CANADIAN MATHEMATICS EDUCATION STUDY GROUP

GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

PROCEEDINGS / ACTES
2021 ANNUAL MEETING /
RENCONTRE ANNUELLE 2021

Virtual Meeting
June 11 – 13, 2021

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PROCEEDINGS OF THE 2021 ANNUAL MEETING OF THE CANADIAN MATHEMATICS EDUCATION STUDY GROUP / ACTES DE LA RENCONTRE ANNUELLE 2021 DU GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES MATHÉMATIQUES

44th Annual Meeting
Virtual Meeting
June 11 – 13, 2021

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INTRODUCTION

Peter Liljedahl – President, CMESG/GCEDM
Simon Fraser University

With COVID-19 continuing to make meeting face-to-face impossible, the CMESG/GCEDM executive decided that, for the first time, the CMESG/GCEDM meeting would held virtually. When we, the executive, announced this, we were not at all sure how it would turn out. So much of what makes CMESG/GCEDM such a great meeting is that it is a meeting—a time and a place for people interested in mathematics education to come together, to discuss, and to think, and to learn. And although an online meeting can never replace that in-person experience, the 2021 virtual meeting was, by all accounts a great success.

By necessity, the program had to be much compressed with no topic sessions and no gallery walk. What we kept, however, is central to what is at the core of the CMESG/GCEDM meeting is. We had an enlightening plenary by Sarah Mayes-Tang who presented on powerful and emotional moments in teaching at the undergraduate level. Later in the program, we were able to meet in small groups to discuss the plenary before engaging in our customary Q&A with the presenter. We also had five exceptionally well-organized working groups that met for a total of six hours over three days. And because of the lack of a meeting in 2020, we had many new PhD presentations. For these, we used a different format than we had in the past. Each new PhD submitted a video of their presentation. These were released to the meeting participants several days in advance with the expectation that they be viewed ahead of time. The new PhD time slot was then used for discussion about the presentation and the research. This was hugely successful and something we will carry into the future. Finally, there were two well attended social events—a Gathertown event and an Eat and Drink Local event. Although the sum of this program could never replace the depth of engagement that comes with an in-person meeting, it still allowed for us all to connect scientifically, collegially, and socially.

This virtual meeting could not have happened without the hard work of the executive—Manon LeBlanc, Sara Dufour, Patrick Reynolds, Lisa Lunney Borden, Bernardo Galvão-Sousa, and myself. Although we contributed to this effort, we were led by Lisa Lunney Borden who now has the honour of being the only person in the history of CMESG/GCEDM to have hosted two consecutive CMESG/GCEDM meetings (2019 and 2021). On behalf of the CMESG/GCEDM membership I would like to thank the contributors to this proceeding as well as the proceeding editor, Jennifer Holm, for their dedication in creating a written record of our virtual meeting. As you read these proceedings, I hope that you will be able to relive some of the stimulating conversations that takes place when over so many dedicated people rise above the challenges of COVID-19 to bring us together as a community.

Horaire

Lundi **7 juin**	**Jeudi** **10 juin**	**Vendredi** **11 juin**	**Samedi** **12 juin**	**Dimanche** **13 juin**
Panel Session conjointe avec la SMC lundi, 7 juin 14h00 – 16h00	13h00 – 15h00 **FLM Q&R** (fanciennement connu sous le nom Amis de FLM)	13h00 – 15h00 **Plénière** **Présentation des groupes de travail** **Introduction**	13h00 – 15h00 **Groupes de travail**	13h00 – 15h00 **Groupes de travail**
	15h00 – 16h00 Pause café			15h00 – 16h00 **Séances ad hoc**
		16h00 – 18h00 **Groupes de travail**	16h00 – 18h00 **Nouvelles docteures**	16h00 – 18h00 **Petits groupes – Discussion de la plénière** **Q&R avec la conférencière** **Rapports des groupes de travail**
		Activité sociale : Manger et boire local	**Activité sociale : Gather Town**	**AGA**

Schedule

Monday **June 7**	**Thursday** **June 10**	**Friday** **June 11**	**Saturday** **June 12**	**Sunday** **June 13**
Panel Joint session with CMS Monday, June 7 2:00 – 4:00	1:00 – 3:00 **FLM Q&A** (formerly friends of FLM)	1:00 – 3:00 **Plenary** **Working Group presentations** **Introduction**	1:00 – 3:00 **Working Groups**	1:00 – 3:00 **Working Groups**
	3:00 – 4:00 Break			3:00 – 4:00 **ad hoc sessions**
		4:00 – 6:00 **Working Groups**	4:00 – 6:00 **New PhDs**	4:00 – 6:00 **Small groups – Plenary discussions** **Plenary Q&A** **Working Group reports**
		Social activity: Eat and drink local	**Social activity: Gather Town**	**AGM**

Plenary Lectures

Conférences plénières

TEACHING ON EMPTY: COVID-19 PANDEMIC AND ITS IMPACT ON CANADIAN UNIVERSITY MATH INSTRUCTORS' COMPASSION FATIGUE

Sarah Mayes-Tang
University of Toronto

We examine the contributors to teacher resiliency and Compassion Fatigue among Canadian math instructors working in higher education during the time of the COVID-19 Pandemic. Using Bobek's (2002) model for Teacher Resiliency, we present several factors that will impact the Teaching Resiliency of these instructors, paying particular attention to the time (the COVID-19 pandemic) and place (Canadian Universities) they are situated. We find that Canadian math instructors face challenges unique to their environment that will tend to decrease their Teaching Resiliency and increase their risk of Compassion Fatigue. The paper concludes by providing two suggestions for the CMESG or other national math organizations, tailored to this time and place: focus on hospitality and work on archiving pandemic stories.

ABSTRACT FOR SUMMER TALK

TEACHING ON EMPTY: TRAUMA, ACHIEVEMENT, AND WHAT'S NEXT IN OUR MATH EDUCATION COMMUNITY

As math educators we are exhausted. The pandemic has demanded so much of us, from adapting to teaching online to virtually welcoming students into our homes to bearing the weight of students' own pandemic sufferings. With all of our energy drained, where do we go from here? In this talk we will take some time to recognize our exhaustion and the work that we have done in the pandemic that has led us there. Then we will talk about how we can leverage the work that we have done over the past year going forward. I will discuss the questions: What is the role of individual math educators? What is the role of the Canadian math community as a whole?

At the time of the 2021 CMESG meeting, the COVID-19 pandemic had completely changed the Canadian math education community's working environment and life for over one year. I was honoured to receive the invitation to speak to CMESG, an organization that I admire. The most natural choice would be to speak about something that I knew well, closer to course design or students' knowledge.

But I was situated in a specific time and place. Usually, speakers ask us for more in some way, whether it is to learn more about a subject, do more in our classrooms, or simply reflect more about our experiences. I could not imagine telling any teacher to do more or to think about things differently in the summer of 2021. I knew I need to speak both to those who remained relatively untouched by the pandemic *and* to those who were deeply impacted. My singular goal was to convey that this was not a time for *more*. Topics that acknowledged the pain of the pandemic—like trauma, community, and healing—were only ones that I could imagine highlighting at CMESG. So, I nervously stepped into unfamiliar waters and dove into topics that were at the periphery of my expertise.

The scope of my talk—touching on student trauma, faculty trauma, community, moving forward, and future themes in education—is far too large to capture in a scholarly paper, so I will focus on faculty trauma, the theme that inspired the need for the talk. I will describe a web of three interrelated concepts—Trauma, Compassion Fatigue, and Faculty

Resilience—as they relate to the experience of Canadian math educators in higher education. The conceptual framework of teacher resiliency comes from Bobek (2002).

To develop the discussion and themes, I used a concept mapping process. I created a concept map of Canadian math instructors' experiences during the pandemic and how they were related to factors contributing to teacher resiliency. I then summarized some of the themes in the paper and in Tables 1, 2, and 3. The concept maps that I used (with some connections and elements removed for clarity) are in the Appendix (Figures 4 and 5). As I built the concept maps, I chose elements that emphasized *time*, *place*, and *subject area* because I wanted to capture the experiences of Canadian mathematics educators. Elements that speak to Canadian math educators are highlighted in pink/red and connections that were made due to the pandemic are given in green.

Before I begin the academic discussion, I share some of my personal experience as a professor experiencing symptoms of Compassion Fatigue and how this paper relates to my address to the CMESG. Next, we discuss the theories of Trauma, Compassion Fatigue, and Teacher Resiliency. The analysis in the paper unpacks the concept maps and discusses some of the notable patterns. The final section discusses the need for recording stories of Trauma and resilience during the pandemic and makes two suggestions for the CMESG: first, that it should be involved with recording stories from Canadian math educators and second, that it focus on purposefully developing hospitality amongst all of its community members and leaders.

BACKGROUND

MY EXPERIENCE: EMOTIONAL, SUPPORT, AND CURIOSITY DRAINING

I will share my own experience through the perspective of the different vessels or tanks that we carry as mathematics educators. The title of the talk—*Teaching on Empty*—was intended to remind the audience of these different tanks.

In mid-2021, my own 'emotional tank' was worn by supporting students going through the various losses of the pandemic—beginning with family job losses, home loss, the loss of friends, and the loss of an anticipated university experience. It was worn by direct and indirect accusations from staff at my institution that I was not doing enough to support these students. It was worn by seeing losses throughout my community and the businesses that I loved close. It was worn by the parts of my job that I enjoyed that were now gone (see Figure 1). But I knew that I was fortunate: my family and close friends had been largely safe from the greatest losses of life in the pandemic. That changed weeks before the CMESG when our community was shocked by the loss of a close colleague at the University of Toronto, Alfonso Gracia-Saz.

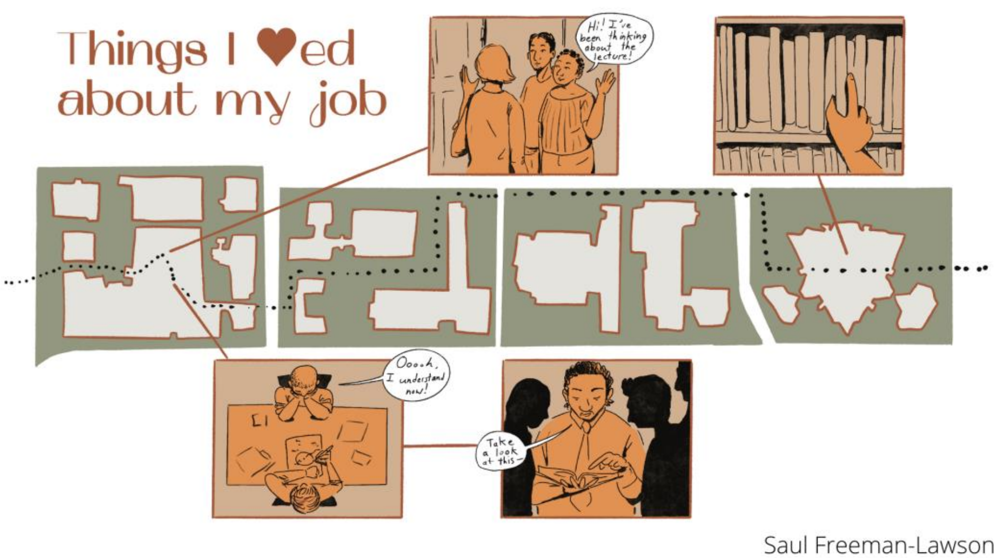

Figure 1. Illustration showing some everyday aspects of a typical professor's job that I—and many others—loved and were now absent or challenging during the pandemic. Image citation: Freeman-Lawson (2021b).

As educators we also have a 'support tank': how free are we to do our work? How supported do we feel in doing our work? Is that support vocal? Do we feel as though there are resources around that we have not yet drawn on? Is there promise of a future in our work? Do our students support our work? Colleagues? Superiors? During the pandemic educators were spurred to action—pushed to do more and more and more! But the level of support varied. With the loss of Alfonso, I also lost a major advocate in my department. Suddenly, the future did not look as certain. Once again, I knew that I was in a fortunate place compared with others…but it was like the floor dropped out from under me.

Our work as educators should be powered by 'curiosity tank'. We aim to fill the curiosity tanks of our students with mathematical questions and insights, and we function best when our curiosity tanks are full of curiosity about our students. Apathy and hopelessness are enemies of curiosity. For the first time in my career, I sometimes had fewer questions than I cared to investigate. Whereas each year for the past several years I read more research papers than the year before, this year I read less. Instead of relishing my research time, I dreaded it.

So, completely unexpectedly, this topic that I felt compelled to speak about became much more personal than I expected. In my summer CMESG talk, I advocated for a gentler approach to our work as math educators and math education scholars. Rather than taking advantage of the moment to 'do more', perhaps it could be an opportunity to do less. While I anticipated that I would be speaking about the empty tanks of others—other teachers, professors, and students—I was becoming the subject of my talk. By the end of 2021, my own tanks were fully depleted.

THEORY & IMPACT OF TRAUMA AND COVID-19

The word 'Trauma' is one that I will use throughout this paper and will usually be capitalized to indicate that its use with purpose according. On their website, CAMH (2022) writes that "Trauma is the lasting emotional response that often results from living through a distressing event." According to this definition and other widely accepted definitions there is no such thing as a *traumatic event* because the same experience that is not traumatic for one person can be traumatic for another. I am using this word and concept because it accurately describes the experience of the COVID-19 pandemic for many, and the frameworks that academics have developed for Trauma are on useful to describe the experiences of faculty during this pandemic.

Trauma has measurable impacts on the brain, particularly in three areas: the prefrontal cortex, the hippocampus, and the amygdala (Bremner, 2006). The formation and recall of memories is overwhelmed by the traumatic stress response, and executive functioning can be fatally hampered by the recollection of a traumatic memory (Northcutt, 2021). As the hippocampus struggles to differentiate between when a traumatic event occurred and the amygdala is wrapped up in a fight or flight response, the prefrontal cortex has all but abandoned the rational thinking. These impacts and others can make learning and succeeding in an university environment so much more difficult for someone who has experienced Trauma.

When I asked on Twitter about Trauma and the COVID-19 pandemic in academia, the result was unanimous: look at the work of the Twitter handle @IrningSanctuary, belonging to neuroscientist Mays Imad. Imad has written several articles and appeared on well-known podcasts for professors and administrators in higher education, discussing the impact of Trauma on both faculty and on students. Her main message is the COVID-19 pandemic is experienced as a deep Trauma for many in our universities (see, for example, Seven recommendations for helping students…Imad shares practical strategies for working with students and colleagues, and for personally handling the trauma as professors (Imad, 2020)). She quotes Rumi:

> *Keep walking, though there's no place to get to.*
> *Don't try to see through the distances.*
> *That's not for human beings.* (Rumi, n.d.)

Imad is not the only one who writes about COVID-19 as a trauma. In fact, most of the literature that I reviewed for this project framed the pandemic and its continued lockdowns as an event that could be experienced as a traumatic event for some people. A lot of the work that has been done on the pandemic and Trauma focuses on K–12 education, while the higher education literature on the pandemic tends to focus on emergency online teaching.

'Trauma-informed teaching' gained prominence throughout the Pandemic. It may include strategies such as being thoughtful about relationship building, promoting predictability and consistency, teaching specific cognitive strategies to help calm anxious minds, giving supportive feedback, recognizing students' individual areas of strength, and limiting exclusionary practices (Minahan, 2019). As Darryl W. Stephens (2020) writes in the context of religious education, referring to trauma-informed teaching "This 'risky teaching' practice, …is no longer optional; the question is how to engage it effectively (Harrison et al., 2020)" (p. 2).

The fact that we—as Canadian math educators—experienced the COVID-19 pandemic primarily in terms of loss was evident in the CMESG annual conference in June 2021. At one point, I asked attendees to share one challenge they faced as a math educator during the previous year on a Padlet. Reviewing the answers now, I can see that many members shared what they were missing: seeing their students, connecting with their colleagues, the energy of a classroom, motivation to teach, and physical spaces on campus. Many others shared feelings of inadequacy: feeling as though they could not connect with their students (especially with their cameras turned off), feeling helpless to help their students through mental health crises, feeling as though their research was pointless, and feeling increasing perfectionism. In all, very few of the responses could be viewed as logistical concerns or frustration.

BURNOUT AND COMPASSION FATIGUE

'Burnout' is a familiar term in North American culture today and almost as many definitions for burnout exist as the authors who have written about it. Herberg Fredenberger's original definition (Freudenberger, 1975) included three symptoms:

1. emotional exhaustion (getting stuck in an emotion, or constantly being exposed to an emotion);
2. depersonalization (not caring any more about your job); and
3. decreased feelings of accomplishment (not feeling as though you are moving forward on anything).

According to Nagoski and Nagoski (2020), 20-30% of professors had moderate to high levels of burnout before the pandemic, and women experience burnout differently than men.

One of the insidious facts about a global crisis that impacts everyone is that instructors not only experience their own trauma but are also impacted by the increased trauma of their students. Compassion fatigue is sometimes described as a special type of burnout that happens to people working in the 'helping professions', but Urmy (2020) corrects: "burnout causes emotional exhaustion; emotional exhaustion is one of the causes of compassion fatigue" (p. 14).

Teachers are in a constant cycle of going to a stressful work environment where they experience high-need students who require teachers to tap into their own emotional reserves. Educators get too little rest each day before they are thrown into the same atmosphere the next day, leading to a cycle of symptoms. According to Lombardo and Eyre (2011) quoted in Urmy (Urmy, 2020) signs of compassion fatigue can include everything from muscle tension to heart palpitations (physical symptoms), depression to poor concentration or judgment (emotional symptoms), and avoidance or dread of certain work-related situations (work-related symptoms).

The symptoms of compassion fatigue can be devastating to the person experiencing them and completely rob them of the joy they usually find in work, ultimately leading them to leave teaching professions (Urmy, 2020).

CONCEPTUAL FRAMEWORK: TEACHING RESILIENCE

Masten et al. (1990), quoted in Pfleging (2021), defines resilience as "the process of, capacity for, or outcome of successful adaptation despite challenging or threatening circumstances" (p. 17). The Faculty Resilience conceptual framework that I will use in this paper was developed by Bobek while working with K–12 teachers (Bobek, 2002). It outlines five key elements that contribute to an individual faculty member's resilience. In brief, they are

1. Productive professional relationships: To what extent does the faculty member have supportive relationships around all aspects of their job, including ongoing and professional teaching and research, if applicable?
2. Competence and skills: Is the faculty member equipped with all the skills that they need in their career? Are they able to perform them in a coherent way that is aligned with their own values?
3. Ownership and advancement: Are there other possibilities for the faculty member, if they want to do so? Does the faculty member feel as though their work matters, both outside and inside of the institution that they work for?
4. Sense of accomplishment: Are there signsthat the faculty member has done a 'good job'? Is the evidence external or internal?

5. Sense of humour: Do others in the institution share their sense of humour? A sense of humour among colleagues and students helps a faculty member get through tough days and lightens their load.

RESULTS

OVERALL RELATIONSHIPS BETWEEN COMPASSION FATIGUE, TRAUMA, AND RESILIENCE

The big picture concept map in Figure 3 of the Appendix (also see Sharifian, 2017) depicts relationships between the main concepts discussed in this paper.

Situating these factors in our time, place, and context

To examine how math instructors at Canadian universities have been impacted by the COVID-19 pandemic I will use Bobek's (2002) model of Faculty Resilience. Using my own concept map to guide the narrative, I will discuss some of the multi-directional relationships between: the factors that build faculty resilience; the experience of teaching within mathematics departments at Canadian universities; and, the stressors and impacts of the COVID-19 pandemic. See Figures 4 and 5 in the Appendix for this more detailed concept map.

THEMES IN THE CONCEPT MAPS

Unique experiences for math instructors

Recently I was with my first-year seminar in the University Archives, and one of my students was shocked to see that the same calculus classes were offered in the 1970s and that even the course descriptions appeared to be the same or similar. The nature of mathematical knowledge is a major reason why the math curriculum in many universities has remained relatively stagnant over the past fifty years. Mathematics is seen "as part of the very structure of the world, containing truths which are valid forever, from the beginning of time" (Davis & Hersh, 2013, p. 1). Of course, this belies the fact that mathematical knowledge *can* be separated from the fact that mathematics is a human activity and humans change. To take a more skeptical view, as bell hooks writes: "Complicity often happens because professors and students alike are afraid to challenge, because that would mean more work. Engaging pedagogy is physically exhausting." (hooks, 1994, p. 160). Viewing course material as stagnant and non-negotiable will greatly decrease ownership and advancement, one of the five factors contributing to resilience in teaching.

During the COVID-19 pandemic there was a drive to rejuvenate courses across colleges and universities, driven by the online delivery format, the resulting necessity of new assessment formats, and the effectiveness of messaging about redesigning courses. Mathematicians—many for the first time—began to imagine mathematics courses that looked different than the ones that they themselves took as students. Can this initiative continue and evolve courses that serve students better and build better mathematicians? Popa (2020) writes about course changes due to the COVID-19 pandemic like this: "The audacity of this much-needed action, which is set against the foil of real suffering, can be felt most immediately in its ambition: Can we transform this crisis into a renewing wave? Will it help bring about a better education system, a better society, and a better world?" (p. 3). When individuals decide to make changes in courses it will increase ownership and advancement in their teaching, one of the contributors to resilience in teaching. It is one of the ways that we see a potential positive contribution to resilience in teaching due to the COVID-19 pandemic (also see Hughes et al., 2020).

