, cours sur les fonctions, l'algèbre, la logique symbolique, les connaissances informatiques, la modélisation, l'histoire des mathématiques ou encore des cours de mathématiques pour « liberal arts ».

Deux questions ont guidé notre discussion :

- *Est-possible pour quelqu'un d'invoquer un comportement en lien avec la numératie et ne pas faire appel aux mathématiques?*
- *Quelle serait la caractéristique d'un problème ou scénario qui évoque uniquement un comportement en lien avec la numératie, et aucun comportement mathématique, chez des étudiants et/ou des citoyens?*

Le défi a été abordé de manière directe durant la rencontre, mais la tension entre les mathématiques et la numératie n'a pas été résolue dans le peu de temps que nous avons eu. Diverses situations du monde réel (telles que la configuration d'une antenne parabolique) et des tâches spécifiques à l'école (problème de pari non résolu, partage de l'achat « achetez-en un et obtenez-en un gratuit », partage des frais d'anniversaire de grand-père entre enfants aux revenus inégaux) ont été discutées… avec animation. En raison d'un public nombreux, beaucoup n'ont pas été entendus. Comment penser à la numératie? Les participants ad hoc ont suggéré ce qui suit :

- Peut-être que, pour plusieurs, il faudrait renommer la numératie « numérotation » pour mettre en évidence le lien important avec les nombres et avec la maîtrise du travail avec les nombres, comme le calcul mental.
- On peut approfondir la compréhension de la différence entre les mathématiques et la numératie en distinguant que les mathématiques sont l'étude abstraite des structures tandis que la numératie est enracinée dans les contextes réels.
- Une caractéristique qui permet de distinguer la numératie des mathématiques est qu'elle permet d'élargir la discussion sur les problèmes examinés, en l'intégrant dans le domaine de l'éthique lié à la prise de décision (par exemple, diviser en parts égales ou équitables et ce que cela peut signifier); dans ce cas, se peut-il que la numératie puisse être enseignée dans les cours de sciences sociales?
- Il est possible qu'une différence réside dans les comportements et les intentions : un mathématicien typique serait intéressé par la construction de modèles qui tentent de fournir des réponses précises aux situations ou aux problèmes présentés, alors qu'une réponse en lien avec la numératie serait moins intéressée par la précision. On peut penser au « back of an envelope calculations », de Fermi.

Il est clair qu'une grande partie de la communauté du GCEDM s'intéresse à ce sujet.

SUGGESTIONS FOR FURTHER READING (NOT WORKS THAT INFORMED THE DISCUSSION) / SUGGESTIONS POUR CONTINUER LA RÉFLEXION ET NON CE QUI A INFORMÉ LA DISCUSSION DU GROUPE

Griffiths, G., & Stone, R. (Eds.). (2013). *Teaching adult numeracy: Principles and practice*. Maidenhead, England: Open University Press.

Steen, L. A. (Ed.). (2001). *Mathematics and democracy: The case for quantitative literacy*. Princeton, NJ: National Council on Education and the Disciplines.

SAMPLE OF A POLICY PAPER / EXEMPLE D'UN DOCUMENT D'ORIENTATION :

Moffatt, M., & Rasmussen, H. (2016). Big idea: A Canada-wide transformation of numeracy skills. *Canada2020*. Retrieved from http://canada2020.ca/numeracy/

MATHEMATICS A PLACE OF LOVING KINDNESS AND…

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Taking a cue from Singh's (2017) one-word chapter titles, we invite readers to complete the sentence, 'I imagine/want mathematics to be a place of loving kindness and...' with their own word, phrase, or image.

What would it mean to ground our mathematics education practices in loving kindness? What would doing so require us to commit to and enact? What associations might arise? What would a more kind mathematical practice look like? These were the focal points for the discussion initiated to determine whether the idea was worth pursuing. The short conversation brought concerns like Understanding, Creativity, Acceptance, Joy and being kind to ourselves to the forefront. We have used the space to develop a proposal for a themed journal issue and hope to explore the idea further in the future.

Twentieth century mythological frameworks about the teaching, learning, knowing and doing of mathematics across the lifespan and in a variety of educational settings are giving way to emerging constellations and sensibilities seeded by a re-invigorated emphasis on human(e) values grounded in intentional and explicit practices of mindfully curating attention, awareness, and action with loving kindness. This mythopoetic work is emerging from the amplification of activities and connections among popularizers, proselytizers and policy-makers enabled by networked technology.

Loving Kindness, in its various interpretations and instantiations across traditions, shares what we believe is an explicit and active opposition to incarnations and material practices of human cruelty, violence, humiliation, shaming and brutality. Love and Kindness can be learned and practiced in mathematics at any level and remain part of the ongoing dynamics of mathematics education. We suspect that some of the resonance arises from having begun to work through traumas, violations, hate, pain, anger, loss and sadness in mathematics and the first articulations of difficult knowledge of learning to share with our communities. We see this invitation as an opportunity to begin to tell stories and to nurture new myths that heal in the disciplines and, in particular, the discipline of mathematics—the source of much pain, anxiety and unkindness. We appreciate the risk of sharing our vulnerabilities. We believe it is a risk worth taking in this moment as we seek to create more hospitable places of learning in mathematics.

An intention is to provide an opportunity to engage and connect with each other's work so as to reduce "connection gaps" (Bruce et al., 2017, p. 143) that limit language, discourse and imagination across scholarly communities of practice.

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ONE APPROACH TO CODING AND TEACHER EDUCATION

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The 16-month BEd program offered at the University of Ontario Institute of Technology (UOIT) includes a 36-hour coding and pedagogy course for all teacher candidates (TCs). Inspired by arguments that coding is an important literacy in the 21st century, the course has three thrusts: 1) why teach coding in K-12; 2) introducing core coding concepts to TCs; and 3) exploring how coding can support teaching and learning across the curriculum with a focus on math. With more jurisdictions mandating coding in K-12 education, we believe it is critical to discuss successes and challenges of different approaches to teaching coding in preservice teacher education.

One of biggest challenges within the UOIT coding course is justifying why the course exists. Most TCs never experienced coding as a K-12 student, nor is coding currently in the Ontario curriculum. Some view it as insufficiently linked to their subject matter specialities, and there is concern that the push for coding is 'coming too soon' (i.e., before a curriculum). This challenge is explicitly addressed throughout the course with readings, viewings and discussions. TCs also work with ministry documents supporting coding in K-12 education. As a result, the initial pushback is slowly replaced with understanding the importance of coding.

In tandem with the focus on the rationale for coding in K-12 education, the UOIT course aims to develop TCs confidence and capability with coding. Most TCs have never coded before and approach the course with considerable anxiety, and feelings of being overwhelmed with learning new content when it is felt that attention should be focused on learning new pedagogy. These are some of the motivations behind why the UOIT course starts with Scratch, a block-based coding environment, which stresses exploration and problem-solving with engaging media. The use of a drag and drop environment resulted in considerable discussion from ad hoc participants, some of whom argued for the use of 'real' coding language. We described how we tried introducing Python, a text-based language with simple syntax, half way into the UOIT course and found that in a 36-hour course, introducing Python worked against many of our goals: it limited which core concepts we could develop and to what depth, it introduced further anxieties for TCs, and it shifted the balance away from important pedagogical discussions. Wanting to incorporate coding to teach concepts and practices, especially as related to mathematics, we decided to minimize the learning curve of the coding environment—Scratch provides a good balance between ease of learning and capability to support learning through coding.

At UOIT, the potential of learning through coding is integrated throughout the course via activities that support procedural writing and media studies in English; dissect and explore geometric concepts and relationships; promote inquiry, generalizations and abstractions; and emphasize connections between big ideas in mathematics. Many TCs get excited when their guided inquiry leads them to new ideas, methods, and insights. However, great care needs to be taken when making didactical decisions so as to avoid provoking more anxiety in already nervous coders. Thus a coding course for teachers is a balance of developing understanding of coding and its potential, developing confidence to code, and of course, developing strategies through which to teach the very things TCs are learning. Questions about how to achieve this balance require ongoing attention.

TEACHING REAL NUMBERS IN ELEMENTARY/MIDDLE SCHOOL: CAN WE PRESENT THEM AS ANSWERS FOR DAILY LIFE PROBLEMS?

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I am referring to a study of mine about teaching real numbers, $\mathbb{R}$, by a problem-based approach where learners are expected to mobilize previously acquired concepts, preparing themselves to later be introduced to the new the concept definition. It is a study that avoids the teaching sequence: definition→properties→examples→exercises. The answer to the question on title depends mainly of two aspects: first, the problems must raise from learner's intuitive background; second, it must be explained what I mean by preparing learners.

Supported by Duva'sl (1999) registers of semiotic representation theory and Tall and Vinner's (1981) concept image notion, I try to design teaching sequences—problem situation→representations→patterns (properties) →mathematical object (definition)—aiming to teach some specific property. The idea is to propose several tasks, in different contexts, that allow, or lead to, representation of acquired concepts in suitable representation systems, such as natural language (with specific terms), number line and decimal fractions. Learners would be prepared to learn the real numbers concept definition after knowing a specific set of properties for the set $\mathbb{R}$ and having the ability to represent them through treating representations and converting representations between the different representation systems I have just indicated.

In this teaching process, it is fundamental to have in mind the specific property of $\mathbb{R}$ to be taught. I will give a brief idea on how I would design a task for the fact that between two real numbers there are infinitely many others. The study of musical notes seems to be a good situation for good problems that could have musical instruments as a mathematics manipulative: they could be a piano and a monochord. A task could be 'Play two notes. Then find a third note to play, but it must be between the first two notes. Can you do this? Can you find more than one note between the first two notes? What is the maximum number of such notes? Is it possible to find two notes with no other note between them? When?' And learners should perform the task for the two instruments. Then the activity could continue with learners being asked to register the instruments and the notes. It would be interesting to know what mathematical representations they would use. Would they present different answers for the different instruments? If asked, would they be able to associate notes on a piano with natural numbers? Would the learners associate notes on a monochord with natural numbers? It would be interesting to observe whether learners would use terms that could be related to real numbers, such as varying, changing or comparing a note, or musical interval. Would they use terms like 'the next note' in the monochord?

There are many other interesting situations that appear, or could appear, in school, such as length measurement, displacement, space of colours, images and scales, shadows, estimating temperature, sun position and hour, carpentry and construction. For now, this study is about teaching activities, but I am working to turn it into research material.

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REPORT OF WORKING GROUP F: MATHEMATICS IN SITU

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At the 2003 CMESG meeting at Acadia University, "the executive approved a last-minute request from a group of 'rebels' to create a fourth … 'ad-hoc' working group" (Dubiel, 2004, p. ix). While not completely unprecedented, nothing like this had occurred since the 1991 meeting at University of New Brunswick, when a 'splinter' group formed out of one of the planned groups. At the 2018 meeting, once more a group of 'rebels' proposed to create an 'ad hoc' working group, in light of the unexpectedly high number of participants in the conference, which some felt might overload the planned working groups. The executive did not approve this last minute request, but we went ahead anyway.

The reports of 'ad hoc' working groups have been handled differently in the proceedings each time. In 1991, the report of the 'splinter' group appeared in the proceedings instead of the report of the original working group. This occurred as the original working group organisers did not submit a report, but the splinter group did. The Table of Contents still lists the original group name and organisers, but when one turns to the page indicated, one finds a report with a different title and author. In the 2003, the report of the 'ad hoc' group appears along with all the others, with only the President's comments in the introduction (quoted above) to indicate its unusual status. In 2018, the 'ad hoc' working group was given the usual single page for ad hocs in the proceedings. The full report of working group F is available online, at http://www.cmesg.org/wp-content/uploads/2018/11/Report-of-Working-Group-F.pdf

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Mathematics Gallery

Gallérie Mathématique

LOGICAL IMPLICATIONS AND ENTAILMENTS OF FOUR METAPHORS OF MATHEMATICAL CREATIVITY

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In my poster I presented my thoughts regarding the question, *What can be learned from the process of developing and refining an emergent definition of collective creativity for the elementary mathematics classroom?* based on my analysis and interpretation of the (co)acting, and interacting of a group of sixth-grade students while they worked on an assigned mathematical task. This design-based research study was initially framed by seven metaphors of creativity: (1) overcoming obstacles; (2) divergent thinking or thinking outside the box; (3) assembling things in new ways; (4) route-finding; (5) expanding possibilities; (6) collaborative emergence; and (7) birthing, producing, originating, or making something new (Aljarrah, 2016).

In my analysis and interpretation of the data, I concentrated on the (co)acting and interacting within the group and how such collaborative practices (Martin, Towers, & Pirie, 2006) contribute to the emergence of the new (e.g., new steps, new ideas, new strategies, new paths, new understandings, etc.). I refined and (re)developed my initial seven metaphors over successive iterations of data analysis and interpretation until I ended up with four metaphors to describe the experience of creativity with(in) the collective: (1) summing forces, (2) expanding possibilities, (3) divergent thinking, and (4) assembling things in new ways. I believe that my four metaphors of creativity were significant to understanding students' creative acts while they were working collaboratively on assigned mathematical tasks.