Another way in which the nature of mathematics as a field of study contributes to resilience in math educators is its depth. While Pedagogical Content Knowledge (PCK; see Van Driel and Berry, 2010) is present in every field of study, math teaching at every level is distinguished by the fact that you can always keep digging deeper: from the operation of subtraction in arithmetic to the Fundamental Theorem of Calculus to the Sylow Theorems possess a literally endless depth. This means that PCK of instructors can keep developing throughout our lives as we look not only *further* in mathematics but also *deeper*. As an indicator of resilience in teaching, Competence can continue to develop: when we are motivated instructors can learn new facets of their subject, new ways of seeing their students' understanding, and new ways of representing/translating mathematical concepts. The presence of compassion fatigue—all too present during the COVID-19 pandemic—cut away at the desire to develop PCK or other teaching-related knowledge bases.

Mathematics teaching is arguably more embodied than many other subjects, meaning that emergency remote teaching demanded a new kind of PCK from educators. How do you portray a dynamic phenomenon when you cannot move across the classroom as you might have before? How does a Teaching Assistant (lacking a tablet) show the development problem solution when they cannot easily write mathematics notation in real time? Competence decreased due to emergency remote teaching the pandemic. Taqiyuddin et al. (2022) specifically identifies the use of tablets as an obstacle to the transition to emergency online teaching for Indonesian math tutors.

Students leave high school with firmly held views of mathematics as a subject, both positive and negative. They might see math as a subject that they will not go near, just need to pass, or one that they love because they just need to memorize a collection of rules and deploy those rules at the correct time (Geisler & Rolka, 2021). There is also evidence that math anxiety is independent from test anxiety (O'Leary et al., 2017). First-year instructors (of which there are many) wage an up-hill battle against students' opinions, especially when we aim to show them a different view of our subject—one that is beautiful and better but might, at first glance, disappoint because it is more difficult to grapple with. There is emerging evidence that students' static view of math as a subject that has right or wrong answers only gets more firm during university and their dynamic view of calculus weakens. Students' strongly held views of math can rub away at our Sense of Accomplishment and might be even more diminished during emergency remote teaching when we do not see much of the positive development.

Related to students' views of math is their mathematical preparation when entering university: do they have the competencies to be able to handle our first-year courses? This has always been an issue of concern, but it is one that has been exacerbated by the COVID-19 pandemic. Projections of the precise impact of learning loss due to school closures vary, but most agree that the learning loss is equal to a time that is greater to the time schools were closed (Engzell et al., 2021). Without minimizing the devastating impacts of this on students, not knowing what preparation students might have has increased Uncertainty in faculty (one of the top stressors of the pandemic) and also decreased their Competence, Ownership, and Sense of Accomplishment, three contributors to Resilience in Teaching.

	Professional Relationships.	Competence & Skills	Ownership & Advancement	Sense of Accomplishment	Sense of Humour
Math knowledge as stable			–		
Re-thinking math courses for online delivery			+		
Depth		+			
Embodied nature of math		–			
Students' strong view of math				–	
Students' loss of learning		–	–	–	
Women's under-representation	–	–	–	–	–

Note 1. A + indicates that the characteristic in the left-hand column impacts the factor in a positive way (to increase resilience) and a – indicates that the characteristic in the left-hand column impacts the& Adv = factor in a negative way (to decrease resilience). Highlighted cells indicate a factor specific to the COVID-19 pandemic and its impacts.

Table 1. Factors contributing to teaching resilience of math instructors: the factors that each characteristic of mathematics teaching impacts.

Men outnumber women more than two-to-one in most mathematics departments in Canada when you consider faculty and graduate students (Government of Canada, 2021b; Corporate Planning and Policy Division, Natural Sciences and Engineering Research Council of Canada, 2017). The level of this underrepresentation is just as important as the fact of the underrepresentation: for every unit increase in underrepresentation of women in an academic department, the odds that a woman experiences gender harassment increase 1.2 times (Johnson et al., 2018). Women in these environments are also more likely to have their expertise questioned, and to be passed over for professional advancement (Chaney et al., 2018).

All these facts together make women in math less likely to develop Productive Professional Relationships with the people in their department, to be seen to have the Competence and Skills to develop, to feel as though they have Ownership in their Advancement through the Department, and to be able to engage in friendly relationships in the

Department, enjoying in a Sense of Humour. Further, their own accomplishment may be questioned: women hired in STEM are often thought to be hired 'just because they are a woman' (Johnson et al., 2016). In summary, every single factor of their Resilience in Teaching is impacted by our identity as a woman in mathematics. Further, we now know unquestionably that the COVID-19 pandemic has had outsized impacts on women, particularly women with intersectional identities from marginalized groups, who experienced more Trauma and other negative consequences from the pandemic (Government of Canada, 2021a; Kaplan, 2020).

<u>Canadian issues and challenges</u>

As the concept map reveals, there are pre-pandemic and pandemic factors that impact the *instructional* resilience of Canadian math instructors' differently than their counterparts from other countries. Here I do not claim that these factors impact every instructor at every institution or impact effected instructors in the same way, but rather that there are elements of our institutional structures that make it more likely that a factor will apply to a given Canadian instructor than our international or United States counterparts.

One of the main factors that differentiates Canadian universities from United States higher education is the relative homogeneity of its publicly funded institutions. Within this, there are two factors that strongly influence class size: the overall large size of the institutions and internationalization of the student body (see, for example, Usher, 2021b). While official figures for class size are difficult to come by, math courses at Canadian universities are large compared to their United States counterparts, even at the upper-year levels. Sometimes, classes are split into separate sections, resulting in 'teaching teams' of various levels of organization. Teaching as part of a team—whether formal or informal, as a new instructor or as a leader—develops a unique kind Productive Professional Relationships, a part of resilience in teaching. They also require different Competencies and Skills than teaching on your own.

A rising trend in Canada is the 'teaching professor', a professor on the level of a tenured professor department that focuses their job on teaching and is usually located within the mathematics department. The lack of smaller liberal arts institutions in Canada mean that positions recognized to be professionally equal to research-track professors were not usually present in Canada. Based on my own experience in one of these positions, I believe that the jury is still out whether the social or cultural status of teaching professors is truly equal to that of our research-focused counterparts. However, the very existence of these positions (in their best light) elevates teaching and can contribute to the Ownership and Advancement quality of everyone.

Another vital consideration in Canada's mathematics instructorships is the widespread and growing use of Sessional Faculty. Stakeholders do not agree on the best ways to describe the growth in the employment and teaching responsibilities of faculty and graduate student instructors within Canada before (Field et al., 2014; Foster & Birdsell Bauer, 2018; Usher, 2021a). What is clear is that increasing student enrolment and budget pressures result in the greater use of Sessional Faculty. Being a Sessional Faculty member and seeing your position as temporary will decrease an instructors' view of their Ownership and Advancement.

COVID-19 pandemic in Canada

During the COVID-19 pandemic, emergency remote teaching introduced unique challenges in working with remote teams. For example, in 2020–21 I led a team of instructors, many of whom I never met in person. We faced a major crisis with one instructor who never attended our virtual meetings and whom we were simply never able to connect. Research suggests that these challenges are typical—the very culture of a team is challenged with the shift to online work, even when team members have worked together (Comella-Dorda et al., 2020). Once again, we see a new set of Competencies and Skills being demanded from the typical Canadian math instructor during the COVID-19 pandemic that may not have been as prevalent in the United States or among instructors teaching other subjects.

The experience of the COVID-19 pandemic also played out differently in Canada than it did in the United States. Some of the main differences in Canada included: greater public health measures and lockdowns; fewer overall deaths; less contention early in the pandemic; and later vaccine access. The loss of life was more than 10 times greater in the United States than in Canada (*Covid-19 world map: Cases, deaths and global trends*, n.d.).

We also experienced different national mournings. While the murder of George Floyd took place in the United States, the unmarked graves of over 1000 Indigenous children were found at former residential schools in Canada in the spring and summer of 2021. This additional trauma impacted us differently (see the Trauma section below).

	Professional Relationships	Competence & Skills	Ownership & Advancement	Sense of Accomplishment	Sense of Humour
Team Teaching	different	different	+		
Growth in Teaching Professors			+		
Growth in Sessional Faculty		+	–		
Shift of teams to online		–			

Table 2. Factors contributing to teaching resilience of math instructors in Canada: the factors that are unique to being in Canada versus United States institutions.

The Role of Communities

The COVID-19 pandemic shifted the roles of community. Local communities play a larger role in teaching than in research. 'Hallway' conversations (in corridors, around conference tables, while sharing an elevator, or going up the stairs to a building), play a significant role in how we see our teaching and how we develop as educators. Further, local factors, such as the students we teach, institutional policies, and campus resources make talking to colleagues at the same university about teaching easier or more natural than reaching out to colleagues from across the country. When the COVID-19 pandemic forced us to work at home instead of online, these happen-stance encounters vanished, and we were distanced from the local communities that grounded our teaching. As de Boer (2021) writes: "by decreeing everyone to work from home, the university is not the meeting place between people and ideas that it should be, despite all good intentions" (p. 106). Within the framework for resilience in teaching, holding Productive Professional Relationships became significantly more difficult than before.

At the same time, the pandemic meant that were tethered to Zoom (https://zoom.us/) and developing relationships online became much more comfortable than it had before. To remember how far we have come, I recall asking a local administrator if we could get a department license for Zoom in February 2020. They quickly dismissed the idea—after all, they asked, how many people in the department *actually* want to collaborate with people at other Universities over video instead of through email or by phone?

The COVID-19 pandemic provided opportunities for national higher education math communities to develop. Perhaps most notably, the First Year Math and Stats in Canada (FYMSiC, n.d.) initiative that began in late 2017 found a home during the pandemic. It connected educators from across the country who were facing similar challenges first to complete the 2019-20 academic year and then to re-structuring the classes for Fall 2020 online delivery. We might be tempted to conclude that the location of teaching communities simply shifted from a local to national core but this is likely not true for the vast majority of educators (especially sessional instructors) who simply saw their teaching community vanish or potential communities never form.

Even when some campuses, such as my own St. George campus at the University of Toronto, were back in person in the Fall of 2021, a number of public health restrictions, including limits on our office visitors, relocations of campus services, and the closure or limitation of restaurants continued to create distance between us and made it difficult to see others in person and maintain relationships. Face masks, which make telling emotion, gender, age, and identity difficult (Fitousi, 2021), provided a particular challenge inside of the classroom. We could not see if students smiled in response to our jokes and we could not trust visual cues for understanding, decreasing our Sense of Accomplishment and of Humour. The public health measures, therefore, had a negative impact on resilience for teaching.

Some members of our teaching networks were lost due to COVID. My colleague Alfonso Gracia-Saz died unexpectedly. I lost a close Productive Professional Relationship. He was a source of near-constant professional support that provided a Sense of Accomplishment, and also a future Undergraduate Chair that provided an assurance of future Professional Advancement. My personal grief was, in many ways, the 'straw that broke the camel's back' before compassion fatigue overwhelmed me. In Imad's words, it felt as though I could not "keep walking" in my work for a time (Imad, 2020).

The overall interplay between broadening communities due to new ways of communicating and contracting communities due to staying at home can both be held true at the same time. While some of the examples are specific to us as mathematicians, we have learned that there is nothing like being in person at the same time—and this has little to do with us as mathematicians, but simply providing common threads with the people around us.

	Professional Relationships	Competence & Skills	Ownership & Advancement	Sense of Accomplishment	Sense of Humour
Decrease in local community	–				
Growth in national communities	+ (for some)	+ (for some)			
Public health mandates	–			–	–
COVID pandemic deaths	–		– (for some)		

Table 3. Factors contributing to teaching resilience of math instructors: the factors that are related to changes in community during the COVID-19 pandemic.

CELEBRATING OUR SUCCESSES

Celebrating together is a component of building strong communities, and celebrations during times of crises require intentionality.

The concept map in Figures 4 and 5 notes several successes related to Teaching Resilience. The first two were mentioned above already. Emergency remote teaching showed many math instructors that the math courses that they teach do no need to look exactly like the ones that they took as students (Hughes et al., 2020). In a moment where many realized that they did not know anything about teaching in a new format, they were more open to listening to evidence-based teaching practices. We do not yet know how much of this will 'stick', but I cannot help but think that basic changes like finding out what students think and situating learning goals ahead of assessments will fundamentally change the way that math is taught in many universities.

Second, as discussed above, we have now established broader national mathematics communities, both large and small. Emergency remote teaching got us used to talking to others over distances, but it took work from mathematicians to establish and sustain these communities. Sustaining these communities is, in itself, another success!

Third, conferences—including the CMESG—successfully moved to online formats during the pandemic (Blanco & Wit, 2020). While many people wish for an in-person format to come back, conference and meeting formats have developed to be incredibly innovative in a short amount of time. The long-term impact of this move will be to create more equitable conference opportunities across the mathematics community, making conferences available to people who might not be able to travel due to affordability, disability, parental status, or location-based responsibilities. This will make our community stronger over the long-term.

Fourth, we were collectively responsible for delivering mathematics courses to tens of thousands of students who were in an emergency. We successfully modified our own pedagogies to deliver these courses online. We helped to train teams of Teaching Assistants and instructors in online teaching.

Fifth, we should celebrate the individual successes that each of us has had during the pandemic with gentleness towards ourselves (See Figure 2). Success could be big or small: getting through a degree program, completing a research project, writing a book, developing a new course, starting a family, or making a difference in a single student's life.

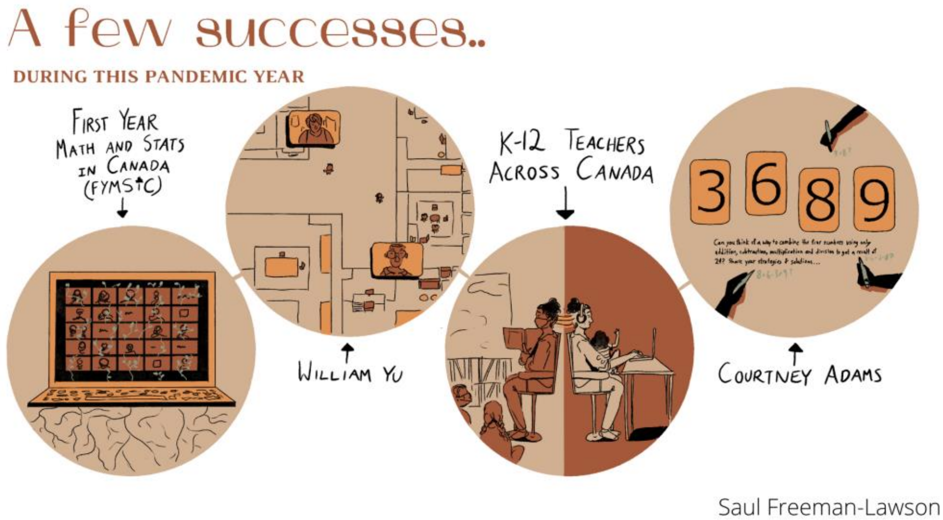

Figure 2. Illustration showing some successes of the 2020-21 academic year by the Canadian math education community. Image citation: Freeman-Lawson (2021a).

Celebrating with our math education communities is a vital part of acknowledging the pain, loss, and trauma of the pandemic. It says that, despite it all we have survived together.

CONCLUSION AND RECOMMENDATION

Examining resilience in teaching from a Canadian math education perspective using Bobek's five-factor model as a framework, I identified four main themes: math context, national context, the role of community, and success. It is unsurprising that the resilience of Canadian post-secondary math educators is greatly impacted by the COVID-19 pandemic. What might be surprising, however, is the extent to which our national context and discipline impact the mechanisms by which these operate.

LOOKING TOWARDS THE FUTURE AND THE CMESG

At the end of 2020, deep into the pandemic but before widespread vaccine availability, Mays Imad (2020) advised faculty to just "keep walking" (title). Throughout this piece she tells us to remind ourselves why we do what we do as educators and to tell ourselves that we matter, tell a colleague "you matter", and tell our students that they matter.

This advice is as important now as it was then, but as we near and find ourselves more firmly into the recovery stage of the pandemic we need to make sure to continue purposeful 'meaning making' work. Walsh (2020) writes that to make meaning can be a family experience: "shared attempts to make sense of the loss, put it in perspective to make it more bearable, and, over time, integrate it into personal and relational life passage" (p. 905). This description can also apply to a community—like the mathematical community at large or the CMESG—and, when enacted, has the effect of increasing resilience for teaching and of normalizing compassion fatigue ("normal in an abnormal time" (Walsh, 2020, p. 905)).

What role could the CMESG play in this recovery? One possibility is that the CMESG could help facilitate the sharing and archiving of math educators' stories from the pandemic. Telling stories can help both individuals and the community as a whole make meaning at this point in the pandemic and beyond. There is also a mountain of information learned during the pandemic that could be lost if we do not preserve it. It includes best practices for running online classes (from facilitating individual class sessions to administering multi-section courses), hybrid teaching strategies, and helping students deal with trauma and loss.

Second, the CMESG should take concrete actions to be the most hospitable community for mathematicians, math teachers, and mathematics educators in the country. Francis Su (2020) paints a vivid picture of hospitable math communities in *Mathematics for human flourishing*:

> *Hospitable math explorers will bend over backward to reassure newcomers that they are welcome at any stage of development.…They will publicly assign competence to others, by recognizing the things they've done well. Those who wield power in mathematical communities must remember that they have great responsibility in setting norms for how to welcome newcomers. Hospitable math explorers will strive to become excellent teachers of mathematics, able to reveal the joy of mathematics even to those who are new to the community.* (p. 192)

What might this look like in our current context? The 2003 CMESG Introduction describes working groups as follows:

> *working groups form the core of each CMESG meeting. Participants choose one of several possible topics and, for three days, become members of a community that meet three house every day to exchange ideas and knowledge and, through discussions that often continue beyond the allotted time, create fresh knowledge and insights. Throughout the three days, the group becomes much more than a sum of its parts – often in ways totally unexpected to its leaders.* (Dubiel, 2004, p. ix)

The CMESG's activities and structures are a great start to hospitality, but it is not enough: a hospitable community is composed of hospitable people. Conversely—and more importantly—it only takes one encounter for someone's experience to be poisoned.

Let us not be tempted to discuss all of the potential 'opportunities' of the pandemic without overarching acknowledgement of the Tragedy and Trauma. Whatever might come, it will not be 'worth it': the death, illness, permanent disability, burnout, compassion fatigue, mental health disease, loss from the profession, lost relationships, lost experiences, and so much more. Acknowledging where we are, let's keep walking *together* as Canadian math educators.

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APPENDIX

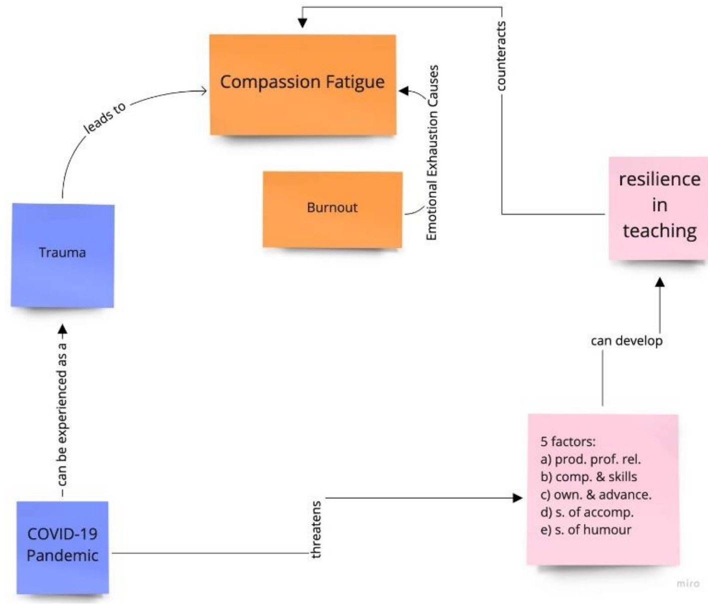

Figure 3. General Concept Map showing relationship between trauma, compassion fatigue, resilience in teaching, and the COVID-19 pandemic.

For Figures 4 and 5:
Note 1. Colours of added elements correspond to the five factors contributing to resilience in teaching, outlined in the pink box.
Note 2. Boxes outlined in pink-red are factors specific to Canadian math educators. Green arrows indicate contributors from the COVID-19 pandemic.