What do learning and teaching look like with(in) each metaphor of creativity? Based on my description of each metaphor, I speculated on some of the logical implications and entailments of it for teaching and learning. For example, I used the 'summing forces' metaphor to describe students' ways of coacting that summed to enable productive steering of the group towards an understanding "that [was] not simply located in the actions of any one individual but in the collective engagement with the task posed" (Martin et al., 2006, p. 157). Teaching under this metaphor can be understood as nudging students by offering rich mathematical tasks that open up "possibilities for many different mathematical responses and pathways of action" (Martin et al., 2006, p. 177). Productive steering through the summing of forces should not be thought of as zeroing in on a final end product or conclusion. Rather, it should be understood as effective learning acts—(co)actions and interactions—prompted, conditioned, and oriented by a common purpose (Davis & Simmt, 2003). Similarly, the other three metaphors of collective creativity each have distinct entailments and implications for teaching.

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MOVING ACHIEVEMENT TOGETHER HOLISTICALLY (M.A.T.H.): AN INDIGENOUS APPROACH TO MATHEMATICS EDUCATION

Ellen Carter, Evan Throop-Robinson, & Lisa Lunney Borden
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Moving Achievement Together Holistically (M.A.T.H.) is a project designed to evaluate the impact of the Mawikinutimatimk Framework (Lunney Borden, 2010) on both teacher pedagogical practices and student learning and engagement. *Mawikinutimatimk* means 'coming together to learn together', and is rooted in *L'nuita'simk* (Mi'kmaw ways of knowing), highlighting a verb-based and spatial reasoning rich approach to teaching mathematics in a way that honours the holistic views of the Mi'kmaw community. Enacting a framework designed in and with community, this research also includes full partnership with school and community leaders. Specific research questions to study of the implementation of the framework include

- How does the professional learning support teacher development? How does the implementation vary across different contexts?
- What types of learning and assessment tasks help to improve student learning across these contexts?
- How do students respond to this approach to learning mathematics in the varying contexts? Does this approach improve engagement and attitude toward learning mathematics?
- How does the involvement of Elders and community knowledge holders in professional learning and classroom teaching support teacher learning and student learning across these contexts?

To implement the Mawikinutimatimk Framework effectively, and address the research questions, the authors provide ongoing professional learning opportunities and classroom-based supports for teachers of grades primary through six. Researchers and teachers collectively generate student learning and assessment materials in cooperation with community Elders and knowledge holders. Each task is developed with process and spatial awareness in mind, implemented with students, assessed, and refined. Data is collected from a wide variety of sources to gain a full perspective of the impact of the work.

One such data collection tool asked students, given their age, to draw a picture of their math class, as well as a picture of them doing math. Students shared details of their drawings, and the resulting pictures were shared with staff to elicit teacher views of student perceptions. Teachers are engaged in ongoing interviews to address each of the research questions. Students participate in small group and individual conversations to share their perspectives throughout the year.

M.A.T.H. is a multi-year project, of which the methodology and data from the pilot year were shared at the CMESG Gallery Walk. We invited our peers to critically examine the work to date and specifically respond to the data collection tools and results. This worthwhile activity produced new insights which will be submitted to *for the learning of mathematics* in the future.

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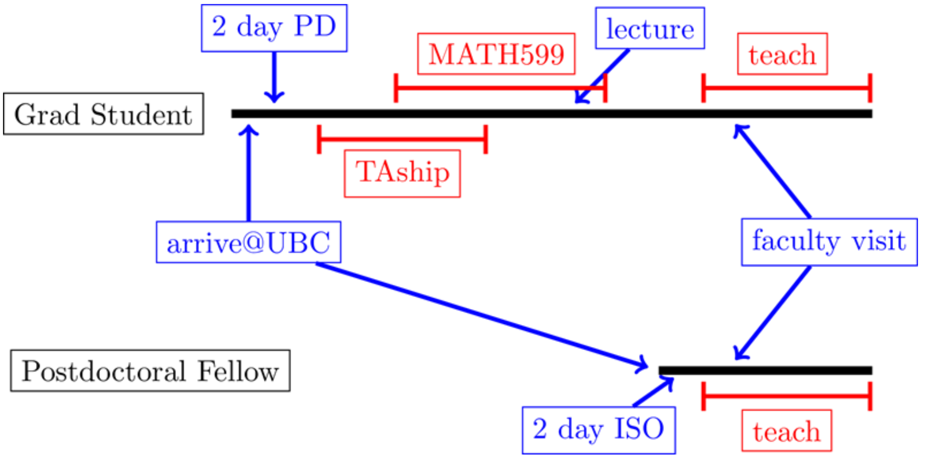
PROFESSIONAL DEVELOPMENT FOR GRADUATE STUDENT AND POSTDOC INSTRUCTORS

Matt Coles
University of British Columbia

THE FACTS

Graduate students and postdoctoral fellows teach a large number of first- and second-year mathematics courses each year at the University of British Columbia (UBC). In particular, *about 50% of first-year course sections* in 2017/2018 were taught by either a graduate student or postdoc. *Graduate students teach at most one course per year* and are encouraged to do so once or twice during their degree. That said, not all graduate students will teach. *Postdocs*, on the other hand, *teach 2 courses per year* with most teaching immediately upon arrival to UBC.

LONG-STANDING PROFESSIONAL DEVELOPMENT



During or after Teaching Assistantships, which include training (two-day professional development), *graduate students* wishing to teach can take *MATH 599*: a course on teaching mathematics which culminates with students giving a live lecture in a real calculus class. Graduate students are then given courses to teach based on the recommendation of the instructor of MATH 599. Postdocs attend *a two-day Instructional Skills Orientation* (ISO) before teaching in the department. Incoming postdoc experience ranges widely. *All new instructors* in the department receive a *faculty visit* in the first two weeks of class. The faculty ensures the instructor is suitable to teach and provides the instructor with some feedback.

RECENT ADDITIONS TO PROFESSIONAL DEVELOPMENT

We now have two new programs to provide additional support for novice instructors: the *Instructor Support Group* for interested novice instructors, and the *PDF Coffee Groups* for new postdocs.

The *Instructor Support Group* is a community of practice for graduate student and postdoc instructors currently teaching in the department. Together the group practices *goal setting with reflection*, as well as *peer review of teaching with feedback*. Participants reported that the group provided ideas for new techniques and improved implementation.

We support all new postdocs by administering an *early in-class student feedback form*. Instructors answer reflection questions about the feedback and bring their responses to the *PDF Coffee Groups*. Together, with a facilitator, they set goals about their own development for the rest of their course.

See the full poster here, https://www.math.ubc.ca/~colesmp/teaching/CMESG2018.pdf, for more info!

UNDERGRADUATE STUDENTS' SENSE OF EMPOWERMENT IN PROGRAMMING-BASED MATHEMATICAL EXPLORATIONS

Sarah Gannon
Brock University

Papert (1971) emphasized the importance of constructionism-based approaches for learning mathematics. By integrating computer programming in mathematics courses, students can employ computational thinking practices in explorations of mathematical phenomena (Buteau, Muller, Mgombelo, & Sacristán, 2018). In so doing, students may gradually engage in a community of practice through legitimate peripheral participation (Lave & Wenger, 1991) as they take up the practices of professional mathematicians (Buteau, Muller, Marshall, Sacristán, & Mgombelo, 2016).