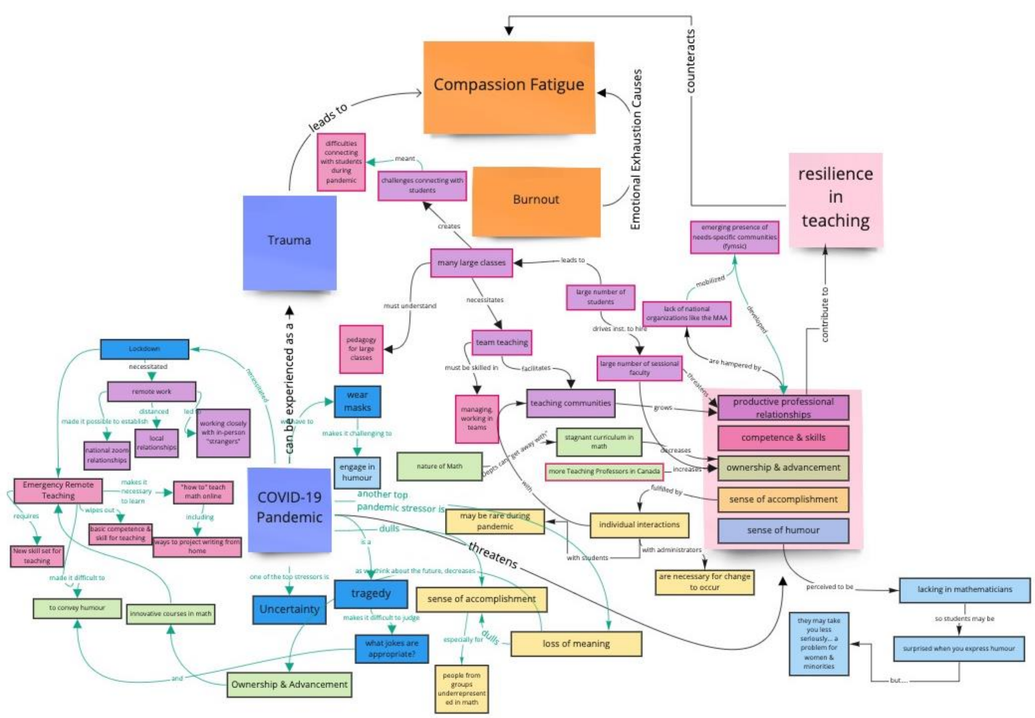

Figure 4. Overview (not readable) of concept map showing how concepts in Figure 3 apply to Canadian math teachers.

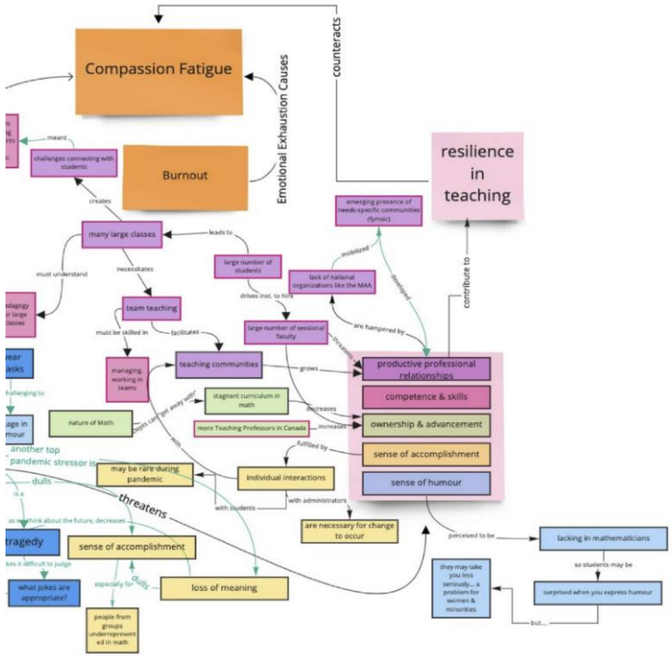

Figure 5a. Concept map showing how concepts in Figure 3 apply to Canadian math teachers (broken into two pages).

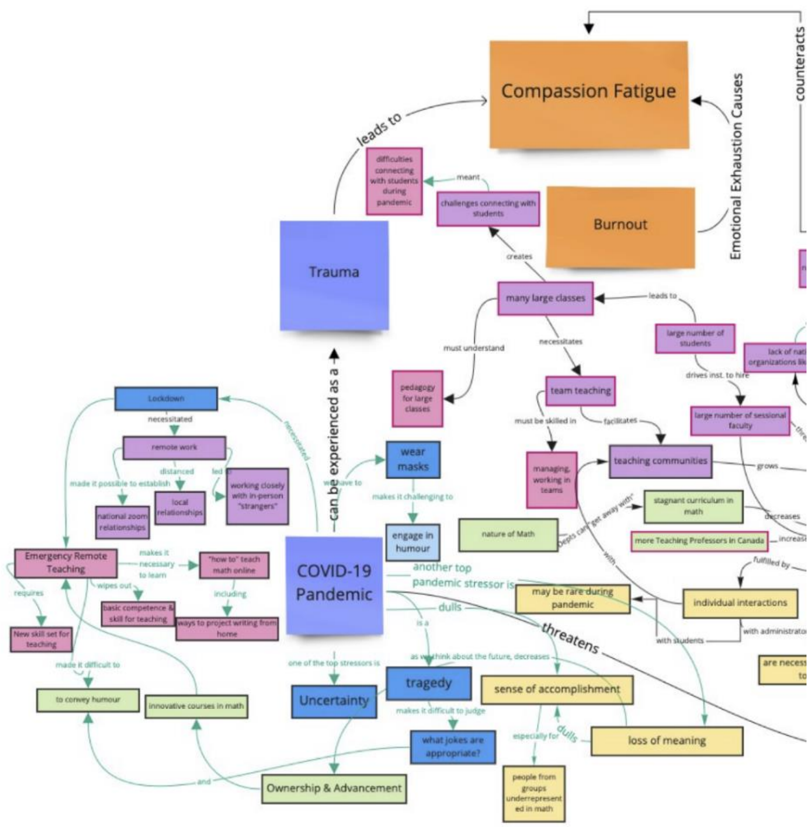

Figure 5b. Concept map showing how concepts in Figure 3 apply to Canadian math teachers (broken into two pages).

Working Groups

Groupes de travail

LEARNING THEORIES

THÉORIES (DE L') APPRENANT

Joyce Mgombelo, *Brock University*
David A Reid, *University of Agder*

PARTICIPANTS

Yasmine Abtahi	Susan Gerofsky	Cynthia Nicol
Ayman Aljarrah	Anjali Khirwakdar	Dionysia Pitsili Chatzi
Alayne Armstrong	Martha Koch	Robyn Ruttenberg-Rozen
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Lisa Lunney Borden	Olivia Lu	Evan Throop Robinson
Canan Gunes	Dominic Manuel	Jo Towers
Claudia Corriveau	Josh Markle	

The working group advertised itself in the program with this abstract:

> *There are many kinds of theories in mathematics education ranging from local theories (of learning a specific topic) to global theories (of cognition). In this working group we are interested in global theories of learning, and how they learn and change. We consider Lakatos' Methodology of Scientific Research Programs as a starting point for our discussions of learning theories and how learning theories learn. We consider both the hard core of a theory that defines it (its organisation in Maturana & Varela's, 1987, terminology) and the 'protective belt' that allows a theory to learn (its structure): "It is this protective belt of auxiliary hypotheses which has to bear the brunt of tests and get adjusted and re-adjusted, or even completely replaced, to defend the thus-hardened core" (Lakatos, 1978, p. 48).*

By 'global theories' we[1] meant what Carolyn Kieran (2019) has called "Grand theoretical frames", theories that address learning in a wide context, not only in mathematics. We[2] theorised that it should be possible to identify differences in the cores of global theories used in mathematics education. We[3] planned to make use of the experiences and expertise of the working group members to identify and elaborate these differences.

[1] The organisers.
[2] Still the organisers.
[3] We will just let you know this pronoun changes its referent, okay?

We asked the participants to introduce themselves and to say what global theories they felt knowledgeable about. This immediately revealed that we did not have a random sample of participants. For some reason an unusually large number indicated that they were knowledgeable about, or wanted to become knowledgeable about, enactivism.

We began the work of the group by dividing into small groups, with each one given the task of comparing the cores of two theories the participants knew well. To focus this discussion, we suggested six aspects of the core to consider:

- The metaphor used for, or definition given to, 'cognition';
- The ontology of the cognising agent (i.e., what cognises?);
- The ontology of the world (i.e., what is the nature of the object of cognition?);
- The methods and methodology (of learning/teaching and/or of research);
- The research objects (what do we look at in this theory of learning, where should we focus our research, what we attend to);
- The epistemology (metaphor of learning).

The groups looked at the following theories:

- Enactivism (Groups 1, 2, 4);
- Behaviourism (Group 1);
- La théorie des situations didactiques (Group 2);
- Indigenous knowledge systems (Group 3);
- Embodied Cognition (Group 3);
- Commognition (Group 4);
- Interdisciplinary approaches (Group 5);
- Discourse Analysis (Group 5).

The Theory of Developmental Instruction was added to this list on Day 2.

The reader may wish to pause at this point and consider how they would characterise one of the theories (or another one not listed) in terms of the aspects we suggested.

The combined table of the groups' characterisations is given in Appendix 1. The discussion, however, focussed on the difficulties the groups faced in trying to complete the activity. The main one the groups encountered was that the aspects we suggested were not a good fit for some theories. Indigenous knowledge systems, Embodied Cognition, Enactivism, La théorie des situations didactiques, Interdisciplinary approaches and Discourse Analysis seemed to be especially hard to pin down, although one of the groups that chose enactivism managed to fill in all the boxes.

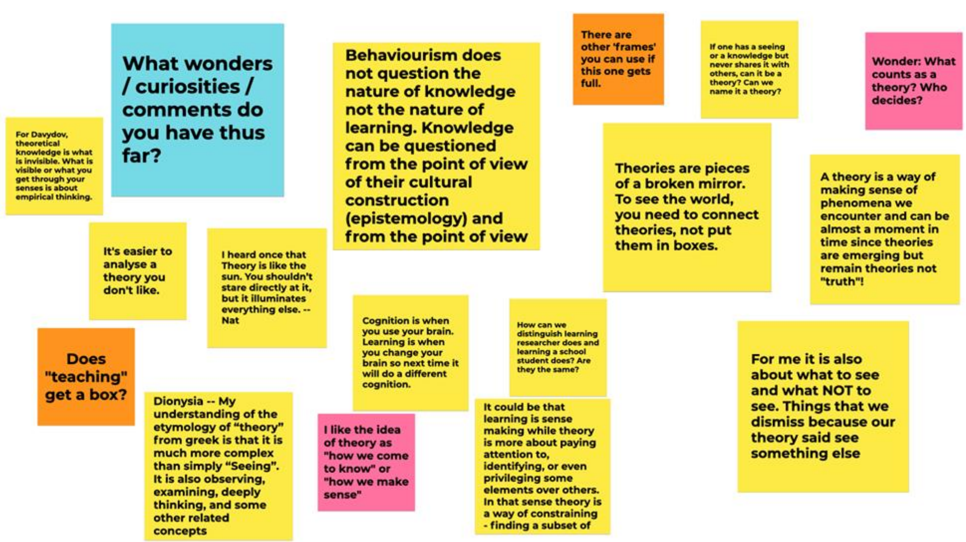

Figure 1. Jamboard 2 (start of Day 2).

For some the structure was limiting, but for the group it was a liberating constraint (Davis & Simmt, 2003). We[4] discussed the nature of the aspects, which reflected the organisers' meta-theory of theories, and which might not fit theories of learning the organisers were not thinking of. We also discussed whether some of the theories considered might not be theories of learning, or might not be grand theories. These discussions inspired others, parts of which were recorded in a Jamboard (see Figure 1).

The discussion continued with the question: 'What counts as theory?' or 'What is theory?'. There were a number of themes that emerged from the discussions. One of the themes was about theory as a way of seeing. One participant noted, from the etymology of word 'theory'—"theory is to see, to speculate, to consider". There were a number of ideas that came out from this theme including theory as not just seeing but also what not to see. The discussion about theory as a way of seeing shifted when it was suggested that maybe a better word would be "perception" rather than "seeing". One participant noted that "understanding of the etymology of 'theory' from Greek is that it is much more complex than simply 'seeing'. It is also observing, examining, deeply thinking, and some other related concepts". This discussion about theory as perceiving led to the discussion about theory as sense-making. Some ideas around theory as sense-making included

> *"theory as a way of making sense of the phenomena we encounter."*
>
> *"…sense-making involves all the senses—not just sight."*
>
> *"…sense making…can come from many modalities, experiences, reflect different time-frames etc."*
>
> *"Maybe an interesting etymological parallel: 'wisdom' shares its root in the latin 'sapere,' or to taste…speaks to knowing drawing on all the senses/being sensuous."*
>
> *"Each of the senses has its own phenomenology and epistemology. It is very different to create images through hearing/listening, smelling, moving, touching, etc. than to see with the eyes. Vision is the only sense that makes it possible to perceive from a distance, with the (illusion of) not being involved."*

Another theme that emerged was around the metaphors used to describe theory. The metaphors included theory as sun, "You shouldn't stare directly at it, but it illuminates everything else", theory as a way of navigating through an activity. Other ideas that emerged from the discussion included: theory as reflection "theories are providing us with ways of thinking and reflecting"; theory as a worldview; theory as ways of being.

Another parallel discussion occurred about the meaning of 'learning' and how it is related to 'knowledge' 'understanding' and 'cognition' within different theories. This was related to the two aspects 'metaphor of cognition' and 'metaphor of learning' in the tables from Day 1. It was also reflected in a diagram on Jamboard 3 (see Figure 2).

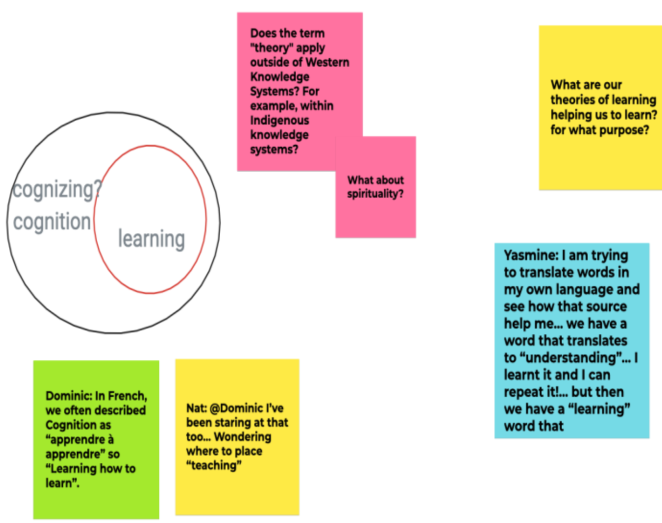

Figure 2. Jamboard 3.

[4] The whole group this time.

The group members wondered if learning is a kind of cognition, or if cognition is a kind of learning, and whether this depends on the language you speak or the theory you apply. In quick succession, the varied perspectives of situated cognition, embodied cognition, behaviourism, psychological cognition, social learning, social-cultural-cognition, and collective cognition were mentioned, but none were explored in detail. They were sufficient to show the importance of theory in establishing the meanings of these concepts.

Yasmine's comment in the Jamboard provoked the most extended discussion. It continued: "…but then we have a 'learning' word that metaphorically relates to swimming. That means being able to navigate in it" and this comment resonated within the group. It was connected to Galperin's definition of knowledge as being able to navigate within an activity and referring to 'immersion' in an area. "Swimming requires attending to the water (the environment or the place or I guess situation) as well." Swimming "makes me think of Dreyfus's skillful coping or skillful know-how in enactivism…an embodied, partially pre-reflective engagement in the world. Maybe that is a distinguishing feature between learning and cognition…?"

On Day 3 we explored what learnings we might take from our experiences in the working group. These were expressed in spoken and written comments throughout the discussion, gathered on a Jamboard (see Figure 3), and at the end of the session several participants shared their thoughts in writing.

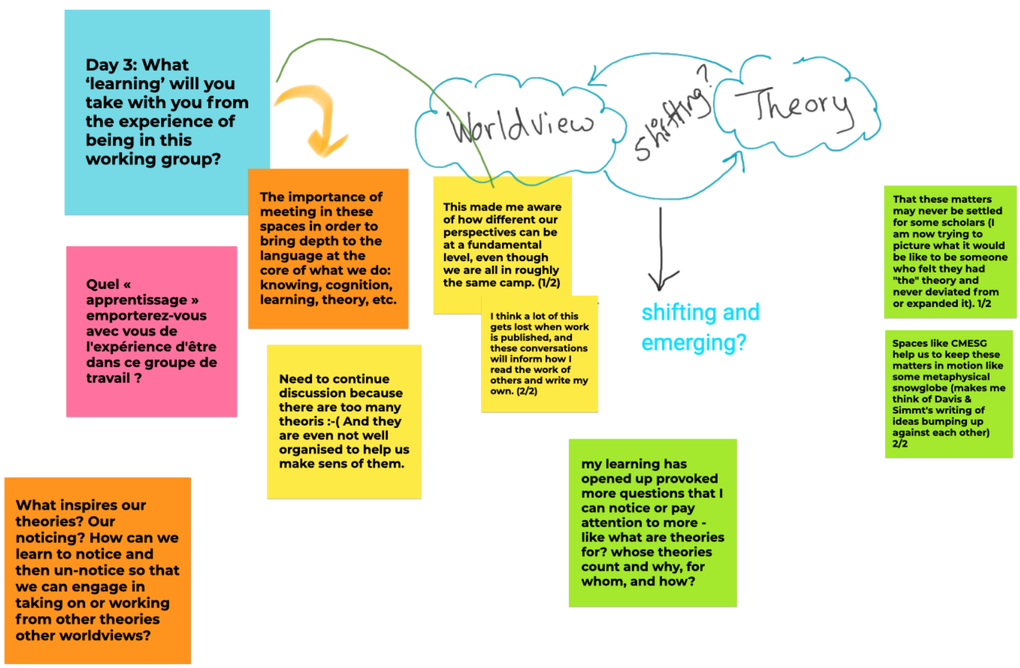

Figure 3. Jamboard 4.

A few comments gleaned from the discussion:

> *"If and how learning and cognition relate to one another depends upon the theory of learning one is referring to. Theories of learning each define these ideas in different ways and also decide if noticing is a 'thing' that 'matters', if learning is change, and/or learning is identity formation, etc."*
>
> *"Yes, we are unlikely to form a new learning theory today! We can explore the ways that learning and cognition relate to one another in various learning theories and share our sense of how we think they could relate to one another. Also true of noticing, whether the object of learning matters, whether awareness is central, etc."*
>
> *"We might not articulate our worldviews, but somehow we live as if certain things are the case..?"*
>
> *"I think it is possible to have more than one worldview—Robin Wall Kimmerer speaks of needing to move between her Indigenous world view and a scientific worldview"*
>
> *"Jack Whitehead speaks of us as teachers working within our classrooms and theories that we are 'living contradictions of ourselves'"*

"Does the act of writing constrain us to think of worldviews as stable (text as static)?"

"Maybe when a theory becomes very general it moves toward becoming a worldview."

"Yes, some theories have become worldviews or at least share a name with a worldview. For instance, post-structural is often seen as both a theory and a worldview. Seems to come back to David's point this morning that the theory and/or worldview you ascribe to frames these other elements."

PARTICIPANT'S WRITINGS

GROUP 1: ANJALI, YASMINE, CLAUDIA, DOMINIC, CHRISTINA

Worldviews and theories -ontology and theories

Different ideas about the worldviews and one's theories were shared—we did not talk much about theories and ideas behind specific theories.

Are we aligning theory with the worldview or one particular context?

My way of seeing world changes the theory or the other way round?

Which perspectives lead us to view theory to make sense of the worldview? (Which theories make most sense for sense-making for us in the moment—for the phenomena that came to the fore through more broad underlying theoretical perspectives and worldviews?)

When thinking about the differences between constructs like learning, understanding, cognizing, it is necessary to ask according to which worldview/grand theory. Language helps us to communicate ideas efficiently but at the same time it can be misleading as the same word might refer to different phenomena for different people with different worldviews.

NAT'S LEARNING...OR IS IT NAT'S COGNITION?

I am struck with the reciprocity of theory-theorizer.

We organize our experiences through them, but they also come to organize our experiences.

I think conversations like this help to break this sort of self-amplifying loop. I appreciated sitting (at a distance) with people's understandings along with the permission to descend into the very core of the words we use in our work (theory, learning, cognition, etc).

The experience of discussing the theories felt a little like chasing your own shadow. We were looking to 'shine light' on theory both inwardly and outwardly, but every time we did, the shadow (the object / word we were looking at) seemed to distort itself so as to avoid capture.

In the end, we are all linked to the shadows (theories) we are chasing.

This working group planted many seeds for theories to explore and new lenses through which to observe.

SUSAN'S MUSINGS

Here are some questions I take away from our discussions:

Do particular learning theories only take account of particular kinds of learning? I can immediately think about differences among:

- Rote learning (say, memorizing a poem, to be considered over time)
- Learning to name something
- Learning to notice something

- Taking noticing and awareness to the place where we observe and ask questions about something—maybe something we were unaware of before
- Learning to do or make something
- Integrating new awareness, doings and ideas with ourselves, our communities and places
- Exploring a (metaphorical) landscape or field that is reconfigured each time the human and greater-than-human world integrates new learning
- Learning a 'fact' or proposition that may or may not be examined
- Learning how the world seems to another
- Learning through dreams, associative images, stories and tales and other non-linear, non propositional experiences (which may not be in our conscious control)
- Learning through rational, logical trains of thought about something
- Learning by noticing variation and difference, and asking why
- Learning by experiencing 'the same thing' through different sensory modalities, forms of representation (including talk and writing, diagrams and pictures, symbolism, whole body movement, music, colour, watching a film, …) and being able to notice and express how these might all represent 'the same thing'.
- Having many-faceted, multimodal, multisensory approaches to 'a thing or process' over time, and recognizing equivalence or similarity—to have a deeper understanding of that 'thing / process'
- Coming into a richer or at least different view of the world through integrating those things that have become familiar to us through learning (with others and with the world, as well as on our own)

ANONYMOUS COMMENTS

What is a theory was a big topic of discussion with many underlying parts I did not expect.