Since 2001, Brock University's Mathematics Integrated with Computers and Applications (MICA) course sequel has taught undergraduate students to utilize computer programming for mathematical explorations. Project assignments, called "exploratory objects" (EOs), enable students to work as mathematicians by posing mathematical questions and exploring them using computer programs (Buteau, Muller, & Ralph, 2015, p. 1). My poster presented data from participants in the first-year course (MICA I) collected as part of the larger study *Educating for the 21st Century: Post-secondary Students Learning Computer Programming for Mathematical Investigation, Simulation, and Real-World Modelling* (Buteau, Mgombelo, Sacristán, & Muller, 2017-2022, S.S.H.R.C. #435-2017-0367).

In MICA I, students have the opportunity to be empowered through the creation of shareable programs, allowing them to contribute to their knowledge, and possibly that of their peers, about their chosen mathematical phenomena (Buteau et al., 2016). The first EO challenges students to pose and explore a conjecture about primes or hailstone sequences through the design and use of a computer program. My poster featured the following excerpts from participant interviews following this assignment, and asked CMESG attendees to consider which quote they thought best illustrated a sense of empowerment to explore mathematics through programming.[1]

> **Hannah**: "Finishing my program made me feel like a professional, like I could have a job creating software to do or teach mathematics."
> **Ashley**: "I told a friend about my data because I was really excited about the patterns I found. I kept testing values until it crashed."
> **Jim**: "I want to build on what I've learned. I know this is the beginning of what programming can do, and I want to see where it goes."

It is our view that these excerpts demonstrate students' sense of excitement, potential, and desire to share with others as they begin to learn programming for mathematical explorations.

ACKNOWLEDGMENT

Thanks to Wendy Forbes and Kirstin Dreise for their help in data collection.

[1] Hannah's quote was most often selected as best illustrating a sense of empowerment.

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PLAYING WITH MATHEMATICS WITHIN DYNAMIC GEOMETRY ENVIRONMENTS IN ELEMENTARY SCHOOL: THE TRANSFORMATIONS PROJECT

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Simon Fraser University

Featherstone (2000) proposed that play could provide school-age children with a venue for approaching the more abstract side of school mathematics. She found elements of play, as identified by Huizinga (1955), to be present to various extent in elementary mathematics classroom—these elements or characteristic features being aesthetics, imagination, rules, purpose, and being voluntary. Featherstone (2000) inquired in what ways play might expose children to aspects of mathematics that may not ordinarily be visible to them (p. 16).

This pilot project seeks to identify elements of self-initiated play within student activity in a digitised upper elementary mathematics classroom and attempts to answer two questions: how self-initiated play helps students overcome obstacles they encounter during construction process within Dynamic Geometry Environment (DGE), and in what ways play might free children "from the dictatorship of concrete objects and enable them to behave in accordance with meaning", as Vygotsky (2016, p. 19) put it.

Here, I examine several instances of student engagements with the Web Sketchpad (Sinclair, 2018), which took place in a Grade 6/7 classroom in a high-density, affluent neighbourhood elementary school in Canada. Isometric transformations provided the context for these engagements, with the sub-topic of reflectional symmetry being chosen as a starting point.

First, the whole class worked with two symmetry web sketches: Discrete Symmetry and Continuous Symmetry (https://www.sfu.ca/geometry4yl/symmetry.html). Then, the students were shown a picture of the mirror machine, designed to encode Leonardo Da Vinci's writings, and invited to re-create it in Basic Geometry Tools web sketch (http://www.sfu.ca/content/dam/sfu/geometry4yl/sketchpadfiles/BasicGeometryTools/index.html).

Playful engagements with mathematics have been observed both during pre-construction phase, when students used a finished mirror machine to draw a variety of shapes within the Continuous Symmetry web sketch, and during construction phase when students were making own mirror machines, while being vaguely guided by the 'black box' of the Continuous Symmetry sketch. Aesthetics and being voluntary were the most evident characteristics of play in students' work.

For the full poster, and to see examples of play with the concept of reflectional symmetry, go to https://drive.google.com/file/d/1moc4bnX__6ce8zKLfrHZY-CD2bhWDNpu/view?usp=sharing

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LAUNCHING A DYNAMIC MATHEMATICS CURRICULUM NETWORK: A PARTICIPATORY APPROACH TO ENHANCING MATHEMATICS TEACHING & LEARNNG

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In this study, funded by the Social Sciences and Humanities Research Council of Canada, we are collaborating with teachers and other mathematics educators and researchers to re-conceptualize mathematics curriculum as a dynamic, web-enabled network. The Network represents concepts and processes as connected nodes, which makes the connections within school mathematics more visible. The website for the Network (http://dynamicmathcurriculum.ca/) also enables educators to share teaching approaches and resources with others.

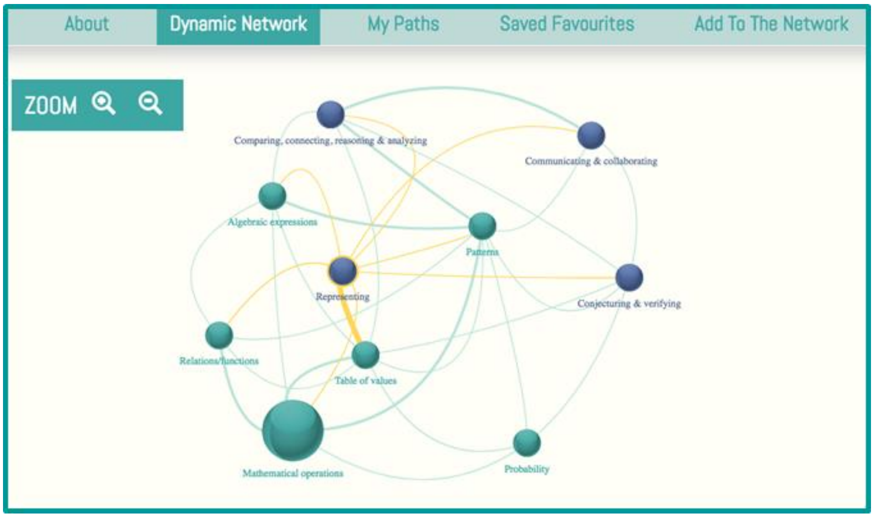

Figure 1. The Network.

Our research draws on complexity thinking, participatory approaches to curriculum design, and non-linear models of mathematics learning. We selected algebraic thinking as an initial area of focus and developed an interactive research design that engages educators in collaborative problem solving and discussion of the mathematical concepts that emerge. Further, participants co-construct physical models of the ways these concepts are connected. Four video-recorded data gathering sessions of this kind took place with educators in Ontario, Quebec and Manitoba. Our analysis of the nodes and connections in the six models created by these participants resulted in an initial network with 10 nodes and 31 connections. Working with a technology provider this analysis is represented as a dynamic, web-based interface (as shown above).

At the CMESG gallery walk, we summarized our research process, displayed a segment of the Network from the first phase, and invited CMESG participants to explore the website and share their views. We continue to gather feedback and suggestions on the nodes and connections as the basis for the next iteration of the Network. We are also continuing to analyse data from the study to focus on the ways mathematics educators work collaboratively on problem solving and think about and make connections among concepts and processes. Our study contributes to existing literature on participatory approaches to enhancing mathematics teaching and learning and provides insights into the use of digital tools to represent mathematics curricula.

STUDENTS' PERSONALIZATION OF THEIR UNDERSTANDING OF STATISTICAL CONCEPTS BY WRITING DIALOGUE FOR A STORY-BASED TASK

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The purpose of this study was to investigate the effectiveness of a series of innovative story-based tasks used to explore statistical concepts. The focus of the intervention was to provide students in a first-year post-secondary statistics course the opportunity to investigate in-depth problems through the use of stories. The stories in the intervention have a clear beginning and end and tell a sequence of events with a character that is driving the events towards a solution to a problem or conflict (Egan, 1986). They have characters, plots, context, conflict, imagery, emotions, and humor (Zazkis & Liljedahl, 2009). The stories are fictional but are set in realistic situations. The stories are left intentionally incomplete at key points. To complete the story, the students were prompted to write dialogue on behalf of the characters to explain the results of a statistical analysis or to elaborate on a statistical concept.

From the qualitative data analysis, one theme that arose was the personalization of students' understanding of statistical concepts. To illustrate, two participants' dialogue for a prompt that asked the students to interpret the outliers (if any) in the context of the story are presented in Table 1. In this story, three accounting articling students are tasked with determining whether an automated inventory system is working at a bike shop.

Jolene: **So by the looks of the data given in the BoxPlot, we don't have any outliers to deal with!** *Bart:* <u>Sooo, that's a good thing, right? I know they are kinda bad or at least important in some way, right?</u> *Jolene:* <u>Ya Bart that is correct, it is a good thing!</u> *They are meant to show us points in the data that have diverged from the rest of the data pattern*, and in this case we have none! *Bart:* Oh that's sweet, less work then!		*Franca:* **Based on the boxplot alone, it is evident that there are no outliers present within our data as they would be visible on the extremes.** *Bart:* I thought that there would always be outliers in boxplots? *Jolene:* No, like Franca said, *outliers only exist when there are extremes or when something doesn't fit in with the rest of the data.* <u>This just means that Bob's inventory system wasn't undervaluing the products with huge differences, that the amount the products were being undervalued by was consistent. That's why we have no outliers, everything fits in and nothing extremely unusual is happening with the inventory. Sure it is undervaluing the inventory, but not by an insane amount, so that is why there are none present in this data.</u>	
Legend:	*Definition of outlier*	**How to find outliers**	<u>Interpretation of outliers</u>

Table 2. Two examples of story-dialogue to illustrate personalization.

The participants personalized their understanding by providing different definitions of outliers, explaining how to find outliers in different ways, and interpreting what the outliers mean in different ways. Though only two pieces of dialogue are presented, they are representative of the dialogue seen throughout the analysis.

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WHAT DO YOU REMEMBER FROM YOUR MATHEMATICS STUDIES?

Ofer Marmur
Simon Fraser University

> *"It's impossible to forget those lessons, especially since in one of them the instructor stopped the lesson and marked a huge X in red color on top of the problem he had just solved, and many shouts of 'noooooo!!!' were heard all over the class." [A senior undergraduate student recalls a strong memorable event from her first semester]*

Affect in mathematics has traditionally been divided into three categories consisting of emotions, attitudes, and beliefs, where the order of these indicates a decrease in affective involvement and intensity and an increase in stability (McLeod, 1989). While attitudes and beliefs are relatively stable attributes of affect, emotions are associated with rapidly-changing, in-the-moment psychological states that occur in learners during mathematical doing (Goldin, 2014). The transition between local and global affect, and how the former shapes the latter, is an ongoing research endeavor; as explained by Goldin (2014), "There is much to learn about how in-the-moment emotions that students experience during mathematical activity contribute to longer term effects and about how teachers may skillfully influence them" (p. 393).

Marmur (2018) has utilized the notion of memory as an essential characteristic of human consciousness and existence and focused on the memorability of experienced mathematical events as an explanatory lens on how mathematical affect is shaped and formed. While in the literature it is acknowledged that the more stable attributes of affect are often a result of a lengthy and gradual process of experiencing repeated emotional states, this view has been challenged by findings suggesting that even a single intense experience may be sufficient in having a lasting effect on a student's attitudes and beliefs (e.g., Weber, 2008). These findings indicate that not all experienced local emotional states are of equal significance in the long term. As suggested by Marmur (2018), focusing on those crucial emotions experienced during one's memorable mathematical events, may provide us with a more accurate indication of how the transitional affective process occurs.

The current research-in-progress inquires into memories of particularly meaningful events from one's experience of studying mathematics and examines how these relate to subsequent engagement with the subject. The data originates from four population groups: undergraduate STEM students; mathematics teachers; mathematics education researchers; and mathematicians. Initial themes that emerged include personal feelings of success or failure; a teacher's attitude towards mathematics or the student; and experiences of self-discovery of mathematical properties.

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Appendices

Annexes

Appendix A / Annexe A

WORKING GROUPS AT EACH ANNUAL MEETING / GROUPES DE TRAVAIL DES RENCONTRES ANNUELLES

1977 *Queen's University, Kingston, Ontario*

- Teacher education programmes
- Undergraduate mathematics programmes and prospective teachers
- Research and mathematics education
- Learning and teaching mathematics

1978 *Queen's University, Kingston, Ontario*

- Mathematics courses for prospective elementary teachers
- Mathematization
- Research in mathematics education

1979 *Queen's University, Kingston, Ontario*

- Ratio and proportion: a study of a mathematical concept
- Minicalculators in the mathematics classroom
- Is there a mathematical method?
- Topics suitable for mathematics courses for elementary teachers

1980 *Université Laval, Québec, Québec*

- The teaching of calculus and analysis
- Applications of mathematics for high school students
- Geometry in the elementary and junior high school curriculum
- The diagnosis and remediation of common mathematical errors

1981 *University of Alberta, Edmonton, Alberta*

- Research and the classroom
- Computer education for teachers
- Issues in the teaching of calculus
- Revitalising mathematics in teacher education courses

1982 *Queen's University, Kingston, Ontario*

- The influence of computer science on undergraduate mathematics education
- Applications of research in mathematics education to teacher training programmes
- Problem solving in the curriculum