Creating even more questions:

- What is the relation to methodology and theory

  - How is theory defined by language?
  - Seeing as a theory
  - Cognition relationship with learning and theories
    - Variation theory
  - How to describe learning and theory
  - What are the views outside the western or eurocentric of theory?

MARTHA'S THOUGHTS

For me, theory is helpful as we try to make sense of phenomena around us, it is a way of privileging, paying attention to or foregrounding some aspects of that phenomena instead of others, it includes definitions of terms in identified ways though in many theories those ideas may shift or emerge over time as theories are further developed. Theories can serve descriptive and/or explanatory functions which might even include prediction.

A dimension of theory we did not discuss as much in the group but that is important is the distinction between a theory and other forms of conjecture. This distinction was present in a slide David shared where the element of research methodology was foregrounded. A key point is that a set of ideas becomes a theory as an individual or group develops a set of conjectures further through processes that might include ongoing observation or other forms of empirical evidence as well as philosophical 'argument'.

I also very much agree with David's observation that if/whether we even characterize what we do as 'theory depends a great deal on our worldview—or at least I think that was one of David's suggestions.

Learning theories then help us to make sense of the phenomena of learning by identifying aspects deemed to be worth paying attention to, defining and redefining those ideas, often grouping the ideas together into metaphorical bundles and…

Within a given learning theory…we can then define cognition, learning, noticing, and whatever else might matter…

EVAN'S THOUGHTS

Thinking about theories opens doors to multiple ways of knowing, being and doing.

Theories inform worldviews that necessarily grow, change, 'learn'.

Worldview emerges from theory.

How do worldviews interact / integrate? intersect?

How are theories 'learning'? Thinking about the verb here. Theories 'bumping' other theories nudges thinking in this context and that context. How these are taken-up by others speaks to viability and sustainability.

Think a lot about sense-making, especially from non-western perspectives

What do we 'privilege' as theory?

Do we need to call it theory? In practice, what we do with it to make sense of experiences allows others to see the world differently.

Theories of learning help us make sense of experiences.

All experiences are unique to the 'learner'.

Learning as change

Collective learning within a system

How do different theories help us make sense differently from shared experiences?

Where/how does 'grounded' theory fit / help / muddy learning?

LISA'S THOUGHTS

Is it about 'Maybe'? Theories might be situated in contexts. Perhaps theories speak to each other, inform each other. Theories can be used in interconnected ways to help bring meaning.

I am thinking about what it means to come to know, to come to understand…I am thinking about what it means to learn something.

As I think about experiences and making sense of experiences—what I might call learning—I wonder if it works in all cases. Maybe we need different theories for different contexts. Maybe bringing multiple theories together in an interwoven way can strengthen the work we seek to do.

Several theories presented and discussed offer pieces that fit for me and pieces that do not. I am still wondering about whether or not 'theory' applies to non-Western knowledge systems. I will have to think more about that.

I am also wondering if theories are more about networks and interconnected ideas than hierarchies (see Figure 4).

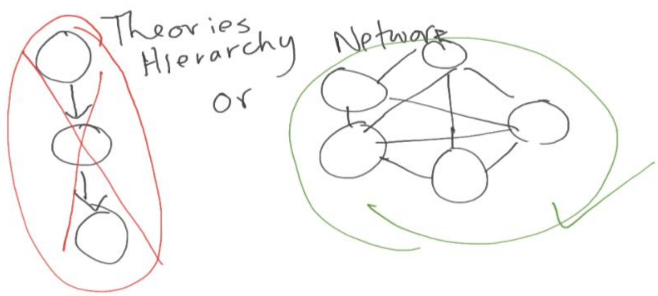

Figure 4.

DIONYSIA'S THOUGHTS:

The working group mostly generated as opposed to answered questions for me—which is nice. It has been quite confusing (but also kind of comforting) to think that not only 'learning' but also the term 'theory' itself changes based on what theory one works with / from. To me, this highlights (and also provides some meaning to) the idea that a theory can only be situated in a context. This is comforting because it challenges the abstractness as a value when theorizing or doing academic work—this relates a bit to my next 'to-be-thinking take-away':

> Lisa drew our attention to how the term 'theory' itself might be a Western, colonial construct, which might not be applicable in non-Western settings. This (along with my discomfort with 'Western' as a uniform category) makes me think how 'theory' is a very value-laden term in a 'Western world' as well, because it valorises certain ways of knowing while deeming others as less valuable or profound. To me, it is interesting to think of 'theory' in the context of the capitalist distinction between intellectual and manual labor. We might, for example, have a 'theory of law', but there is no "theory of plumbing". So, this makes me wonder how the concept of "theory" might partake in the reproduction of this hierarchical distinction between different kinds of labor. In the context of education and learning, this makes me wonder whose work counts as theory, whose work is a "conceptual framework", and whose might be something else.

WE (THE ORGANISERS, AGAIN)

Did we learn how theories learn? No. Did we learn how the grand theories current in mathematics education differ from each other? To some extent. Did we learn that our meta-theory of theories reflected our own interests, biases and structures? Yes. Our understanding of theories was expanded, enriched and complexified. Our understanding of learning, which we at-the-same-time recognised as theory-dependent and tried to use as part of our meta-theory, was similarly challenged and enhanced. For example, our understanding of learning as a cognitive phenomenon, was challenged and enhanced by other understandings or ways of knowing (e.g., learning as a linguistic phenomenon).

The comments we have assembled above reflect imperfectly the differing perspectives and experiences of the participants. As often occurs in working groups, we ended up working on different things that we had anticipated and ended up learning things we were not expecting to learn.

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APPENDIX 1

Appendix 1: Learning Theories / Théories (de l') apprenant

	Behaviourism	Indigenous knowledge systems	Embodied Cognition	Enactivism 1	Enactivism 2	Commognition	La théorie des situations didactiques	Interdisciplinary approaches	Discourse Analysis	Theory of developmental instruction
Cognition	Paired association between a stimulus and a response	L'nuita'simk		Knower is connected to other knowers		*Individualized form of communication		The term 'cognition' is already problematic in some approaches (like critical pedagogy theories). In doing interdisciplinary work, we reconsider what might be meant by learning, understanding, meaning,... Every theory is too narrow and is problematic. Each discipline has its interesting assumptions, and working across disciplines highlights this.	Understanding the world and our interactions in the world.	At least two types of thinking: theoretical and empirical
Ontology of the cognising agent	Little focus on individual difference, blank slate, cultural-historical not paid attention to"I teach, you learn"	relationships	Whole body		Knower / environment is created through knowing	Language, acting (Communication with other)	Element of a triad Learner teacher content	The researcher has to be able to move fluidly among worlds and learn quickly	The researcher is part of the discourse - the researcher cannot be outside of it observing. They can only speak from inside	Acting is oriented by knowledge
Ontology of the world	Realist,	Flux, Flow, movement, everything is in motion		"Bringing forth" The world is brought forth through our interaction with it	Brought forth by knower	Historically and socially established through interaction with others	Milieu-- institution knowledge pre-exists Discipline	The world does not fit any singular theory. It may take a confluence or aggregate of several theories to understand something new and worthwhile	Language shapes the world	The knowledge (to learn in school) exists in a more or less stable form. It is a product of a historical process of human-nature-culture interaction.
Methods & methodology	Practice, repetition, break object of learning into small chunks, reinforcing/rewarding to create paired associations that are learning Focus on frequency counts, successful responses must reach mastery level	Relational, decolonizing, storywork,				Discourse analysis, (mathematical discourses, things said and done by individual learners in direct interactions with others, the perceptible – visible or audible – discourse)	Anticipating activity Exploring, interact within the milieu Investigating the activity Didactic engineering. Ground in school Work of the teacher	It is important to immerse oneself in the writings, written and other works and communities to really grasp new theoretical ideas in their complexity	Narratives, routines, gestures, artefacts Grammar, syntax, phonetics, phonology Lexis, genres, dialect/ variety, register Semiotics Pragmatics (language in use), metaphor Diagrams, drawings, written language, symbols	Teaching experiment Epistemological analysis of the area of knowledge in question.
Research objects	Can't see inside the black box, focus on visible behaviours seen as indicator of "learning"	Knowing, being and doing, relationships, relationality, language	Physical interaction with the environment			Conversation, actions, learners	Teacher-student- content triade	The researcher may be identifying new objects based on curiosity and observation.	Commognition Language in use	The subject of knowledge-mathématiques Learning processes in students
Epistemology (metaphor of learning)	Mechanism, Input/outcome,	Ever changing, evolving, cyclical, stages of life				Developing a (new) discourse (socially established) / routines	Transition from a-didactic to didactic situation Game Devolution: Action (accept challenge - find rules) Situation of communication Situation of validation (conclude and prove) Institutionalization (explicit teaching)	New spaces are created in between theories for new ways of conceiving of learning	Question/ quandary: Does linguistic fluency and immersion in "the language of…" some area of mathematics signal learning or understanding? Socializing oneself in the discourse (Lave & Wenger)?	Learning is making theories of own expérience and available information.

POUR OU CONTRE LES TESTS : EST-CE LA BONNE QUESTION ?

TO TEST OR NOT TO TEST: IS THIS THE QUESTION?

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INTRODUCTION

The focus of this working group was to interrogate the use of tests for educational purposes. Tests are a common way of creating data regarding student learning in mathematics from primary school to post-secondary. Testing students and teachers in mathematics/numeracy at the provincial/national level has gathered momentum in several jurisdictions, including Ontario and Australia. International tests like the Programme for International Student Assessment (PISA) offer tests that are used to compare countries and provinces. We framed this working group around the following questions: (Table 1).

Ce groupe de travail visait l'examen de l'utilisation de tests à des fins pédagogiques. Du primaire au postsecondaire, les tests sont un moyen courant de créer des données sur l'apprentissage des élèves en mathématiques. L'évaluation des élèves et des enseignants en mathématiques et en numératie tant à l'échelle provinciale, nationale ou internationale a pris de l'ampleur dans plusieurs juridictions, notamment en Ontario et en Australie. Des tests internationaux comme le Programme international pour le suivi des acquis des élèves (PISA) offrent des tests qui sont utilisés pour comparer les pays et les provinces. Nous avons articulé ce groupe de travail autour des grandes questions suivantes :

Who are the tests designed for?	À qui les tests s'adressent-ils ?
For what purpose?	À quelles fins ?
Is that the best way to gather that information?	Est-ce le meilleur moyen de recueillir ces données ?
And if not a test—what else?	Et si le test ne convient pas, quelles sont les autres alternatives ?

Table 1. Questions framing working group.

Day 1: While participants were introducing themselves and what brought them to this working group, we asked them to identify if they were interested in discussing classroom student tests, large-scale student tests, or provincial teacher tests. We wrestled with some terminology (Test, testing, exam, evaluation and assessment) and the linguistic duality that we often deal with in Canada. We exposed the nuances of terms like *didactics / didactique*, *assessment* (in English and no equivalent in French), and *evaluation / évaluation*.

Les premières conversations étaient d'un ordre étymologique et sémantique. Nous avons discuté de la notion de ***didactique***, très utilisée en français, mais pas en anglais, et de ***assessment***, très utilisée en anglais, mais pas en français. Le terme *évaluation* en français est utilisé à la fois pour *evaluation* et *assessment* en anglais.

The larger group was split into two smaller groups for day one's discussion: one group focused on the testing of teachers and the other group focused on the testing of students (local, provincial, international). The group that discussed the testing of teachers considered the notions of the difference between Tests and tests. There are tests for teachers that have implications for certification and for graduation—those would be considered Tests: ones that have significant implications or consequences attached to the results of the test. Abrams, Pedulla, and Madaus (2003) identified three levels of stakes with respect to the consequences of the stakes for the group either taking or implicated by the results of the test. The levels of stakes they identified for teachers and students were high stakes, moderate stakes, and low stakes. High stakes tests are those where the consequences impact accreditation or graduation. This description of each of the identified tests for teachers or pre-service teachers is applicable.

All tests that have been implemented for teachers are considered high stakes as either teacher certification or degree completion depends on the results of the tests. In Australia, "All students undertaking initial teacher education programs [must] demonstrate they have personal literacy and numeracy skills in the top 30 percent of the Australian adult population ('the test standard') prior to graduating from their program" (Australian Council for Educational Research, 2017, p. 1). In Ontario, in order to become certified to teach in Ontario "applicants are required to score 70% or higher on both the mathematics content and the pedagogy components of the [Mathematics Test of Proficiency] in the same attempt" (Queen's Printer for Ontario, 2021, para. 3).

Participants in the Teacher Testing group made the following comments regarding the testing of teachers:

> *I do not believe testing preservice teachers helps them learn the math they have missed. I think they need the math background, first. Then they also need to have skills to be able to help students learn math, from their Image or Visual model that they have themselves to the Algorithms to do the math operations.*
>
> *I think we have to remember that 'math knowledge for teaching' is more than curriculum content knowledge and pedagogy. I argue (as do Ball and others) that 'specialized' content knowledge IS mathematics—but is an applied branch of it that is deeply related to teaching.*

These comments show that the 'what' that is being tested on teacher tests does not necessarily align with research regarding the kinds of mathematics knowledge that is needed for effective teaching of mathematics to students. Sometimes just knowing the mathematics content is not enough to be able to teach it well.

In the discussion regarding testing of students, we looked at a sample of mathematics test questions that really brought forward for participants the notion of what is truly being tested and for what goal. Some participants saw end of course tests as being one way to show how prepared students are for the next level of mathematics. Issues that were noted was that externally mandated examinations (like Grade 12 Diploma Examinations in Alberta) crept into the daily practice of teachers in that they modelled their classroom assessment practices after that examination and used only that type of assessment in their classrooms. This backwash to teacher practice was seen as a major negative influence that the examinations had on the education system.

Discussion also revolved around the intended purposes of tests and the unintended uses of test results. For example, teachers are judged on their teaching based on student results on the test. The intent of the test is not on teacher evaluation but is used to highlight successful teachers and successful schools. The Fraser Institute reports a ranking of schools based on student performance on standardized exams (*School Performance*). Housing around schools that have high rankings has seen an increase in value due to the rankings of the schools in that area. Each of these unintended consequences of the importance some society members places on exam results can be far reaching and impact the local economy.

When we came back together as a whole group, similar ideas came forward from both groups with respect to the concerns about testing (Testing and testing). Concerns were raised about biases presenting in the design, development, and administration of tests. There was a common theme that assessing student mathematical understanding was important but with alternatives other than what we have commonly known as a test. In addition, there was a common theme that teachers should have a certain level of mathematical understanding but there was inconsistent thinking about what that might look like. One participant commented

> *I am for testing teachers BEFORE they enrol in a teacher education program IN ORDER TO direct targeted supports which are resource limited in the current environment OR for them to determine the best means to improve their understanding of mathematics and mathematics for teaching. My purpose here is to encourage growth, not as a punitive measure.*

Pour la discussion du premier jour, le groupe de travail a été divisé en deux sous-groupes : un groupe s'est concentré sur les tests visant le personnel enseignant et l'autre groupe sur les tests visant les élèves (au niveau local, provincial, international). Le groupe qui a discuté des tests pour le personnel enseignant a tenu à faire une différence importante entre les tests qui ont des implications pour la certification et l'obtention du diplôme et les autres tests. Ces premiers tests seraient considérés comme des ***Tests*** avec un « T » majuscule : ceux pour qui les résultats du test ont des implications ou des conséquences significatives. Tous les tests visant le personnel enseignant sont considérés comme des épreuves à enjeux élevés, car la certification des enseignants et l'obtention d'un diplôme en dépendent.

Lors de la discussion sur l'évaluation des élèves, le sous-groupe a examiné un échantillon de questions de tests de mathématiques qui ont vraiment mis en évidence, pour les participants, ce qui est réellement évalué et l'objectif visé par ces tests. Certains participants ont vu les tests de fin de cours comme un moyen de montrer à quel point les élèves sont préparés pour le prochain niveau en mathématiques. Les participants ont noté que les examens imposés par les gouvernements influencent la pratique quotidienne du personnel enseignant qui parfois modèle ses pratiques d'évaluation en classe sur ces examens. Cette influence négative sur la pratique pédagogique a été considérée majeure pour les systèmes éducatifs.

This conversation led us to our focus for Day 2: what is assessment and how can we advance the notion of assessment to be more broadly conceptualized and enacted in classes at all levels.

DAY 2

We explored sample math test questions from a few jurisdictions and discussed their purposes and philosophies behind them. We advanced the notion of what assessment is and to consider assessment as 'to sit beside' and to use assessment as a way to support students in an ontological way of becoming who they want to be and can testing fit into that space or way to think about assessment. We refined the notion of assessment as being a continual process of collecting information in order to make learning decisions and that evaluation was analyzing the collected information in order to make a judgement about student learning and future mathematics possibilities. These refinements helped to frame the day's conversations.

Après avoir fait un peu de mathématiques en utilisant des échantillons de plusieurs juridictions, le groupe de travail a exploré en profondeur l'importance de l'esprit de la notion de « assessment » ou d'évaluation visant à améliorer l'apprentissage. Il y a eu ici l'idée d'une relation réflexive entre l'évaluation et l'apprentissage.

We split into three groups to consider the following questions: Can we have education without testing (considerations for/against testing)? If not, what are the alternatives? Could testing be reimagined to incorporate our philosophy of teaching and learning? The first group focused on testing students at the K–12 level, the second group focused on testing teachers, and the third group focused on testing students at the post-secondary level.

Pour la deuxième journée, le groupe s'est divisé en trois sous-groupes pour réfléchir aux questions suivantes : peut-on éduquer sans tester (considérations pour ou contre les tests) ? Sinon, quelles sont les alternatives ? Les tests pourraient-ils être repensés pour intégrer notre philosophie de l'enseignement et de l'apprentissage ? Le premier groupe s'est concentré sur le test visant des élèves de la maternelle à la 12e année, le deuxième groupe sur le test de compétence visant le personnel enseignant.

The different groups agreed that we can have education without what we have traditionally known as tests. They also agreed that there needs to be a rethinking of what a test is and how tests are operationalized at the different levels being considerate of the audience. Groups noted the following considerations for why testing is considered a valuable source of information: *"A way to measure if we are getting the concepts. Students can be misled into thinking they understand a concept if they do not have an opportunity to demonstrate the skill in a 'test environment'; Check that the concept being tested is a concept that fits a testing environment and needs a test; and Check that the test format (MC, short answer, long answer, oral, two stage, group, pass/fail) fits the concept being tested."*

Additionally, groups offered ways to adjust what we know tests to be so that they do less harm and gather evidence of student learning. "*Tests could be done without the assigning of grades—used for feedback and formative assessment purposes rather than summative assessment purposes.*" Participants gave many reasons that testing is not an effective way to gather information about student learning and also provided ways in which testing may be more harmful than is worth the kinds of data that is collected about student learning.

> *Anxiety can interfere with the information that the test results provide. How can we minimize test anxiety? Can an introduction to testing in a low stakes environment help? Are we helping our high anxiety students by removing testing from their assessment experiences, or does this increase their anxiety?*
>
> *Need to ensure that the test grade is not the only motivation for learning the skills: how to help students care less about the grade/score and more about the learning? (This consideration is not limited to testing: it applies to every assessment that has a grade connected to it.)*

Many alternatives to testing were offered by the groups. Some of the alternatives that were offered are: observations as evidence of learning, interviews, conversations with students, oral defense or explanation of a solution to a rich math problem, and portfolios. The group focusing on assessment in post-secondary mathematics offered the following list:

- Investigative problems, approached in groups or individually;
- Broaden the definition of tests;
- Ungrading https://www.jessestommel.com/ungrading-an-faq/;
- Giving students a choice in the assessment they will provide;
- Alternative assessment (alternative forms of 'tests' work too!);
- Two-stage tests: start as individual, then have a chance to work with a group on the same problems--score is blended;
- Mini lectures followed with short in-class homework assignments, each student gets their own problems, and can help each other, with immediate feedback;
- Portfolios or bodies of work encompassing longer time periods (a chapter or a unit or perhaps entire semester).

Other questions group participants had were

> *When we give a test, we are making assumptions about our students, treating them as a uniform whole, that may not be true. Any suggestions to making our tests more fair for everyone?*
>
> *How do we mitigate the potential harms of testing (anxiety, inequity) without losing the potential benefits (feedback for further learning/growth/development, thresholds to ensure expertise in contexts of health & safety)?*

A theme that appeared in all three groups was one of trust of the education system, at all levels, to assess and provide evidence of student learning so that accountability measures are not needed. One participant stated:

> *There are lots of other things that can replace written tests. One of the things I struggle with is how to convince myself, students, their parents, the teacher who teaches the next level, and sometimes the administration and counsellors that the students are ready, i.e., they have all the tools they need to move on to the next level. I think that written tests are a way to "check" that students have these tools, but written tests are definitely NOT the only nor the best way to do things. But I often come back to the question of how to convince others.*

Les différents sous-groupes ont convenu qu'il est possible de penser une éducation sans tests comme nous les connaissons traditionnellement. Ils ont également convenu qu'il faudrait repenser ce que sont les tests et comment ceux-ci sont mis en œuvre aux différents niveaux en tenant compte du public cible. Ils ont proposé des moyens d'ajuster les tests que nous connaissons afin qu'ils fassent moins de mal et qu'ils permettent de recueillir des preuves d'apprentissage des élèves. Les tests pourraient être effectués sans attribution de notes—utilisés à des fins de rétroaction et d'évaluation formative plutôt qu'à des fins d'évaluation sommative. Les participants ont donné de nombreuses raisons pour lesquelles les tests ne sont pas un moyen efficace de recueillir des informations sur l'apprentissage des élèves et ont également indiqué comment les tests peuvent même être nuisibles comparativement aux types de données qu'ils permettent de recueillir sur l'apprentissage des élèves.

Les trois sous-groupes ont proposé de nombreuses alternatives aux tests. Parmi les alternatives proposées, nous pouvons mentioner : les observations comme preuve d'apprentissage, les entretiens, les conversations avec les élèves, la défense ou l'explication orale d'une solution à un problème mathématique complexe et les portfolios. Le groupe qui s'est concentré sur l'évaluation des mathématiques au niveau postsecondaire a proposé la liste suivante :

- Problèmes d'investigation, abordés en groupe ou individuellement;
- Élargir la définition des tests;
- Ungrading : https://www.jessestommel.com/ungrading-an-faq/;
- Donner aux étudiants le choix dans les méthodes d'évaluation;
- Évaluation alternative (les formes alternatives de « tests » fonctionnent aussi !);
- Tests en deux étapes : commencer individuellement, puis avoir la possibilité de travailler avec un groupe sur les mêmes problems—la note est mixte;
- Mini-exposés suivis de courts devoirs en classe, les étudiants ont leurs propres problèmes et peuvent s'entraider, avec un retour immédiat;
- Portfolios ou compilation de travaux englobant des périodes plus longues (un chapitre, une unité ou peut-être un semestre entier).

Au final, le manque de confiance dans le système d'éducation a été à l'origine des exigences en matière de tests et de l'émergence d'entités telles que l'Office de la qualité et de la responsabilité en éducation (OQRE) en Ontario, Educational Testing Service (ETS) aux États-Unis et le Programme international pour le suivi des acquis des élèves (PISA) à l'échelle internationale. Une question est et demeure : comment pouvons-nous convaincre les autres que les tests, qu'ils soient administrés en classe, à l'échelle provinciale ou internationale, ne sont ni la seule, ni la meilleure façon d'évaluer l'apprentissage des élèves ou des étudiants ?

This notion of convincing others and the challenges of pushing back against societal norms and expectations came out strongly at all levels and for all audiences. The lack of trust in the education system has been an impetus for testing requirements and the emergence of testing organizations like the Education Quality and Accountability Office (EQAO) in Ontario, Educational Testing Service (ETS) in the United States, and Programme for International Students Assessment (PISA) internationally. The question remains unanswered - how can we convince others that tests, whether they are classroom, provincial, or international, are not the only, nor the best, way to assess student learning?

DAY 3

We gave participants a test to illustrate test wiseness skills to focus on what we pushed what a 'test' could look like and feel like from a student experience and considered potential principles of assessment practices that could be considered when designing/implementing tests/assessments. We had participants review and comment on the *Principles of fair assessment practices* (Joint Advisory Committee, 1993) document. Though this document is fairly old and has some outdated language, the principles hold in today's context.

Nous avons lancé les discussions avec un test un peu loufoque, de la perspective de plusieurs. Nous avons discuté des aspects liés à la langue—problématique des langues premières et secondes. Nous avons ensuite exploré les principes à garder en tête dans le contexte de bonnes pratiques d'évaluation. Nous avons demandé aux participants d'examiner et de commenter le document intitulé *Principles of fair assessment practices* (Joint Advisory Committee,1993). Bien que ce document soit assez désuet et comporte un langage dépassé, les principes restent valables dans le contexte actuel.

When we asked groups to consider principles and guidelines for good and successful assessment/evaluation practices, one group developed the list and comments in the table below.

Assessment/Evaluation Practices	**Comments/considerations**
The quality of items, especially in higher stakes assessments, is crucially important	Challenges of getting beyond procedural skill and definitions, and into deeper understandings
Low stakes especially in this environment, more group work	
Pre-service teachers will have a lot of power over vulnerable populations (e.g., students). At what point should/must there be minimal criteria for their initial capacity? Is there an obligation to ensure that there must be some baseline?	
Allowing multiple attempts at an assessment, with an assessment result only recorded when it shows an improvement in skills. Use similar but not identical questions and provide immediate feedback so students can identify where they may have misconceptions or knowledge gaps.	Feasible as an online assessment. Likely an administrative overload if not online.
Multi part assessments: students first complete work independently; then connect in groups to compare; final opportunity to modify individual work based on the group discussion.	Determining 'fair groupings' can be tricky.
Ideally we can devise some method to check for individual skill development while also promoting creativity, collaboration, and problem solving.	Allowing students to progress (next grade level, next course) without verifying prerequisites can defer trauma from one grade/course to a future grade/course.
Retrieval practice and interleaving. Use low stakes or no-stakes tests to achieve the benefits of retrieval and interleaving.	Retrieval does not need to occur in a test environment and ideally interleaving becomes more common outside a testing environment. (Interleaving is already common within most testing environments.)
Vary assessment practices to help all students.	Discussions centered on students who do not flourish within traditional high stakes exam environments. Should we consider those students who enjoy a rush of stress and are excited by being tested? Or is this a personality trait to discourage?
Match the skill being assessed to the assessment method. Does multi-step problem solving belong in a multiple choice question?	We saw many examples of reasonable questions becoming poor questions by forcing them into a multiple choice setting.
Acceptance that some deviations in grading will occur, depending on who is marking, and the emotional state of that person while marking. This is true of all assessments, and rubrics are not magic.	If test results are used for comparison, recognition of this grading range is essential. It can be minimized, but not eliminated.
Build security nets into our assessments, so that students who have not (yet) learned a skill have opportunities to develop the skill.	Our improved skill set with online resources makes this feasible.

CONCLUSION

Participants summarized their learning from this working group in the following ways:

> *I liked the distinction between the point-gathering paradigm and the paradigm of gathering evidence that learning has taken place. The ultimate goal is for students / teachers to learn and the purpose of the assessment is to guide learning.*

> *I was not aware of the sociopolitical implications of testing especially when it is used to determine a student's future. Now I am. And I have to decide how I want to change my testing to make it fairer for students.*
>
> *I have learned that poorly designed standardized tests and misuse of these test results has created test trauma. This has helped me better understand why testing has become stigmatized within education.*

Les participants ont résumé leur apprentissage de ce groupe de travail de la manière suivante :

> *J'ai aimé la distinction entre le paradigme de la collecte des scores et celui de la collecte de preuves d'apprentissage. L'objectif ultime est que les étudiants/élèves/enseignants apprennent et le but de l'évaluation est de guider l'apprentissage.*
>
> *Je n'étais pas conscient des implications sociopolitiques des tests, surtout lorsqu'ils sont utilisés pour déterminer l'avenir d'un élève (ou d'un enseignant). Maintenant, je le suis. Et je dois décider de la manière dont je veux modifier mes tests pour les rendre plus équitables pour les élèves.*
>
> *J'ai appris que des tests standardisés mal conçus et une mauvaise utilisation des résultats de ces tests peuvent avoir créé des traumatismes liés aux tests. Cela m'a aidé à mieux comprendre pourquoi les tests peuvent devenir stigmatisants en éducation.*

What we came to was a realization that…

- Testing causes harm and maintains systems of privilege and oppression.
- The ultimate goal is for students / teachers to learn and the purpose of the assessment is to guide learning.
- The Working Group made participants think more about how test results are often used for things they were not intended for, i.e., comparing different teachers or school systems (political reasons) rather than a way for students to understand better how they are assimilating the material.
- Assessment methods should be clearly related to the goals and objectives of instruction and be compatible with the instructional approaches used.
- "Testing without teaching is unethical": ensure that concepts on a test match the concepts that have been taught.

Au final, nous avons réalisé que…

- Les tests causent du tort et maintiennent des systèmes de privilèges et d'oppressions.
- Le but ultime est que les élèves, les étudiants et les enseignants apprennent et l'évaluation a pour but de guider l'apprentissage.
- Le groupe de travail a fait réfléchir les participants à la façon dont les résultats des tests sont souvent utilisés à des fins autres que celles pour lesquelles ils ont été conçus, par exemple pour comparer différents enseignants ou systèmes scolaires (pour des raisons politiques) plutôt que pour permettre aux élèves de mieux comprendre la matière.
- Les méthodes d'évaluation doivent être clairement cohérentes aux buts et objectifs d'enseignement, et être compatibles avec les approches pédagogiques utilisées.
- « Tester sans enseigner n'est pas éthique ». Il faudrait s'assurer que les concepts d'un test correspondent aux concepts qui ont été enseignés.

And we discovered that or are still wondering about…

- How can we support change throughout the system to reflect our new thinking about assessment / testing?
- What can I change in my practice—where do I start?
- How did we get here? Why is there such a difference between conceptualizations and enactment of assessment between teachers, researchers, parents and others?
- Is testing teachers causing more harm than not testing them which may lead to harms to generations of students?
- We have reviewed poor test questions (which are poor questions in any assessment environment), discussed the student trauma connected with grades (which is independent of how grades are attained, in a testing situation or from another assessment strategy), and we have focused on testing as a tool for grade collection. Have we missed an opportunity to consider methods of testing that can be beneficial for student learning? Or has testing gathered such baggage as an assessment method that just saying the word 'test' will create division amongst us?

- Some participants are still wondering about the challenge / suggestion that a high stakes test (for teachers) might be inequitable. Their (new) parallel wondering is that if we allow an 'out' for some, or dismiss the whole idea as inequitable, that this is even worse because it ALLOWS the inequity to perpetuate.They would rather work hard to support not-yet-prepared pre-service teachers so that they CAN become prepared. This, to me, creates equity.

Et nous avons découvert que… Ou nous nous interrogeons encore sur…

- Comment pouvons-nous provoquer un changement dans l'ensemble du système pour refléter notre nouvelle façon de penser l'évaluation/les tests ?
- Qu'est-ce que je peux changer dans mes pratiques ? Et par où commencer ?
- Comment en sommes-nous arrivés là ? Pourquoi y a-t-il une telle différence entre les conceptualisations et la mise en œuvre de l'évaluation entre le personnel enseignant, les chercheurs, les parents et autres ?
- Le fait de tester le personnel enseignant est-il plus nuisible que de ne pas les tester ? Le fait de tester le personnel enseignant pourrait-il nuire à des générations d'élèves ?
- Nous avons examiné les questions de tests mal formulées (elles sont de mauvaises questions dans n'importe quel environnement d'évaluation), discuté du traumatisme de l'élève lié aux notes (qui est indépendant de la manière dont les notes sont obtenues, dans une situation de test ou à partir d'une autre stratégie d'évaluation), et on s'est concentré sur le test comme outil d'attribution et de collecte de scores. Aurait-on manqué l'occasion d'envisager des méthodes de test qui pourraient être bénéfiques pour l'apprentissage des élèves ? Ou les tests auraient-ils une telle réputation en tant que méthode d'évaluation que le simple fait de prononcer le mot « test » crée des divisions entre nous ?
- Plusieurs s'interrogent toujours sur le défi ou la suggestion selon lequel/laquelle un test à enjeux élevés (pour les enseignants) pourrait être inéquitable. Leur (nouvelle) question parallèle est que si nous permettons à certains de s'en sortir, ou si nous rejetons l'idée dans son ensemble comme inéquitable, c'est encore pire parce que cela PERMET à l'inégalité de se perpétuer. Ils préfèrent travailler dur pour soutenir les étudiants-maîtres qui ne sont pas encore préparés afin qu'ils puissent le devenir. Pour eux, c'est cela qui crée l'équité.

En terminant, nous remercions chaque membre du groupe de travail pour son engagement et son professionnalisme. L'extrait suivant d'un participant est très puissant et résume très bien les discussions d'une grande partie de la session : « *Les tests révèlent à quel point nos engagements envers la vérité et la réconciliation, l'équité, la diversité, l'inclusion et la décolonisation sont intermittents, superficiels et creux dans la pratique* ». Pensez-y-bien…

Thank you to each of the group members for a respectful engagement around an emotionally charged topic. The notions of equity and fairness were paramount in the discussions as well as desiring a broadening of a societal understanding of the purposes of assessment and the value in all kinds of assessment, not just Tests or tests.

RESOURCES USED:

PISA Results

- https://www.oecd.org/pisa/publications/pisa-2018-results.htm (all countries)
- https://www.oecd.org/pisa/publications/PISA2018_CN_CAN.pdf (Canada)

Link to the purpose of testing sites

- Tests for teachers or pre-service teachers
    - • MPT (Ontario)—https://www.eqao.com/the-assessments/math-proficiency-test/
    - California Teacher Certification Exams—http://www.ctcexams.nesinc.com/PageView.aspx?f=GEN_AboutCSET.html
    - Australia Exams for Preservice Teachers (Literacy and Numeracy)—https://teacheredtest.acer.edu.au/about

- Tests for students
    - Grade 12 diploma exams (Alberta)— https://www.alberta.ca/diploma-exams.aspx https://www.alberta.ca/fr-CA/diploma-exams.aspx
    - Grades 6 and 9 Provincial Achievement Tests (Alberta)—https://www.alberta.ca/provincial-achievement-tests.aspx https://www.alberta.ca/fr-CA/provincial-achievement-tests.aspx
    - Grade 3 Student Learning Assessments (Alberta)—https://www.alberta.ca/student-learning-assessments.aspx#:~:text=Student%20Learning%20Assessments%20(SLAs)%20are,2%20provincial%20programs%20of%20study https://www.alberta.ca/fr-CA/student-learning-assessments.aspx
    - Grade 10 Numeracy Assessment (BC)— https://www2.gov.bc.ca/gov/content/education-training/k-12/administration/program-management/assessment/graduation/graduation-assessments/numeracy-assessment?keyword=grade&keyword=10&keyword=numeracy&keyword=assessment
    - Grade 12 Exams (Saskatchewan)— https://www.saskatchewan.ca/residents/education-and-learning/departmental-exams
    - Provincial Exams (Manitoba)— https://www.edu.gov.mb.ca/k12/assess/assess_program.html https://www.edu.gov.mb.ca/m12/eval/eval_prov.html
    - EQAO site (Ontario)—https://www.eqao.com/the-assessments/

"Teacher math tests don't boost student scores, agency finds"—https://www.theglobeandmail.com/canada/article-review-shows-testing-teachers-has-little-impact-on-student-performance/

Education Quality and Accountability Office

Literature review of the empirical evidence on the connection between compulsory teacher competency testing and student outcomes—https://www.eqao.com/en/research_data/communication-docs/report-literature-review-teacher-competency-testing.pdf#search=literature%20review

"The Math Proficiency Test is 'a slap in the face' for student teachers"—https://www.queensjournal.ca/story/2021-05-31/news/the-math-proficiency-test-is-a-slap-in-the-face-for-student-teachers/

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THE REWARDS AND CHALLENGES OF VIDEO IN THE FIELD OF MATHEMATICS EDUCATION: LOOKING BACK IN ORDER TO PREPARE FOR THE FUTURE

LES APPORTS ET DÉFIS DE LA VIDÉO POUR (LA FORMATION À) L'ENSEIGNEMENT-APPRENTISSAGE DES MATHÉMATIQUES : REGARD DU PASSÉ POUR PRÉPARER LE FUTUR

A REPORT IN TWO LANGUAGES / UN RAPPORT EN DEUX LANGUES

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INTRODUCTION

Dans ce groupe de travail GCEDM 2021, nous nous sommes appuyés sur les expériences et connaissances des participant.e.s, afin de partager des idées, des questionnements et des ressources quant à la vidéo dans (la formation à) l'enseignement-apprentissage des mathématiques. Par des échanges en équipe et en plus grand groupe, nous sommes partis de nos préoccupations basées sur nos expériences passées pour ensuite nous projeter dans l'avenir et réfléchir à la place que nous voulons donner à la vidéo dans l'enseignement et la recherche au cours des prochaines années.

Commençons d'abord par situer le contexte dans lequel ce groupe de travail a pris forme. En raison de la situation mondiale liée à la COVID-19, près de 1,6 milliard d'apprenant.e.s (soit plus de 90 %) dans plus de 190 pays du monde, du préscolaire à l'université, n'ont pas été en mesure de fréquenter leur établissement d'enseignement (Organisation des Nations Unies, 2020). Parmi tous les aspects négatifs de cette crise sans précédent, le secteur éducatif fait preuve de résilience, par l'innovation et la mise au point de méthodes d'enseignement alternatives.

L'une des méthodes de formation à distance que l'on retrouve de manière répandue dans plusieurs pays est l'enseignement en ligne. Même si de telles initiatives d'enseignement en ligne existent depuis plusieurs années, c'est un nouveau monde de possibilités qui s'est ouvert pour bon nombre d'enseignant.e.s. Il a fallu apprendre « sur le tas », dans certains cas avec le soutien de formations dédiées qui ont été créées en urgence. Nous avons ainsi pu réaliser que l'enseignement en ligne va bien au-delà du simplement ouvrir la caméra et de parler devant l'écran.

Certains d'entre nous ont tant bien que mal saisi cette opportunité pour revoir notre enseignement universitaire et exploiter l'apport des outils technologiques. En effet, les outils technologiques peuvent amener un changement des pratiques au lieu de soutenir les pratiques traditionnelles, ce qui demande des ajustements autant pour les enseignant.e.s que les étudiant.e.s (Garrison et Akyol, 2013). Parmi les méthodes d'enseignement en ligne possibles, on retrouve bien évidemment la visioconférence, mais aussi les capsules vidéo, les forums de discussion, les quiz, etc. Dans notre groupe de travail, nous avons décidé de nous concentrer sur la vidéo.

Dans le domaine de l'éducation, la vidéo est utilisée de diverses façons en recherche et dans la pratique (p. ex., Duvillard, 2017; Expósito et al., 2020; Fiorella et Mayer, 2018; Poellhuber, 2017). D'autres travaux se penchent spécifiquement sur la vidéo pour soutenir l'apprentissage des mathématiques et le développement professionnel à l'enseignement des mathématiques (p. ex., Borko et al., 2011; Coles, 2013, 2019; Jaworski, 1990; Sherin et van Es, 2009; Towers, 2007; Van Es et al., 2014). Au cours de la dernière année, plusieurs d'entre nous avons dû adapter notre vie professionnelle, qui se déroule maintenant principalement en ligne, ce qui nous a amenés à redéfinir notre rôle d'enseignant.e, d'apprenant.e et de chercheur.e. En conséquence, la vidéo a joué un rôle beaucoup plus important qu'auparavant et, dans ce processus d'adaptation, nous avons essayé une variété de nouvelles approches, nous avons beaucoup appris et nous nous sommes posé de nombreuses questions concernant (la formation à) l'enseignement et l'apprentissage des mathématiques en ligne.

During our first working group activity, participants used one word to describe their experiences with video in the field of mathematics education over the past year. Note that they were allowed to answer the question more than once. We used *Mentimeter* (https://www.mentimeter.com/) to create a word cloud with these words. Thus, the larger words occurred more often than the smaller words (see Figure 1).

Figure 1. Word cloud: Experiences with video described by single words.

Observe that some of these words are fairly positive (e.g., exciting, agréable, fun, interesting, rewarding, invigorating) while others are more negative (e.g., frustrating, difficult, exhausting, time consuming, turbulent). And some words are fairly contemplative (e.g., reflective, pause).

Pour approfondir ces réflexions, nous avons ciblé des questions à aborder dans le groupe de travail.

RETOUR SUR LES QUESTIONS QUI ONT GUIDÉ LE GROUPE DE TRAVAIL

This working group was an opportunity to consider the rewards and challenges of working with video from the perspective of both research and practice. The following questions were identified to guide the working group's discussions:

1. How have we experimented with video and to what purposes? / Quels usages de la vidéo avons-nous expérimentés et dans quelles intentions ?

2. What are some of the challenges we encountered when using video? / Quels ont été les défis rencontrés en utilisant la video ?
3. Comment pouvons-nous utiliser la vidéo pour mieux soutenir nos étudiant.e.s ? / How can we use video to better support our students?
4. Comment les chercheur.e.s/formateurs.trices peuvent soutenir le développement professionnel en enseignement par l'utilisation de la video ? / How can researchers/facilitators use video to support teacher professional development?
5. What are some of the particular rewards and challenges related to using video in the field of mathematics (education)? / Quels apports et défis de l'utilisation de la vidéo sont particuliers au domaine de l'enseignement-apprentissage des mathématiques ?
6. Which aspects of video would we like to develop further once we return to in-person teaching? / Quels aspects de la vidéo voulons-nous continuer à développer avec le retour de l'enseignement en présentiel ?
7. Quels sont les défis à considérer lors de l'utilisation de la vidéo dans la pratique et dans la recherche à venir ? / What are some of the challenges to consider in terms of using video as a tool for future research and practice?
8. Quelles questions de recherche peut-on envisager pour nous aider à mieux comprendre les apports et défis de la video ? / What research questions can we ask to help us better understand the rewards and challenges related to the use of video?

Nous allons revenir sur ces questions, séparées en quatre blocs de deux questions, ainsi que sur les idées évoquées par les participant.e.s.