1983 *University of British Columbia, Vancouver, British Columbia*

- Developing statistical thinking
- Training in diagnosis and remediation of teachers
- Mathematics and language
- The influence of computer science on the mathematics curriculum

1984 *University of Waterloo, Waterloo, Ontario*

- Logo and the mathematics curriculum
- The impact of research and technology on school algebra
- Epistemology and mathematics
- Visual thinking in mathematics

1985 *Université Laval, Québec, Québec*

- Lessons from research about students' errors
- Logo activities for the high school
- Impact of symbolic manipulation software on the teaching of calculus

1986 *Memorial University of Newfoundland, St. John's, Newfoundland*

- The role of feelings in mathematics
- The problem of rigour in mathematics teaching
- Microcomputers in teacher education
- The role of microcomputers in developing statistical thinking

1987 *Queen's University, Kingston, Ontario*

- Methods courses for secondary teacher education
- The problem of formal reasoning in undergraduate programmes
- Small group work in the mathematics classroom

1988 *University of Manitoba, Winnipeg, Manitoba*

- Teacher education: what could it be?
- Natural learning and mathematics
- Using software for geometrical investigations
- A study of the remedial teaching of mathematics

1989 *Brock University, St. Catharines, Ontario*

- Using computers to investigate work with teachers
- Computers in the undergraduate mathematics curriculum
- Natural language and mathematical language
- Research strategies for pupils' conceptions in mathematics

1990 *Simon Fraser University, Vancouver, British Columbia*

· Reading and writing in the mathematics classroom
· The NCTM "Standards" and Canadian reality
· Explanatory models of children's mathematics
· Chaos and fractal geometry for high school students

1991 *University of New Brunswick, Fredericton, New Brunswick*

· Fractal geometry in the curriculum
· Socio-cultural aspects of mathematics
· Technology and understanding mathematics
· Constructivism: implications for teacher education in mathematics

1992 *ICME–7, Université Laval, Québec, Québec*

1993 *York University, Toronto, Ontario*

· Research in undergraduate teaching and learning of mathematics
· New ideas in assessment
· Computers in the classroom: mathematical and social implications
· Gender and mathematics
· Training pre-service teachers for creating mathematical communities in the classroom

1994 *University of Regina, Regina, Saskatchewan*

· Theories of mathematics education
· Pre-service mathematics teachers as purposeful learners: issues of enculturation
· Popularizing mathematics

1995 *University of Western Ontario, London, Ontario*

· Autonomy and authority in the design and conduct of learning activity
· Expanding the conversation: trying to talk about what our theories don't talk about
· Factors affecting the transition from high school to university mathematics
· Geometric proofs and knowledge without axioms

1996 *Mount Saint Vincent University, Halifax, Nova Scotia*

· Teacher education: challenges, opportunities and innovations
· Formation à l'enseignement des mathématiques au secondaire: nouvelles perspectives et défis
· What is dynamic algebra?
· The role of proof in post-secondary education

1997 *Lakehead University, Thunder Bay, Ontario*

· Awareness and expression of generality in teaching mathematics
· Communicating mathematics
· The crisis in school mathematics content

1998 *University of British Columbia, Vancouver, British Columbia*

- Assessing mathematical thinking
- From theory to observational data (and back again)
- Bringing Ethnomathematics into the classroom in a meaningful way
- Mathematical software for the undergraduate curriculum

1999 *Brock University, St. Catharines, Ontario*

- Information technology and mathematics education: What's out there and how can we use it?
- Applied mathematics in the secondary school curriculum
- Elementary mathematics
- Teaching practices and teacher education

2000 *Université du Québec à Montréal, Montréal, Québec*

- Des cours de mathématiques pour les futurs enseignants et enseignantes du primaire/Mathematics courses for prospective elementary teachers
- Crafting an algebraic mind: Intersections from history and the contemporary mathematics classroom
- Mathematics education et didactique des mathématiques : y a-t-il une raison pour vivre des vies séparées?/Mathematics education et didactique des mathématiques: Is there a reason for living separate lives?
- Teachers, technologies, and productive pedagogy

2001 *University of Alberta, Edmonton, Alberta*

- Considering how linear algebra is taught and learned
- Children's proving
- Inservice mathematics teacher education
- Where is the mathematics?

2002 *Queen's University, Kingston, Ontario*

- Mathematics and the arts
- Philosophy for children on mathematics
- The arithmetic/algebra interface: Implications for primary and secondary mathematics / Articulation arithmétique/algèbre: Implications pour l'enseignement des mathématiques au primaire et au secondaire
- Mathematics, the written and the drawn
- Des cours de mathématiques pour les futurs (et actuels) maîtres au secondaire / Types and characteristics desired of courses in mathematics programs for future (and in-service) teachers

2003 *Acadia University, Wolfville, Nova Scotia*

- L'histoire des mathématiques en tant que levier pédagogique au primaire et au secondaire / The history of mathematics as a pedagogic tool in Grades K–12
- Teacher research: An empowering practice?
- Images of undergraduate mathematics
- A mathematics curriculum manifesto

2004 *Université Laval, Québec, Québec*

- Learner generated examples as space for mathematical learning
- Transition to university mathematics
- Integrating applications and modeling in secondary and post secondary mathematics
- Elementary teacher education – Defining the crucial experiences
- A critical look at the language and practice of mathematics education technology

2005 *University of Ottawa, Ottawa, Ontario*

- Mathematics, education, society, and peace
- Learning mathematics in the early years (pre-K – 3)
- Discrete mathematics in secondary school curriculum
- Socio-cultural dimensions of mathematics learning

2006 *University of Calgary, Calgary, Alberta*

- Secondary mathematics teacher development
- Developing links between statistical and probabilistic thinking in school mathematics education
- Developing trust and respect when working with teachers of mathematics
- The body, the sense, and mathematics learning

2007 *University of New Brunswick, New Brunswick*

- Outreach in mathematics – Activities, engagement, & reflection
- Geometry, space, and technology: challenges for teachers and students
- The design and implementation of learning situations
- The multifaceted role of feedback in the teaching and learning of mathematics

2008 *Université de Sherbrooke, Sherbrooke, Québec*

- Mathematical reasoning of young children
- Mathematics-in-and-for-teaching (MifT): the case of algebra
- Mathematics and human alienation
- Communication and mathematical technology use throughout the post-secondary curriculum / Utilisation de technologies dans l'enseignement mathématique postsecondaire
- Cultures of generality and their associated pedagogies

2009 *York University, Toronto, Ontario*

- Mathematically gifted students / Les élèves doués et talentueux en mathématiques
- Mathematics and the life sciences
- Les méthodologies de recherches actuelles et émergentes en didactique des mathématiques / Contemporary and emergent research methodologies in mathematics education
- Reframing learning (mathematics) as collective action
- Étude des pratiques d'enseignement
- Mathematics as social (in)justice / Mathématiques citoyennes face à l'(in)justice sociale

2010 *Simon Fraser University, Burnaby, British Columbia*

- Teaching mathematics to special needs students: Who is at-risk?
- Attending to data analysis and visualizing data
- Recruitment, attrition, and retention in post-secondary mathematics
  Can we be thankful for mathematics? Mathematical thinking and aboriginal peoples
- Beauty in applied mathematics
- Noticing and engaging the mathematicians in our classrooms

2011 *Memorial University of Newfoundland, St. John's, Newfoundland*

- Mathematics teaching and climate change
- Meaningful procedural knowledge in mathematics learning
- Emergent methods for mathematics education research: Using data to develop theory / Méthodes émergentes pour les recherches en didactique des mathématiques: partir des données pour développer des théories
- Using simulation to develop students' mathematical competencies – Post secondary and teacher education
- Making art, doing mathematics / Créer de l'art; faire des maths
- Selecting tasks for future teachers in mathematics education

2012 *Université Laval, Québec City, Québec*

- Numeracy: Goals, affordances, and challenges
- Diversities in mathematics and their relation to equity
- Technology and mathematics teachers (K-16) / La technologie et l'enseignant mathématique (K-16)
- La preuve en mathématiques et en classe / Proof in mathematics and in schools
- The role of text/books in the mathematics classroom / Le rôle des manuels scolaires dans la classe de mathématiques
- Preparing teachers for the development of algebraic thinking at elementary and secondary levels / Préparer les enseignants au développement de la pensée algébrique au primaire et au secondaire

2013 *Brock University, St. Catharines, Ontario*

- MOOCs and online mathematics teaching and learning
- Exploring creativity: From the mathematics classroom to the mathematicians' mind / Explorer la créativité : de la classe de mathématiques à l'esprit des mathématiciens
- Mathematics of Planet Earth 2013: Education and communication / Mathématiques de la planète Terre 2013 : formation et communication (K-16)
- What does it mean to understand multiplicative ideas and processes? Designing strategies for teaching and learning
- Mathematics curriculum re-conceptualisation

2014 *University of Alberta, Edmonton, Alberta*

- Mathematical habits of mind / Modes de pensée mathématiques
- Formative assessment in mathematics: Developing understandings, sharing practice, and confronting dilemmas
- Texter mathematique / Texting mathematics
- Complex dynamical systems
- Role-playing and script-writing in mathematics education practice and research

2015 *Université de Moncton, Moncton, New Brunswick*

- Task design and problem posing
- Indigenous ways of knowing in mathematics
- Theoretical frameworks in mathematics education research / Les cadres théoriques dans la recherche en didactictique des mathématiques
- Early years teaching, learning and research: Tensions in adult-child interactions around mathematics
- Innovations in tertiary mathematics teaching, learning and research / Innovations au post-secondaire pour l'enseignement, l'apprentissage et la recherche

2016 Queen's University, Kingston, Ontario

- Computational thinking and mathematics curriculum
- Mathematics in teacher education: What, how… and why / Les mathématiques dans la formation des enseignants : quoi, comment… et pourquoi
- Problem solving: Definition, role, and pedagogy / Résolution de problèmes : définition, rôle, et pédagogie associée
- Mathematics education and social justice: Learning to meet the others in the classroom / Éducation mathématique et justice sociale : apprendre à rencontrer les autres dans las classe
- Role of spatial reasoning in mathematics
- The public discourse about mathematics and mathematics education / Le discours public sur les mathématiques et l'enseignement des mathématiques

2017 McGill University, Montréal, Québec

- Teaching first year mathematics courses in transition from secondary to tertiary
- L'anxiété mathématique chez les futurs enseignants du primaire : à la recherche de nouvelles réponses à des enjeux qui perdurent / Elementary preservice teachers and mathematics anxiety: Searching for new responses to enduring issues
- Social media and mathematics education
- Quantitative reasoning in the early years / Le raisonnement quantitatif dans les premières années du parcours scolaire
- Social, cultural, historical and philosophical perspectives on tools for mathematics
- Compréhension approfondie des mathématiques scolaires / Deep understanding of school mathematics

2018 Quest University, Squamish, British Columbia

- The 21st century secondary school mathematics classroom
- Confronting colonialism / Affronter le Colonialisme
- Playing with mathematics/Learning mathematics through play
- Robotics in mathematics education
- Relation, ritual and romance: Rethinking interest in mathematics learning

Appendix B / Annexe B

PLENARY LECTURES AT EACH ANNUAL MEETING / CONFÉRENCES PLÉNIÈRES DES RENCONTRES ANNUELLES

1977	A.J. COLEMAN	The objectives of mathematics education
	C. GAULIN	Innovations in teacher education programmes
	T.E. KIEREN	The state of research in mathematics education
1978	G.R. RISING	The mathematician's contribution to curriculum development
	A.I. WEINZWEIG	The mathematician's contribution to pedagogy
1979	J. AGASSI	The Lakatosian revolution
	J.A. EASLEY	Formal and informal research methods and the cultural status of school mathematics
1980	C. GATTEGNO	Reflections on forty years of thinking about the teaching of mathematics
	D. HAWKINS	Understanding understanding mathematics
1981	K. IVERSON	Mathematics and computers
	J. KILPATRICK	The reasonable effectiveness of research in mathematics education
1982	P.J. DAVIS	Towards a philosophy of computation
	G. VERGNAUD	Cognitive and developmental psychology and research in mathematics education
1983	S.I. BROWN	The nature of problem generation and the mathematics curriculum
	P.J. HILTON	The nature of mathematics today and implications for mathematics teaching

1984	A.J. BISHOP	The social construction of meaning: A significant development for mathematics education?
	L. HENKIN	Linguistic aspects of mathematics and mathematics instruction
1985	H. BAUERSFELD	Contributions to a fundamental theory of mathematics learning and teaching
	H.O. POLLAK	On the relation between the applications of mathematics and the teaching of mathematics
1986	R. FINNEY	Professional applications of undergraduate mathematics
	A.H. SCHOENFELD	Confessions of an accidental theorist
1987	P. NESHER	Formulating instructional theory: the role of students' misconceptions
	H.S. WILF	The calculator with a college education
1988	C. KEITEL	Mathematics education and technology
	L.A. STEEN	All one system
1989	N. BALACHEFF	Teaching mathematical proof: The relevance and complexity of a social approach
	D. SCHATTSNEIDER	Geometry is alive and well
1990	U. D'AMBROSIO	Values in mathematics education
	A. SIERPINSKA	On understanding mathematics
1991	J .J. KAPUT	Mathematics and technology: Multiple visions of multiple futures
	C. LABORDE	Approches théoriques et méthodologiques des recherches françaises en didactique des mathématiques
1992	ICME-7	
1993	G.G. JOSEPH	What is a square root? A study of geometrical representation in different mathematical traditions
	J CONFREY	Forging a revised theory of intellectual development: Piaget, Vygotsky and beyond
1994	A. SFARD	Understanding = Doing + Seeing ?
	K. DEVLIN	Mathematics for the twenty-first century
1995	M. ARTIGUE	The role of epistemological analysis in a didactic approach to the phenomenon of mathematics learning and teaching
	K. MILLETT	Teaching and making certain it counts
1996	C. HOYLES	Beyond the classroom: The curriculum as a key factor in students' approaches to proof
	D. HENDERSON	Alive mathematical reasoning