QUESTIONS 1 AND 2

Our first two questions focused on participants' personal experiences and challenges with video:

- Question 1: How have we experimented with video and to what purposes? / Quels usages de la vidéo avons-nous expérimentés et dans quelles intentions ?
- Question 2: What are some of the challenges we encountered when using video? / Quels ont été les défis rencontrés en utilisant la video ?

Let us begin with the first question. In teaching, Working Group participants created content videos for their students. Certain videos addressed specific student misconceptions. Some videos were short 'gap filling' two-minute videos. Others were longer. Some participants 'curated' already existing videos found on the Internet. Most agreed that video is particularly effective for showing graphs, colour-coding, and other visuals. One example of such a video explains Taylor Polynomials (see https://monurl.ca/taylorpolynomials). In our culture, it has become a norm to reach out to *YouTube* (https://www.youtube.com/) to learn more about a wide variety of topics (e.g., how to wax a car, how to sharpen a knife, how to build home furniture, etc), so participants hypothesized that it probably feels natural for students to learn mathematics through watching videos. Some participants asked their students to view videos outside of meeting time thus following the 'flipped classroom' model. Other participants asked their students to create short videos to introduce themselves to their instructors and to their classmates. Students were also asked to create videos to explain their thinking or to practice teaching. These videos were often assessed for clarity, understanding, etc. A few participants recorded videos to give their students feedback on assignments and projects. Participants who work with pre-service and in-service teachers asked learners to observe videos of teachers teaching mathematics in their own classroom. These videos were used as a starting point to generate discussion. Finally, participants watched videos to learn topics, for inspiration and ideas to help prepare their own classes, and to reflect on their own teaching.

In research, Working Group participants used video to collect data (interviews, classroom data, problem solving sessions, teacher professional development); as a starting point for discussion (for example, with teacher professional development); as a tool for storytelling; and as a vehicle to disseminate research ideas and results.

Some participants used videos for professional development. They found that, by creating videos and holding live video events, they could reach a much larger audience (i.e., many more teachers, educators, …).

Figure 2 is a concept map created by one of the breakout groups while brainstorming their uses of video over the past year.

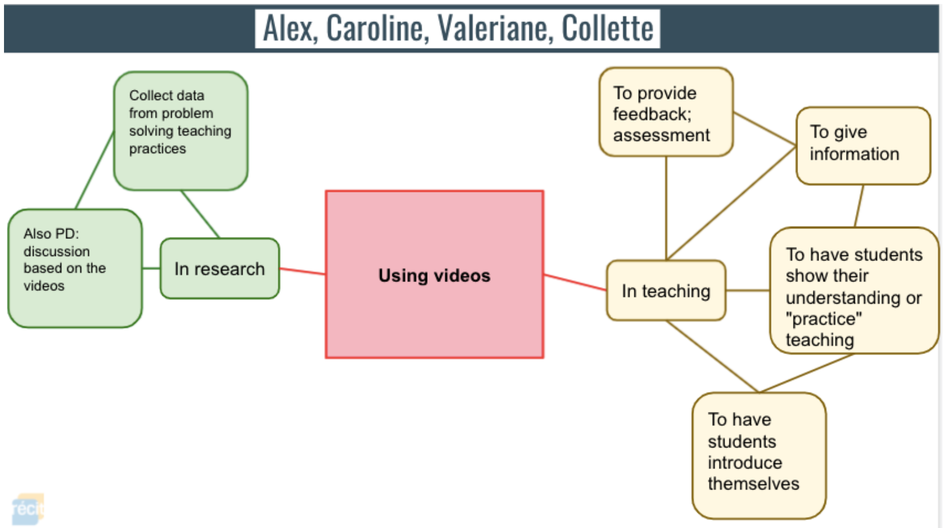

Figure 2. Concept map: Using videos.

Let us now move on to Question 2. Although there are so many ways of using video to support teaching, it comes with its own set of challenges. Many teachers are preparing videos for the first time. For some, it is a difficult transition from teaching in person to creating videos. There are technical issues with creating video. How do you ensure the videos you create are of reasonable quality (this includes quality of sound and good lighting)? What does it even mean for a video to be of 'good' quality? What are the criteria for a 'good' video? And how do we effectively use video as a teaching tool? In addition, creating, editing, and uploading videos is particularly time consuming. Finally, it is easy to worry about the *length* of a video. Many of us believe that our videos should be *short*. But how short is short and where are the best places to segment our material into digestible chunks?

For those who prefer 'curating' videos from *YouTube* and other internet sources for their students to watch, it is sometimes difficult to find good already existing videos, surtout en français! And for those who like to have their pre-service teachers analyze and reflect on videos of mathematics teaching in the classroom, it is difficult to find good secondary school videos, surtout en français!

Once we create or choose videos for our students to watch, how do we know that students are actually watching these videos? And how do we know that students are actually learning from the videos they do watch?

Many teachers and facilitators worry that video is so much more impersonal than in-person teaching, especially when we never have the opportunity to be in physical proximity to our students. Are students more likely to 'disconnect' when viewing a video than they would be listening to a 'live' teacher in a classroom? Also, when students view videos we create for them, we do not see how they react to our teaching, to the examples we use, and to the metaphors we build. The work of creating (or choosing) videos is very different from the (necessarily interactive) performance of in-person teaching. Thus, we inevitably teach differently when we use videos. Once again, how does this affect student learning? Some Working Group participants asked the question: "When does a video replace a teacher?" In other words, once we have created a complete library of videos, will we still be needed? Since books have not replaced teachers, we would like to believe that videos will not either.

Finally, in some learning platforms, it is difficult to organize videos so that they are easy for students to find, and some students do not have access to the necessary technology for watching videos (e.g., some areas of Canada still do not have reliable Internet; some, especially younger, students do not have a computer, smartphone, or tablet).

In research, there are also challenges encountered related to the use of video. It is often difficult to get participants to consent to being filmed for research purposes. When working with video, there are large quantities of data which can be overwhelming in terms of analysis. Lastly, video clips are not always an accepted medium to submit to or include in journal articles, dissertations, and other publications which have traditionally been more print-oriented.

Below, in Figure 3, is a concept map created by one of the breakout groups while thinking about some of the challenges related to the use of video over the past year.

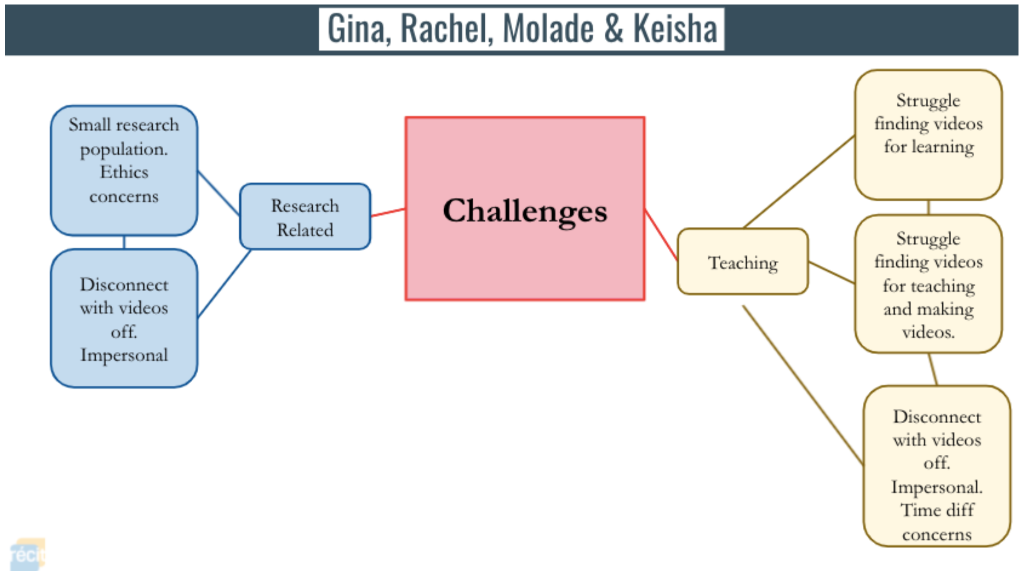

Figure 3. Concept map: Challenges related to the use of video

QUESTIONS 3 ET 4

Les participant.e.s du groupe de travail se sont ensuite penché.e.s sur ces deux questions :

- Question 3: Comment pouvons-nous utiliser la vidéo pour mieux soutenir nos étudiant.e.s ? / How can we use video to better support our students?
- Question 4: Comment les chercheur.e.s / formateurs.trices peuvent soutenir le développement professionnel en enseignement par l'utilisation de la video ? / How can researchers/facilitators use video to support teacher professional development?

Pour aborder ces deux questions, nous avons partagé des idées de ressources dans une banque de ressources collective. Cette banque de ressources a été enrichie au cours des trois journées du groupe de travail. Ce partage collectif a permis de rassembler une vingtaine de ressources visant à mieux soutenir nos étudiant.e.s et à soutenir le développement professionnel en enseignement par l'utilisation de la vidéo. On y retrouve notamment des conseils basés sur la recherche et l'expérience, des outils utiles, des plateformes de vidéo pour (la formation à) l'enseignement des mathématiques ainsi que des chaînes YouTube qui expliquent des concepts mathématiques avancés.

Conseils basés sur la recherche et l'expérience

Tout d'abord, quelques ressources partagées font ressortir des conseils basés sur la recherche et l'expérience. Par exemple, une ressource présente 19 conseils pour préparer, produire et diffuser des capsules vidéo dans le cadre d'un enseignement en ligne. Ces conseils sont présentés sous forme de vidéo (https://monurl.ca/conseilsvideo) et sous forme de texte (Thibault, 2021). La Figure 4 illustre ces conseils visant à favoriser l'enseignement universitaire à distance à l'aide des capsules vidéo.

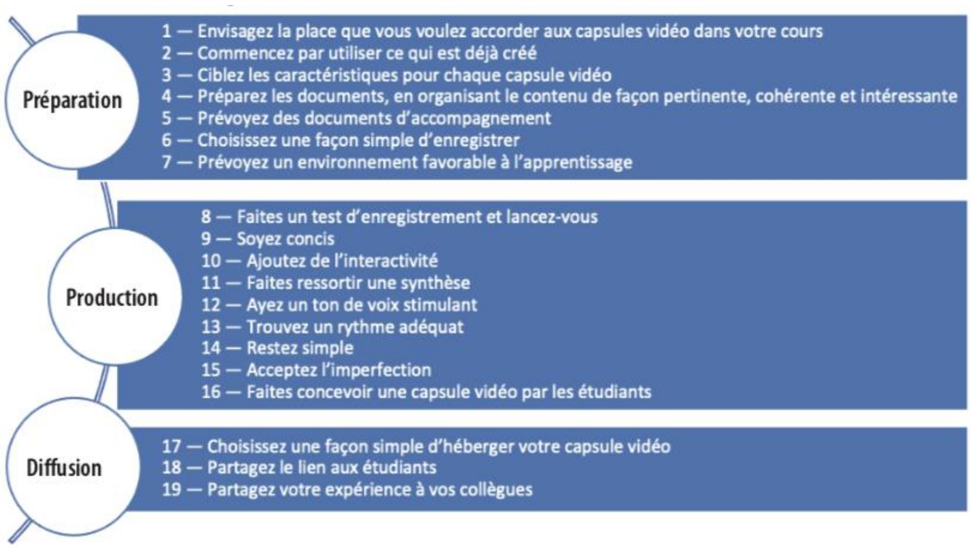

Figure 4. 19 conseils liés aux capsules vidéo (Thibault, 2021, p. 5)

Similar resources also exist in English (e.g., https://monurl.ca/bestpracticesvideo).

Outils utiles

Parmi les autres ressources évoquées par les participant.e.s, plusieurs sont en fait des outils utiles en lien avec l'enseignement à l'aide de la vidéo. EdPuzzle (https://edpuzzle.com/) est une plateforme pour déposer des vidéos et ajouter des questions qui permettent d'offrir une rétroaction (voir https://monurl.ca/tutorieledpuzzle ; see https://monurl.ca/tutorialedpuzzle). Il s'agit donc d'une façon de favoriser l'interaction, en engageant l'apprenant.e à réfléchir et à répondre à des questions pour lesquelles l'enseignant.e peut fournir une rétroaction automatique ou personnalisée.

Flipgrid (https://flipgrid.com/) est une plateforme où les étudiant.e.s peuvent déposer de courts enregistrements vidéo pour répondre à une question, présenter un concept, etc. (voir https://monurl.ca/flipgridtutoriel ; see https://monurl.ca/flipgridtutorial). C'est une façon d'instaurer une communauté de pratique, en amenant les étudiant.e.s à produire des vidéos au lieu de simplement les consulter.

Panopto (https://www.panopto.com/) est une plateforme pour déposer, capturer et modifier des vidéos (versions gratuite ou pro). Dans un environnement numérique d'apprentissage tel que Moodle, un module peut s'ajouter pour déposer les vidéos dans la structure d'un cours.

We also shared resources about using video to give student feedback. In addition, there are peer evaluation tools to help give students feedback from their classmates. *Peer Scholar* (https://www.peerscholar.com) is one of those tools intended to create, assess and reflect, using the power of formative peer-assessment to deepen learning while also exercising and measuring core skills. *Kritik* (https://www.kritik.io) aims to create authentic peer-to-peer interaction, based on a peer-grading platform that distributes fair and accurate assessments by harnessing collective intelligence to simplify workflows and reduce turnaround time on feedback. Various pricing options are available for these platforms.

Video websites in the field of mathematics education

Another resource type is video websites for (training in) mathematics education, which are found in abundance (mostly in English). For example, *Teaching Channel* (https://learn.teachingchannel.com/) is similar to a *YouTube* for teachers, with videos from real classrooms (some of them are full classes). The videos are curated, so quality is supposed to be assured. When purchasing a subscription, full access to resources is provided in each video, such as transcriptions, lesson plans and materials.

Also, *Learning & Teaching with Learning Trajectories* (https://www.learningtrajectories.org/) can help to learn about how young children (from birth to age 8) think and learn about mathematics. More than 800 videos are shared on *Vimeo* (https://vimeo.com/user40320950).

Annenberg Learner (https://www.learner.org/subject/mathematics/) includes a large collection of videos for teaching specific mathematical concepts and teacher professional development (in-service and pre-service).

Moreover, *YouCubed* (https://www.youcubed.org/) offers an abundance of math education and growth mindset resources, including videos, particularly for teacher professional development and teacher education.

Virtual Nerd (https://www.virtualnerd.com/) regroups over 1,500 video lessons covering middle grades math through Algebra 2.

The previous platforms include collections of videos, but there are also individual resources for a specific need. For example, a video of a class lesson about the case of intersecting lines can be used for unpacking discourse moves (see www.mathsolutions.com/MDISC35).

Chaînes YouTube qui expliquent des concepts mathématiques avancés

Toujours dans cette idée de ressources plus ciblées, on retrouve des chaînes YouTube qui expliquent des concepts mathématiques avancés. Par exemple, la chaîne Mathéma-TIC (https://www.youtube.com/user/MathemaTICfr) permet d'accéder à plus de 250 vidéos sur plusieurs sujets abordés au postsecondaire.

Such video channels also exist in English (for example, https://monurl.ca/mat137 or https://www.youtube.com/user/numberphile/). These channels portray math beautifully and also show different mathematicians which is great for representation.

Ce partage collectif pour en arriver à une banque de ressources a permis aux participant.e.s de référer aux collègues des ressources qui ont été utiles dans leur pratique d'enseignement ou de recherche, puis même de trouver des ressources à partir du besoin exprimé par un participant. Ainsi, l'ensemble de ces ressources peut aider à mieux soutenir les étudiant.e.s et à soutenir le développement professionnel en enseignement par l'utilisation de la vidéo.

QUESTIONS 5 AND 6

We then moved on to discuss questions 5 and 6 related to the particular rewards and challenges related to using video in the field of mathematics (education) and to the uses of video once we return to in-person teaching:

- Question 5: What are some of the particular rewards and challenges related to using video in the field of mathematics (education)? / Quels apports et défis de l'utilisation de la vidéo sont particuliers au domaine de l'enseignement-apprentissage des mathématiques ?
- Question 6: Which aspects of video would we like to develop further once we return to in-person teaching? / Quels aspects de la vidéo voulons-nous continuer à développer avec le retour de l'enseignement en présentiel ?

To begin exploring question 5, participants hunted for an example of a video that illustrates some particularities of the field of *mathematics* education. Participants also described characteristic features of their examples. Below are the main ideas that emerged along with several video examples.

Video can be used in a 'traditional' (i.e., "chalk 'n talk") way to replace lectures. A classic example is *Khan Academy* (https://www.khanacademy.org/). Here is a video where powers of 10 are introduced: https://monurl.ca/powers10. One procedure is shown repeatedly with several examples. In the following video, Eddie Woo gives an introduction to calculus: https://monurl.ca/introcalculus. He is entertaining and informative while standing in front of a white board and jotting down some of his ideas. These are both examples of how video can be used to introduce a topic and/or to give an overview of a topic in mathematics. Another classic example of 'chalk and talk' is Herbert Gross from MIT's complex variables: https://monurl.ca/complexvariables. Herbert draws analogies, builds new ideas on top of old ideas, and uses many graphs.

Video can also be coated in sexy graphics but rigid, without context, and show only one way of doing mathematics. These types of videos are often used with elementary school children. As an example, on day two of our working group, the following video was the top result when searching YouTube on the general topic of 'Math Videos' and sorting by number of views: https://monurl.ca/additionad. It is an ad for a math game for kids with 214M views in under a year!

Video can support the visualization of mathematics; it can be used to animate, illustrate, and provide a dynamic environment for understanding math (e.g., https://monurl.ca/essencecalculus). Other, more simple, examples include dynamically illustrating the Pythagorean Theorem or the formulae for volumes.

Video is a natural medium to demonstrate the connections between mathematics and art: one can easily incorporate theatre, humour, music, beauty, …. There are several such examples on the following edutopia website: https://monurl.ca/edutopiafilmfestival. As yet another example, the following video is visual in nature and could be used in a classroom setting to generate discussion using questions such as "What do you see?", "What is happening?": https://monurl.ca/naturebynumbers. In other words, video can be used to 'provoke' discussion in the classroom. Another classic example is the movie *Flatland*: https://monurl.ca/flatland. Finally, videos such as these can be used to change the nature of conversations in the mathematics classroom. For example, after viewing these videos, we can challenge our students to ask questions rather than to give answers. Here is one final example which incorporates many ideas into a very short video. The following video motivates the importance of feedback control systems and explains in an accurate but fun way what control systems are: https://monurl.ca/automation.

La vidéo peut donner une explication qui s'appuie sur le sens et / ou l'expérience des élèves, par exemple : https://monurl.ca/proportionnalite. La vidéo peut aussi donner, à l'aide de visuels, le sens que les mathématiques sont partout autour de nous.

Video gives a platform for providing context to mathematics: historical context, cultural context, or context to the problems themselves. Furthermore, video can be used to represent diversity in mathematics. For example, it can show a variety of mathematicians of different race and ethnicity; it can show cultural diversity in mathematics; or it can show multiple perspectives and understandings from different voices. Finally, video can be used to provoke thinking through the spreading and sharing of ideas. For example, Niral Shaw asks "How much mathematics education should there be?": https://monurl.ca/howmuchmathed. Does more mathematics education lead to more equity and justice or does it lead to McDonalds on Mars and racism on Jupiter?

In mathematics education, in order to better support our students, it is important to have access to their *thinking*. Video can be used to observe students' gestures and/or doodling while making sense of mathematics. These drawings and gestures can then be analyzed in an attempt to understand student thinking. When working with pre-service and in-service teachers, these videos can be a powerful tool to initiate discussion and to reflect on possible teacher moves to best help students.

Let us now move on to question 6 where we discussed the aspects of video we would like to develop further once we return to in-person teaching.

Many of the ideas that emerged during these conversations are summarized in the following two concept maps (Figures 5 and 6).

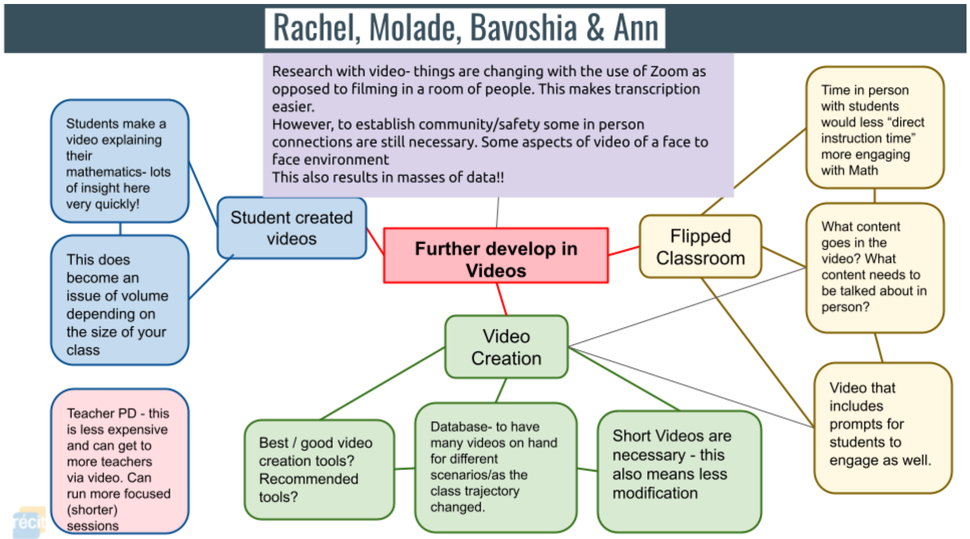

Figure 5. Concept map: Aspects of video we would like to develop further

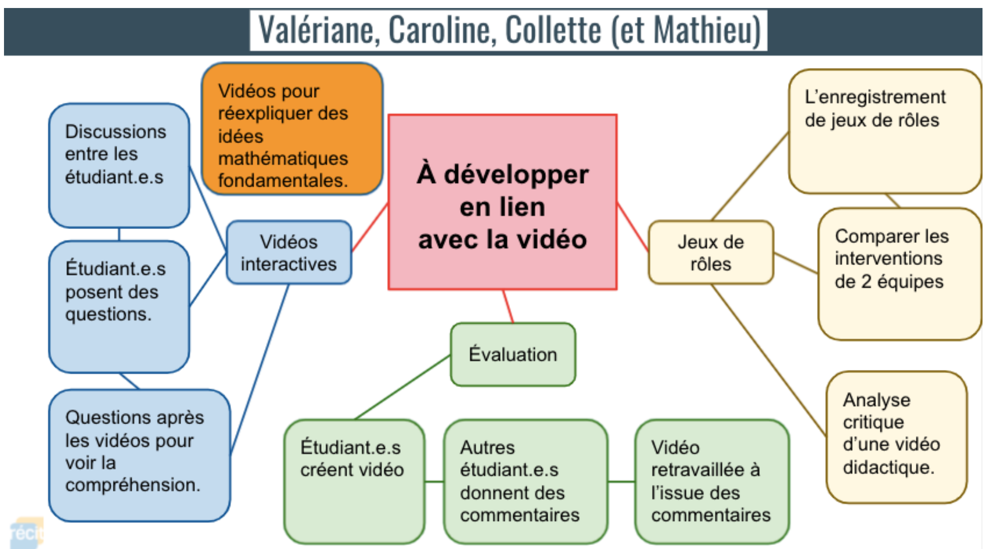

Figure 6. Réseau de concept : aspects à développer en lien avec la video.

Several Working Group members expressed a desire to have their students create more videos. When students explain their thinking, it is easy to assess their understanding of mathematics. Also, these videos can be interactive. Students can ask questions, respond to questions, and comment on each other's ideas. Video can then be improved / remade based on feedback received. Taking this design one step further, role playing can be incorporated into the process by asking students to assume certain positions.

Participants would like to improve the videos they have already made (often in a hurry). They would like to build up their video banks, especially of short videos that explain fundamental concepts in mathematics. This would allow teachers to customize viewing depending on the needs of a particular class and/or to individuals within a particular class. Some participants would like to experiment with using video to help with consistency in large multi-section courses. Participants would like to insert more breaks, questions, and prompts in videos to ensure that their students are watching actively. Participants would also like to use more videos to generate discussion in the classroom, especially when working with pre-service teachers.

Working Group members also expressed a desire to continue exploring the 'flipped classroom' and / or blended learning model by asking their students to watch videos prior to class. This would allow for more time to engage with mathematics during meeting time.

A few participants would like to further explore using video for teacher professional development. Video is less expensive (no need to rent physical space, to provide food) and takes less time (no need to travel) than in-person workshops. Professional development can be broken down into shorter, more focused sessions and can ultimately reach more teachers.

Finally, some members of our working group asked: "What if we do not return to in-person teaching?"

QUESTIONS 7 ET 8

Finalement, nous avons proposé aux participant.e.s de se projeter dans l'avenir en sous-groupes, pour réfléchir à la place voulant être donnée à la vidéo dans l'enseignement et la recherche, en choisissant au moins une de ces questions :

- Question 7: Quels sont les défis à considérer lors de l'utilisation de la vidéo dans la pratique et dans la recherche à venir ? / What are some of the challenges to consider in terms of using video as a tool for future research and practice?
- Question 8: Quelles questions de recherche peut-on envisager pour nous aider à mieux comprendre les apports et défis de la video ? / What research questions can we ask to help us better understand the rewards and challenges related to the use of video?

En ce qui concerne les défis à considérer lors de l'utilisation de la vidéo dans la pratique, il y a des défis déjà relevés par les expériences vécues des participant.e.s (voir question 2) qui pourraient demeurer des défis pour l'avenir. De plus, d'autres défis ont été soulevés, tels que des limites technologiques. En effet, il est parfois complexe de partager des fichiers vidéo de grande taille et de recevoir le soutien informatique à l'université pour répondre aux besoins techniques.

Il y a aussi des défis liés à la confidentialité lorsqu'on diffuse des vidéos où l'on voit des personnes : il faut donc s'assurer d'avoir les droits de diffuser ces productions, en respect de l'éthique et des droits d'auteur. D'ailleurs, lorsque ce sont les personnes étudiantes qui conçoivent les vidéos, plusieurs d'entre elles ne sont pas à l'aise de se voir en vidéo ou d'accepter que d'autres personnes les voient, ce qui peut poser obstacle pour analyser et commenter les productions vidéo.

Un autre défi est de ne pas savoir avec certitude l'utilisation que les étudiant.e.s font des vidéos conçues pour le cours. De plus, un enjeu pourrait être le trop grand recours à la vidéo dans l'enseignement ; si la majorité des apprentissages des étudiant.e.s se réalise sous forme de vidéo, un lacement pourrait survenir.

Dans l'optique de produire de courtes vidéos, il faut être en mesure de trier les contenus et les exemples à présenter, ce qui constitue un défi en soi : quoi aborder et sur quoi mettre l'accent en un temps restreint, de manière à engager les étudiant.e.s dans une réflexion didactique / mathématique.

En ce qui concerne les défis à considérer lors de l'utilisation de la vidéo dans la recherche, la collecte de données sous forme de vidéo amène des enjeux considérables. Par exemple, s'il s'agit d'entretiens où les participant.e.s de la recherche gardent leur caméra éteinte, l'analyse en est affectée étant donné que le langage non verbal nous procure habituellement des informations importantes en didactique des mathématiques. Un autre défi serait de devoir gérer la grande quantité de données produites par des enregistrements video ; les données sont riches, mais le traitement et l'analyse sont alors complexes et coûteux en temps.

Pour la question 8, les participant.e.s ont aussi envisagé diverses questions de recherche pour nous aider à mieux comprendre les apports et défis de la vidéo. En voici quelques exemples :

- How will videos change teaching and learning and research?
- Comment la pandémie et l'utilisation des vidéos ont changé les croyances didactiques ? Est-ce que ça affecte les pratiques ?

- Considering the large amount of data from video, how can we benefit from this affordance for broadening the research spectrum?
- Quelles sont les différences entre la compréhension des étudiant.e.s qui regardent les vidéos ou en personne ?
- Comment est-ce que des évaluations par vidéo changent la démonstration de compréhension des étudiant.e.s comparé à l'écrit ?
- Quels sont les enjeux pour créer des vidéos efficaces (durée, quantité d'information visuelle, …) ?
- Quel type d'engagement cognitif peut-on susciter avec la video ? Par quels moyens ?
- Quels sont les apports perçus des personnes qui visionnent les vidéos (à croiser avec les autres regards) ?
- Are there age groups that use videos to learn more than other age groups?
- Quels sont les éléments qui influencent à avoir recours ou non aux vidéos (même dans un retour à la normale) ?
- Quel est le rôle des vidéos sur les pratiques en stage (analyse réflexive, développement professionnel) ?

Ces questionnements sont variés et portent sur plusieurs objets de recherche complémentaires en lien avec la vidéo. On se questionne notamment sur ce qui fonctionne ou non par la vidéo, tout comme dans les travaux de Fiorella et Mayer (2018).

Certains participant.e.s anticipent un retour à la normale et se questionnent sur ce qu'il va rester du recours à la vidéo quand les personnes enseignantes/chercheures auront la possibilité de revenir à leur pratique habituelle. À ce sujet, certains travaux de recherche (par exemple, Carrillo et Flores, 2020) suggèrent une certaine pérennité pour des méthodes d'enseignement en ligne notamment l'utilisation de la vidéo. Est-ce que la vidéo gardera une place privilégiée ou sera-t-elle plutôt considérée comme un complément de formation, comme le suggère Duvillard (2017) ? Est-ce que la vidéo sera utilisée autant en formation initiale en mathématiques (et à l'enseignement des mathématiques, tel qu'étudié par Towers, 2007) qu'en formation continue (dont le potentiel de développement professionnel a déjà été investigué dans divers travaux tels que Borko et al., 2011; Coles, 2013, 2019; Jaworski, 1990; Sherin et van Es, 2009; Van Es et al., 2014) ? Quel est l'apport de concevoir ses propres vidéos en enseignement, comme l'a questionné Poellhuber (2017) ?

Certains travaux suggèrent des avantages substantiels de l'enseignement à l'aide de la vidéo. Les résultats de Expósito et al. (2020) suggèrent une amélioration de la réussite universitaire. Toutefois, plusieurs des travaux déjà menés en lien avec la vidéo ne sont pas spécifiques en didactique des mathématiques et plusieurs avenues gagneraient donc à être explorées dans ce domaine spécifique.

CONCLUSION

Working Group participants used, worked, and experimented with video more during the 2020–2021 academic year than ever before. This was a consequence of the sudden shift to online teaching and learning due to the COVID pandemic. Participants experienced new challenges and learned much along the way. This working group was an opportunity to look back and reflect on the past year in order to prepare and look forward to the year(s) ahead. Throughout the three days of CMESG, participants shared their struggles and experiences, shared resources, and asked questions related to the use of video in the field of mathematics education. Through this work, participants found inspiration in terms of curating and creating their own videos, ways to better support their future students, to reach out to practicing teachers, and working with large amounts of video during research.

Here are some main ideas that the participants remember most from this working group about video (source: monurl.ca/cmesgvideo):

- piquer la curiosité de l'apprenant.e par une vidéo sans mot;
- façon d'engager les étudiant.e.s de manière originale et flexible;
- videos can be used in many ways, for examples as resources for the classroom, as resources to supplement the classroom or as a support outside of classroom time;
- to add (cultural) context to math video in order to avoid explanations only based on procedural techniques;
- faire ressortir des questions de recherche, telles que : Comment la pandémie et l'utilisation de vidéo a changé les croyances didactiques ? Est-ce que cela a affecté les pratiques des enseignant.e.s ?;

- utilisation de la vidéo dans l'évaluation des étudiant.e.s en formation initiale à l'enseignement des mathématiques, par exemple pour donner de la retroaction;
- importance to bring discussions around possibilities, challenges and pitfalls regarding video;
- use video in research as a way to empower students to tell their own stories in their own ways;
- video can contribute to activist/participatory research by allowing researchers, mathematics educators and students to advance in their mathematics education journey.

The role video plays in our working lives is constantly evolving. The members of Working Group C recommend that, in the years to come, we check in with members of the CMESG community to survey the ways in which they are using video in their research and practice and the big questions they have asked and answered regarding the use of video in the field of mathematics (education) teaching, learning, and research.

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APPENDIX

Ce dialogue rapporte assez fidèlement les échanges entre Scosha Merovitz et Mathieu Thibault lors de la préparation du groupe de travail :

S : Hi Mathieu!
M : Allo Scosha !
S : So, we accepted to co-lead a CMESG Working Group, but so much has happened over the past year…. Our working lives have changed so much…
M : Tu veux dire à cause de la COVID ? Du travail à distance ?
S : Yes, I don't want to just pretend this whole year hasn't happened. I want to *talk* about it. *In* our working group.
M : En fait, notre sujet, c'est la vidéo comme outil pour la recherche et la pratique.
S : Well, did you work with video a lot over the past year?
M : Oui, j'ai conçu plusieurs capsules vidéo pour mes étudiant.es et ils/elles en ont conçu aussi.
S : So, in our Working Group, we could talk about *our own personal experiences*, what *each* of us has done with video over the past year.
M : Oui, on pourrait aborder les usages de la vidéo que nous avons expérimentés et dans quelles intentions.
S : Exactly. And what were some of the major challenges we encountered?
M : Oui : quels ont été nos grands défis ? Et nous devons aussi penser à nos étudiants : comment pouvons-nous utiliser la vidéo pour mieux soutenir nos étudiants ?
S : Yes: How have we or how can we use video to better support our students? And then there's our research. And our work with teachers, math consultants, …. How can we use video to support research and professional development?
M : Oui. Tout ça est très intéressant. Mais nous n'avons pas parlé de mathématiques… et c'est un colloque d'enseignement des *mathématiques*.
S : Yes, of course, it's understood that we're all looking at this from a perspective of mathematics and mathematics education.
M : Oui, mais on pourrait poser les mêmes questions à n'importe quel enseignant ou chercheur…
S : Oh, you mean *in the field of mathematics*, what's special about the role that video plays in *mathematics* teaching, learning, and research?
M : Exactement : qu'est-ce qu'il y a de particulier dans le domaine de (la formation à) l'enseignement-apprentissage des mathématiques ?
S : That's an excellent question; we should definitely discuss that with our colleagues!
M : Quoi d'autre veut-on aborder ?
S : Yes, what else? Well… most of us have used video so much more over the past year. And most of us would like to go back to teaching in person. But we don't just want to throw away this entire year…. What have we learned? What aspects of video would we like to keep using once we return back to in-person teaching?
M : Oui, continuer à utiliser *et* continuer à développer. Quels aspects de la vidéo voulons-nous continuer à *développer* avec le retour de l'enseignement en présentiel ? Et quels seront nos grands défis à considérer lors de l'utilisation de la vidéo dans la pratique et dans la recherche à venir ?

S : Yes, it's always good to consider our major challenges. Wow, we've learned so much over the past year! But there's even more to learn about using video in our professional lives…. We should really take the time to ask ourselves some important questions…
M : Oui, par exemple, quelles questions de recherche peut-on envisager pour nous aider à mieux comprendre les apports et défis de la vidéo ?
S : Perfect: what research questions can we ask to help us better understand the rewards and challenges related to the use of video? What else…
M : Il ne reste plus qu'à attendre les participant.e.s dans notre groupe de travail !
S : Yes, I look forward to sharing perspectives in this working group.

HOW CAN WE BE CREATIVE WITH LARGE CLASSES?

COMMENT COMPOSER AVEC LES GRANDS GROUPES ?

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INTRODUCTION

Large classes are the reality in post-secondary education. Like your uncle at Christmas, they were not invited, nor overly welcome, but now that they are here, they will be staying. À n'en point douter, les grands groupes font désormais partie du paysage de l'enseignement postsecondaire.

Whether it takes the form of a calculus class of 200 students, or a math education or a problem solving class with 60 students, every large class undeniably comes with its own set of constraints and challenges. Rather than considering those constraints and challenges as sound reasons not to embark, we tried in this working group to look at them as potential levers for creativity. We also looked for opportunities that might be afforded by large classes.

Dans ce groupe de travail, nous avons donc considéré comme donnée de départ la présence de grands groupes au postsecondaire et nous nous sommes penchés sur les questions suivantes :

1. Quels types d'activités d'enseignement et d'apprentissage peuvent être efficaces dans les grands groups ?
2. How and to what extent can technology help?
3. La prise en compte des grands groupes pourrait-elle améliorer la formation que nous offrons aux étudiants ?
4. What viable forms of assessment might support our students' learning?
5. Dans quelle mesure les grands groupes mettent-ils à l'épreuve nos théories didactiques ? Pourraient-ils nous conduire à les enrichir ?
6. To what extent does the consideration of online classes change our reply to any of the above questions?

En somme, nous avons cherché à explorer ensemble où il est possible de faire preuve de créativité pour relever les défis qui caractérisent l'enseignement aux grands groupes. Nous avions la chance d'avoir dans ce groupe des passionnés de l'enseignement des mathématiques, qui avaient eu à composer avec de grands groupes et qui étaient prêts à jouer le jeu avec nous.

With the online format of the CMESG / GCEDM 2021 meeting and despite the less-than-large size of our working group, we applied ourselves in emulating some of the conditions and strategies that would be applicable to large classes. This was particularly visible with the different technology tools that we used throughout the three sessions.

SESSION 1—AN ORIENTATION

We started our first day with (a not so) traditional participant introduction. The participants were successively 'deployed' in randomly generated breakout rooms of growing sizes—2, 3, 4, …—where they were to introduce themselves. After three iterations of this meeting scheme, they then had to figure out, in yet another breakout room and using *JamBoard* (https://jamboard.google.com/) to share work, whether it was possible to meet everyone in the working group in this way, and if so, how many iterations would be necessary for that. Besides providing an opportunity to ask deep modelling questions, formulate assumptions and play with the math on an online communication tool, this was a way for each participant to get a feeling of a comfortable number of students in a breakout room for working on a math problem (answer: 4 or 5).

To allow sharing of perceptions and experiences with large classes, while playing with the idea that constraints can be a lever for creativity, we each engaged in writing a haiku poem that summarized our personal vision of teaching to a large class or attending one. As soon as a haiku was finished, their author would post it on a shared *Padlet* (https://fr.padlet.com/), thereby contributing to a dynamic flow of creativity (Figure 1).

Figure 1. Padlet of haiku poems.

The collection revealed an interesting array of emotions, from despair to enthusiasm, with much questioning. A word cloud of these poems showed a strong "How" at its center, trying to link the "you" "to" the "I" (Figure 2).

Figure 2. Word cloud from haiku poems.

To define more precisely the scope of our reflection and the problems to be solved, participants were then asked to characterize in small teams what these large classes entail or mean to them, in terms of the students, the teacher and the math, inside and outside class. That was done by filling a larger version of the table below, on team slides of a shared *Google Slides* presentation (docs.google.com/presentation).

	Students	Enseignant	Math
En classe			
Outside class			

The following summarizes the collective reflection that we assembled from the tables filled by each team.

Math courses associated with classes with a very high number of students are often either service or survey courses, which welcome a wide array of students, few of which are mathematics majors. The impact of class size was said to be potentially influenced by the diversity of students' educational, cultural backgrounds, goals, interests, as there can be a wide range of students, especially in the types of courses that are given to very large groups. The types of students who take a large class are believed to be different from those who enroll in smaller math classes, encountered later on for those who pursue their study of mathematics. The type of instruction common in large classes tends to also differ from that of a small class. As one team noted, the teacher of such a large class aims at speaking to an 'average' student without necessarily knowing that student; the perception is that they often go for the lowest common denominator.

Yet, from the teacher perspective, for a class to be considered large may depend on the setting, especially concerning assignments and the grading they require outside class. A large class could start at 20 students if individualized assessment is involved, 60 students if a common assignment requires personal reflection, or more than 100 or 150 if the assignment is uniform and 'objective', with a greater presence of multiple choice questions, for example. Within the classroom, if the method of teaching is just 'stand and deliver', the feeling is that size does not matter, really.

Du point de vue des étudiants, ils se sentiront dans un grand groupe s'ils perçoivent être entourés de trop de personnes qu'ils ne connaissent pas ou si l'enseignant ne se souvient pas de leurs noms. Cela peut conduire à un sentiment d'invisibilité ou à la crainte d'être observé et jugé, et, dans un cas comme dans l'autre, diminuer la volonté d'interagir et de poser des questions en classe.

Paradoxically, such retreat may alleviate the weight on the teacher who, despite wanting their students to engage in class, may find themselves overwhelmed by the number of questions that can arise within a large group of students.

The use of the chat in a video conferencing environment may serve as valuable help, as some students may be willing to have peers provide quick answers to their questions, or to experience some delay in getting one from the teacher.

With practically no time for individual attention in class, some teachers may (consciously or not) move into teaching to an audience rather than to individuals, and their courses will tend to be tightly structured and to adopt a top-down approach. They will look for ways to make their delivery of the content livelier and more interesting, as ‘performing’ becomes a key element of their teaching. Other teachers will rather look for ways to facilitate individual or group work. They may present starting problems or examples, from where active learning can take place, in a form adapted to the size of the class, e.g., use of small teams or think, pair, but *not* share.

Outside class, the interaction between students and teachers also is likely to be affected by the size of the class. On campus, an asymmetric relation develops as students can recognize the teacher but the reciprocal is not true. Some participants suggested that the physical distance that separates them in a large class may have a student see the teacher in a large class not quite as a fully fleshed human, but more as an abstract arbiter of knowledge or even as faceless and calculating academic police. Some students may feel more intimidated to reach out to the instructor, but others may not; the latter might prove sufficient in number to generate a volume of emails or a number of visits to the teacher’s office that both exceed their capacity to answer them all. The interactions therefore tend to become more transactional, as the teacher struggles in giving students individual time while trying to be fair to all students. In addition, with the increased administrative and organizational work caused by large classes (e.g., Teaching Assistant coordination), more time is spent coordinating the course and less time is available to think how to teach mathematics. In addition, as the stakes may feel higher for these large classes, it may seem too risky to experiment with the teaching, especially in multi-section, or service, or introductory courses. Consequently, teachers tend to be more protective of their time.

Il peut être plus difficile de créer un sentiment d’appartenance à une communauté au sein d’un grand groupe. En revanche, avec un plus grand nombre d’étudiants, il y a également plus de possibilités pour les étudiants de trouver un groupe de soutien, surtout si l’on s’applique à créer des communautés plus petites avec des regroupements d’étudiants. Les enseignants veulent que leurs élèves participent en classe et s’engagent dans leurs apprentissages, et des participants ont fait valoir l’importance de trouver des moyens d’y parvenir dans les grands groupes.