1997	R. BORASSI	What does it really mean to teach mathematics through inquiry?
	P. TAYLOR	The high school math curriculum
	T. KIEREN	Triple embodiment: Studies of mathematical understanding-in-interaction in my work and in the work of CMESG/GCEDM
1998	J. MASON	Structure of attention in teaching mathematics
	K. HEINRICH	Communicating mathematics or mathematics storytelling
1999	J. BORWEIN	The impact of technology on the doing of mathematics
	W. WHITELEY	The decline and rise of geometry in 20th century North America
	W. LANGFORD	Industrial mathematics for the 21st century
	J. ADLER	Learning to understand mathematics teacher development and change: Researching resource availability and use in the context of formalised INSET in South Africa
	B. BARTON	An archaeology of mathematical concepts: Sifting languages for mathematical meanings
2000	G. LABELLE	Manipulating combinatorial structures
	M. B. BUSSI	The theoretical dimension of mathematics: A challenge for didacticians
2001	O. SKOVSMOSE	Mathematics in action: A challenge for social theorising
	C. ROUSSEAU	Mathematics, a living discipline within science and technology
2002	D. BALL & H. BASS	Toward a practice-based theory of mathematical knowledge for teaching
	J. BORWEIN	The experimental mathematician: The pleasure of discovery and the role of proof
2003	T. ARCHIBALD	Using history of mathematics in the classroom: Prospects and problems
	A. SIERPINSKA	Research in mathematics education through a keyhole
2004	C. MARGOLINAS	La situation du professeur et les connaissances en jeu au cours de l’activité mathématique en classe
	N. BOULEAU	La personnalité d’Evariste Galois: le contexte psychologique d’un goût prononcé pour les mathématique abstraites
2005	S. LERMAN	Learning as developing identity in the mathematics classroom
	J. TAYLOR	Soap bubbles and crystals
2006	B. JAWORSKI	Developmental research in mathematics teaching and learning: Developing learning communities based on inquiry and design
	E. DOOLITTLE	Mathematics as medicine

2007	R. NÚÑEZ	Understanding abstraction in mathematics education: Meaning, language, gesture, and the human brain
	T. C. STEVENS	Mathematics departments, new faculty, and the future of collegiate mathematics
2008	A. DJEBBAR	Art, culture et mathématiques en pays d'Islam (IX^{e}-XV^{e} s.)
	A. WATSON	Adolescent learning and secondary mathematics
2009	M. BORBA	Humans-with-media and the production of mathematical knowledge in online environments
	G. de VRIES	Mathematical biology: A case study in interdisciplinarity
2010	W. BYERS	Ambiguity and mathematical thinking
	M. CIVIL	Learning from and with parents: Resources for equity in mathematics education
	B. HODGSON	Collaboration et échanges internationaux en éduction mathématique dans le cadre de la CIEM : regards selon une perspective canadienne / ICMI as a space for international collaboration and exchange in mathematics education: Some views from a Canadian perspective
	S. DAWSON	My journey across, through, over, and around academia: "...a path laid while walking..."
2011	C. K. PALMER	Pattern composition: Beyond the basics
	P. TSAMIR & D. TIROSH	The Pair-Dialogue approach in mathematics teacher education
2012	P. GERDES	Old and new mathematical ideas from Africa: Challenges for reflection
	M. WALSHAW	Towards an understanding of ethical practical action in mathematics education: Insights from contemporary inquiries
	W. HIGGINSON	Cooda, wooda, didda, shooda: Time series reflections on CMESG/GCEDM
2013	R. LEIKIN	On the relationships between mathematical creativity, excellence and giftedness
	B. RALPH	Are we teaching Roman numerals in a digital age?
	E. MULLER	Through a CMESG looking glass
2014	D. HEWITT	The economic use of time and effort in the teaching and learning of mathematics
	N. NIGAM	Mathematics in industry, mathematics in the classroom: Analogy and metaphor
2015	É. RODITI	Diversité, variabilité et convergence des pratiques enseignantes / Diversity, variability, and commonalities among teaching practices
	D. HUGHES HALLET	Connections: Mathematical, interdisciplinary, electronic, and personal

2016	B. R. HODGSON	Apport des mathématiciens à la formation des enseignants du primaire : regards sur le « modèle Laval »
	C. KIERAN	Task design in mathematics education: Frameworks and exemplars
	E. MULLER	A third pillar of scientific inquiry of complex systems—Some implications for mathematics education in Canada
	P. TAYLOR	Structure—An allegory
2017	Y. SAINT-AUBIN	The most unglamorous job of all: Writing exercises
	A. SELDEN	40+ years of teaching and thinking about university mathematics students, proofs, and proving: An abbreviated academic memoir
2018	D. VIOLETTE	Et si on enseignait la passion?
	M. GOOS	Making connections across disciplinary boundaries in preservice mathematics teacher education

Appendix C / Annexe C

PROCEEDINGS OF ANNUAL MEETINGS / ACTES DES RENCONTRES ANNUELLES

Past proceedings of CMESG/GCEDM annual meetings have been deposited in the ERIC documentation system with call numbers as follows:

Proceedings of the 1980 Annual Meeting . . . . ED 204120

Proceedings of the 1981 Annual Meeting . . . . ED 234988

Proceedings of the 1982 Annual Meeting . . . . ED 234989

Proceedings of the 1983 Annual Meeting . . . . ED 243653

Proceedings of the 1984 Annual Meeting . . . . ED 257640

Proceedings of the 1985 Annual Meeting . . . . ED 277573

Proceedings of the 1986 Annual Meeting . . . . ED 297966

Proceedings of the 1987 Annual Meeting . . . . ED 295842

Proceedings of the 1988 Annual Meeting . . . . ED 306259

Proceedings of the 1989 Annual Meeting . . . . ED 319606

Proceedings of the 1990 Annual Meeting . . . . ED 344746

Proceedings of the 1991 Annual Meeting . . . . ED 350161

Proceedings of the 1993 Annual Meeting . . . . ED 407243

Proceedings of the 1994 Annual Meeting . . . . ED 407242

NOTE

There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year.