These reflections led well to what we wanted to address on our second day.

SESSION 2—TEACHING AND LEARNING

The session started with a summary of the characteristics given the day before, seen as either constraints that limit possibilities (less student–instructor interaction, greater uniformity, more rigid curriculum, rote assignments conducive to efficient grading) or affordances indicating possibilities (potential for more student–mathematics interaction, for greater community development, for renewed forms of assignment). Moving from one view to the other may entail rethinking both the way we envision social interactions in teaching and learning and the use of time inside and outside the classroom.

> *Instructors generally favor smaller class sizes because it allows them to work closely and develop a relationship with their students. However, this reasoning does not consider learning that may happen either between students or even outside of the classroom.* (Ake-Little et al., 2020, p. 602)

To see how we could rethink these things in practical terms, participants were broken into small teams within which they had to identify a teaching context (i.e., a specific course given to a large class) for which they would need to be creative in order to help meet the learning goals that they believe ought to be met. Across the four teams, the following courses were selected: two classical mathematics service courses (*Calculus* and *Linear algebra*), an imagined survey course on *Representation and proofs for everyone* (which would welcome both high school and university students) and a course for prospective secondary teachers on *Digital technologies for mathematics teaching and learning*. Interestingly, the learning goals defined by participants for such courses, with 100 students or more in a classroom, were more often expressed as mathematical ways of thinking (e.g., learning to represent in different ways and to connect between representations) than as specifically related to some mathematical content (e.g., manipulating / evaluating limits derivatives).

We then had participants think of a task that would contribute to such learning goals. For that, we offered to turn to Chevallard's Anthropological Theory of the Didactic (ATD; Chevallard, 1998) as framework for situating the task within the institutional context where it is perceived as belonging to the practice. In ATD, a praxeology is described as being made of four components: a *type of task*, *techniques* for achieving it, a *technology* (in the sense of the discourse (*logos*) that justify the techniques (technê)) and the underlying *theory* that supports such discourse (Chevallard, 1998). We encouraged participants to identify the *moment of the study* where their activity would take place: first encounter with the type of task (where an embryo of a technique might emerge), exploration of the type of task and elaboration of a technique relative to that type of task, working the technique, justifying the technique, formalizing the theoretical organization, etc. We also made connections to Brousseau's *Theory of didactical situations* (Brousseau, 1986; Warfield, 2013), with the *action*, *formulation* and *validation* phases. Using that theoretical framework, we stressed the importance of organizing an appropriate didactic milieu, that would allow students to assess on their own the value of what they did, and to modify their action in adapting their strategy.

Ces idées ont été illustrées avec une tâche utilisée avec de futurs enseignants au primaire. En prenant pour milieu une règle en équilibre sur un crayon ou un simulateur en ligne de levier (https://phet.colorado.edu/sims/html/balancing-act/latest/balancing-act_fr.html) où l'on ajoute d'abord quelques masses à gauche du point d'appui, l'objectif est de trouver différentes manières de ramener le système à l'équilibre en ajoutant des masses à droite (*action*). Travaillant en équipe, sur papier ou à l'aide d'un outil de présentation en ligne (ex. Google slides), les étudiants sont ensuite invités à dégager une régularité à partir des différentes solutions et à l'exprimer mathématiquement (*formulation*). Ils peuvent alors retourner dans le *milieu* et *valider* leur modèle naissant avec de nouvelles configurations.

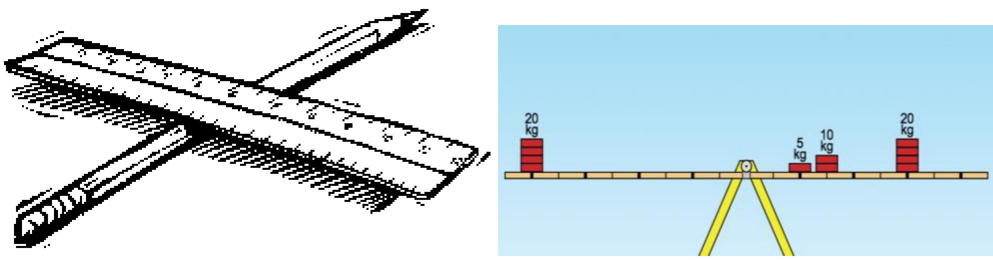

Figure 3. Image on the right from *Action d'équilibrage* (https://phet.colorado.edu/sims/html/balancing-act/latest/balancing-act_fr.html).

For the large class teaching context that they had defined within their team, our participants also went for tasks where students would explore, express, justify, validate among themselves, as they typically relied on the possibility of breaking such class into small teams.

The team on the *Representation and proofs for everyone* course was keen on having students discover theorems on their own, build their own representations and proofs and appreciate the different ways to prove the same thing. They proposed a task where their students would explore "circular number" (or modular arithmetic) with the simple following prompt: "What if we had only 5 digits (0, 1, 2, 3, 4) and 4 + 1 = 0 ?" With this task, they envisioned that the students would create their own jamboard, where the instructor could pop in and comment (but not tell), and that there would be a time for bringing the whole class together for sharing and discussing ideas, particularly looking for connections between the different representations used.

The team with the course *Digital technologies for mathematics teaching and learning* for future secondary teachers went for an "open middle" (Kaplinksy, 2019) type of problem:

> *Using the numbers from 1 to 30, at most one time each, fill in the 6 boxes below to create a system of linear equations for which (3, 2) is a solution.*
> □ *x* + □ *y* = □
> □ *x* + □ *y* = □

Their view was that future teachers working in teams on such a task would be gaining firsthand experience of solving these algebra tasks (that are receiving growing attention), and just like the students, they would go deeper into their algebraic structure. It could also make them more aware of possible misconceptions. The task could also be extended to developing small programs (e.g., in *Desmos*, https://www.desmos.com/) to solve similar problems.

The team on the *Calculus* course aimed with their task at getting students to relate piecewise functions, continuity, and discontinuity to the 'real world' and having them develop comfort with representing and interpreting piecewise functions in multiple modes (graphical, physical, etc.). The *formulation* task that they designed was twofold. Given the definitions and multiple representations of concepts of limits and continuity, students would first look at a piecewise function with jump discontinuities and think of a real-life activity that might be described by the function, and analyze its continuity. Then, they would think individually about another example of a discontinuous function from the 'real world', draw the graph of their function and provide its symbolic expression. Students would then be paired with their neighbor, and later grouped with another pair, to compare their functions, discuss their graphs, and choose one to be posted on a Padlet.

Finally, the team on the *Linear Algebra* course gave themselves the goal of having students understand eigenvectors through building and exploration of a mathematical model. They went for a version of the famous Bernadelli beetle problem:

> *In 1941, H. Bernadelli explored a beetle population that consists of three age-classes. One-half of the females survive from year 1 to year 2, one-third of the females survive from year 2 to year 3. The females reproduce in their third year, producing an average of six new females. After they reproduce, the females die. We're interested in the long-term behavior of the population. So let's see if we can answer the following questions. Suppose that in a given year there are 60 beetles age 1 year, 60 beetles age 2 years and 60 beetles age 3 years. What will the age distribution of the beetles look like in the following year? How about 5 years from now? How about 10 years from now?*

By working in teams with a given set of data, and exploring its behavior for different periods, students would start to see a pattern emerge and might build a matrix to capture the model. Once they have the matrix, a reflection on the long-term behavior of the system could prepare the class for ideas around dominant and subdominant eigenvectors.

The level of engagement in task design had our group pursue its fine-tuning on Day 3. This was done in conjunction with the last topic that we wanted to explore.

SESSION 3—ASSESSMENT

To address assessment, we started by referring to the notion of constructive alignment (Biggs, 2003). The idea is to look for coherence between intention ("I want my students to be able to do *x*"), activity ("Students engage in doing *x*"), and assessment ("How well are students able to do *x*?"). As one participant reacted, this could lead to favoring a focus on techniques and procedures. Yet, if the intentions (or learning goals) include general mathematical ways of thinking (reasoning, proof, modelling, representing) or *habits of mind*, such as "seeking and developing new ways of describing situations" (Cuoco et al., 1996, p. 376), then such alignment should also benefit student's learning, analysis and capacity to adapt.

This could be seen with the work done by the team on *Representation and proofs for everyone*. As they felt the "need to break the thinking that problems have *1* answer and *1* way to solve", they tried to trigger such reflection with questions like "*Do you like your representation? What does it show about the situation? What does it not show about the situation? How does your representation compare / relate to those of others?*" and then proposed a self-assessment in the form of a journal entry, where students would answer the following: "*What mathematical knowledge do you feel you have developed through this experience (combined or not with other experiences)? Is there a representation that a group has used that you did not understand?*"

Le recours à un journal de bord fut aussi l'un des moyens envisagés par l'équipe sur les Technologies numériques pour faire réfléchir les futurs enseignants sur la façon de faire évoluer les conceptions des étudiants. On a estimé qu'en ayant travaillé sur de tels problèmes « à milieu ouvert », les étudiants de ce cours seraient en mesure de créer une plus grande variété de problèmes sur les systèmes d'équations linéaires et de développer une vision plus claire du moment et de la façon de les utiliser pour permettre une telle évolution.

The team on *Calculus* stayed very closely aligned to their learning task with the test question that they designed for assessment. From the picture of a graph of a function representing the amount of electricity consumed throughout a typical day in a typical family home, students would have to determine whether or not the function is continuous or discontinuous at different points in time, label the type of discontinuity (if applicable), explain why using definitions from class, and interpret the meaning of certain discontinuities in the context of the problem. Their choice of assessment confirmed their specific goal of having students develop "comfort with representing and interpreting piecewise functions in multiple modes (graphical, physical, etc.)."

The team on *Linear Algebra* went for a greater challenge in terms of transfer by having their students move from a population model to a geometric one in playing with eigenvectors and eigenvalues: "*Given a letter, and a transformed letter (through a linear transformation): Is the shape getting smaller / larger? What does that tell you about the eigenvalues? What is the transformation? What is the result if we apply the same transformation 20 times? What is a shape that would not change under this transformation?*"

Still, as was expressed by one of the participants in our framapad collective summary of the working group (framapad.org), one may wonder "*what freedom there is for creative assessment when one needs to grade a thousand of them*". That was possibly best answered by another participant on the same document: "*Use more formative, self-assessment so that you limit how much you have to grade while also giving the students an understanding of your expectations in the limited grading that you do.*" This sound approach may not completely alleviate the legitimate concern, also expressed in the comments, that "*large classes [may not be] conducive to individual assessment that demonstrates a deeper level*".

CONCLUSION

En résumant nos travaux et discussions, plusieurs participants du groupe de travail ont déclaré que les séances leur avaient permis de mieux apprécier les complexités et les tensions associées à l'enseignement dans un grand groupe, de même que la nécessité et la possibilité d'y être créatif.

Participants often pointed to the use of online resources as a powerful support for opening the space for such creativity, favoring active learning in the context of large classes and making the experience both enjoyable and profitable. Random breakout rooms (as in Zoom) were described as fun. Jamboard was perceived as an efficient way to share work, online and even in person, that made it easy for the instructor and groups to scan through, group together, etc. Padlets appeared as nice tools to share ideas. Polls and google forms can gain in relevance with large numbers of students. Having technology systematically present also encourages the use of specialized mathematics applications (Wolfram Alpha, Desmos, GeoGebra, etc.) and the design of better assessment tasks.

Some questions remained on the limits on class size and resource availability for reaching the learning goals. When, and under which conditions does a large class become too large to provide the learning experience that students should have? As the reality of higher education is complex and evolving rapidly, we will not attempt to provide a definitive answer to these questions. We rather refer readers to the recent review done by Jerez et al. (2021) who acknowledge, like others before them, that "exploring the real impact of large-group activities [is] a difficult task as a larger group of students implicates more variables influencing the effectiveness of teaching methods" (p. 157). Nevertheless, they ended up identifying five factors that seem to facilitate the effectiveness of large-group activities: interactions (student–student, student–teacher), active teaching and learning methods, classroom management, students motivation and engagement, and use of online teaching resources. These factors include some of the paths that we tried to explore in our working group.

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RETURNING TO OUR ROOTS: EXPLORING COLLABORATIVE POSSIBILITIES FOR RESEARCH AND TEACHING IN MATHEMATICS AND MATHEMATICS EDUCATION

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In our working group description, we reminded CMESG members that the creation of our organization is rooted in a desire for mathematicians and mathematics educators to work together (Coleman et al., 1977, p. ii). Thus, one goal of this working group was to 'return to our roots' and explore these past/present/future collaborations: their challenges and complexities, as well as their successes and outcomes. This was done by sharing, and reflecting on, the participants' own experiences with collaboration between mathematicians and mathematics educators, addressing, for instance, the initiation of the collaboration, the nature of the collaboration, emerging issues and concerns as well as results.

Dans la description de notre groupe de travail, nous avons rappelé aux membres du GCEDM que la création de notre organisation est enracinée dans un désir des mathématiciens et des didacticiens des mathématiques de travailler ensemble. Conséquemment, un des objectifs de ce groupe de travail était de « retourner à nos racines » et d'explorer ces collaborations passes / présentes / futures : leurs défis, leurs complexités, ainsi que leurs réussites et leurs résultats. Cela s'est fait d'abord en partageant et en réfléchissant aux expériences personnelles des participants en ce qui a trait à la collaboration entre mathématiciens et didacticiens des mathématiques. Par exemple, nous avons abordé l'amorce de la collaboration et sa nature, en plus des problèmes, préoccupations et résultats qui en ont découlé.

A key focus of the working group was to break down barriers and stereotypes, and to build up possible collaborations, between mathematics educators and mathematicians. In the working group, we asked participants to reflect on and discuss how collaborations can bring something to both mathematics educators and mathematicians with respect to their own individual (and others') teaching and research, revealing how all perspectives contribute to the collaborations.

L'un des principaux objectifs du groupe de travail a été de questionner les barrières et stéréotypes, et de créer d'éventuelles collaborations entre les mathématiciens et les didacticiens des mathématiques. Le groupe de travail a cherché à réfléchir aux manières dont les collaborations peuvent être profitables aux deux communautés à la fois en ce qui concerne l'enseignement, tout en révélant comment les multiples perspectives contribuent aux projets collaboratifs.

In addition to reflecting on past and present collaborations, we dedicated time to imagining and forming new collaborations and to fostering new sparks of ideas for potential future collaborative projects. We asked questions about, for example, what makes a collaboration between mathematics educators and mathematicians different from a collaboration between mathematics educators and/or between mathematicians? How might barriers or stereotypes between the two communities be negotiated? What are some ideas for how collaborations can be initiated and / or sustained? What type of research and / or research topics call for/can be addressed by collaborations between mathematicians and mathematics educators? How do mathematicians and mathematics educators coordinate (potentially) different research agendas and cultures, knowledge, interests, purposes, goals, languages, and (internal / external) measures of 'success'?

En plus de réfléchir aux collaborations passées et présentes, nous avons consacré du temps à imaginer et à former de nouvelles collaborations, au sein et au-delà de la rencontre, ainsi qu'à favoriser de nouvelles idées pour de futurs projets collaboratifs. Nous avons abordé, par exemple, ce qui différencie une collaboration entre didacticiens des mathématiques et mathématiciens d'une collaboration entre didacticiens des mathématiques et / ou entre mathématiciens. Nous nous sommes également questionnés sur les differences / similitudes entre les deux communautés. Sont-elles mutuellement exclusives ? Comment aborder les barrières ou les stéréotypes entre ces communautés ? Que dire de la façon dont les collaborations peuvent être initiées et / ou maintenues ? Quels types de recherche et / ou sujets de recherche peuvent être abordés à travers des collaborations entre mathématiciens et didacticiens des mathématiques ? Comment les mathématiciens et les didacticiens des mathématiques coordonnent-ils des objectifs de recherche, des cultures, des connaissances, des intérêts, des buts, des vocabulaires et des mesures (internes/externes) de « réussite » (potentiellement) différents ?

Like the rest of the CMESG 2021 meeting, our working group was conducted virtually through Zoom in a condensed format, including three, two-hour meetings spread over the three days of the meeting. An outline of our six hours together is as follows in Table 1.

DAY 1: What does collaboration between mathematicians and mathematics educators look like?	DAY 2: What challenges arise in such collaborations?	DAY 3: Imagining an actual collaboration
• Introducing and positioning oneself and defining 'mathematician' • Recalling collaborations and what made them successful • Reflecting on the work of Artigue (1998): *Research in mathematics education through the eyes of mathematicians*	• To what meaning of the word 'Mathematics' do you subscribe to when you identify yourself as a researcher in your field? • Discuss Nardi's (2008) dialogues and explore possible tensions amidst collaborations	• Brainstorming research topics and forming collaborative groups • Working on Research Grant Proposal • Reflect on connections to Day 1 and Day 2

Table 1. Outline of three days.

Comme le reste de la réunion du GCEDM 2021, notre groupe de travail s'est déroulé virtuellement via Zoom dans un format condensé, comprenant trois réunions de deux heures réparties sur les trois jours de la réunion. Voici un aperçu de nos six heures ensemble :

JOUR 1 : À quoi ressemble une collaboration entre mathématiciens et didacticiens des mathématiques ? • Se présenter, se positionner et définir « mathématicien » • D'anciennes collaborations et ce qui a fait leur succès • Réfléchir au travail d'Artigue (1998) : *Research in mathematics education through the eyes of mathematicians*	JOUR 2 : Quels défis surgissent dans de telles collaborations ? • À quel sens du mot « Mathématiques » souscrivez-vous lorsque vous vous identifiez dans votre domaine ? • Discussions sur les dialogues de Nardi (2008) et exploration des tensions possibles lors des collaborations	JOUR 3 : Imaginer une collaboration • Discussions sur des sujets de recherche et formation de groupes de collaboration • Premiers pas vers un projet de recherche collaboratif • Réflexion sur les liens avec les jours 1 et 2

Table 2. Un aperçu de nos six heures.

In the following pages, we present a more detailed, participant-driven account for each of these three days as they were lived out during CMESG 2021.

Dans les pages suivantes, nous présentons un compte rendu plus détaillé pour chacun de ces trois jours tels qu'ils ont été vécus au cours du GCEDM 2021.

DAY 1

Our first activity was designed as a novel way for everyone to introduce themselves. Participants were asked to share their name and to use one of the following words/terms (or one of their own created words/terms that is not already on the list) to describe themselves: mathematics educator, mathematician, research mathematician, mathematics teacher, mathematics didactician, mathematics teacher educator, mathematics enthusiast, non-mathematician, or mathematics instructor. It was also open for them to share WHY they selected the term that they did. A Padlet (https://padlet.com/), entitled *Describing and Positioning Oneself* (see Figures 1 and 2), was created in advance of this session which had each of these words/terms as headings and then everyone was provided the opportunity to place their name under their preferred heading (or headings, as it turned out) while reading and reflecting on others' responses throughout the Padlet.

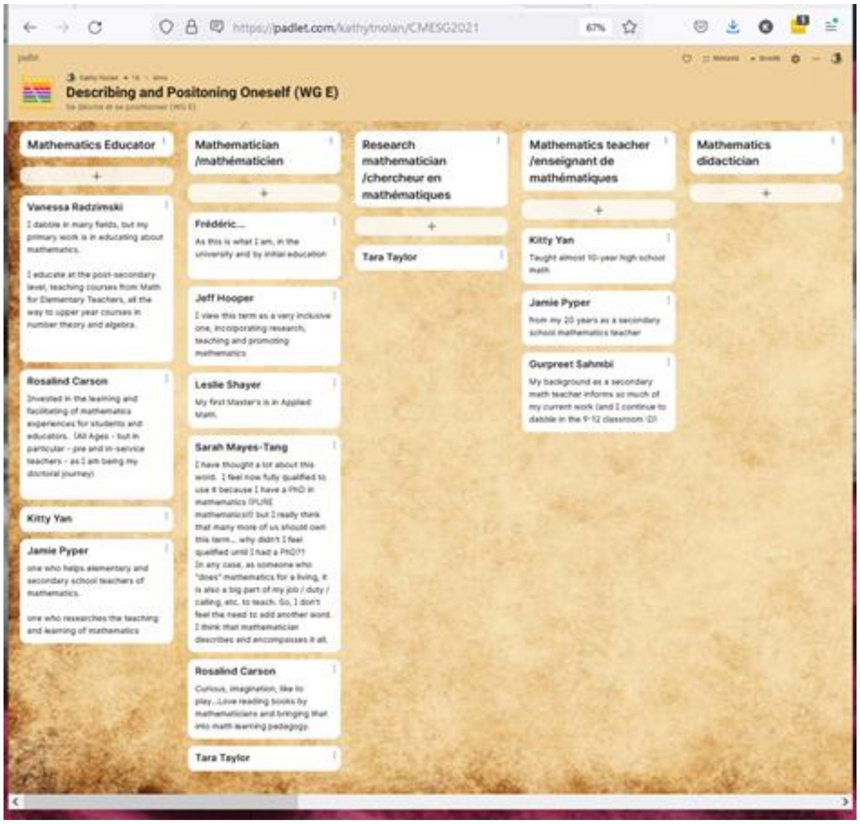

Figure 1. Describing and Positioning Oneself, Padlet Part (a).

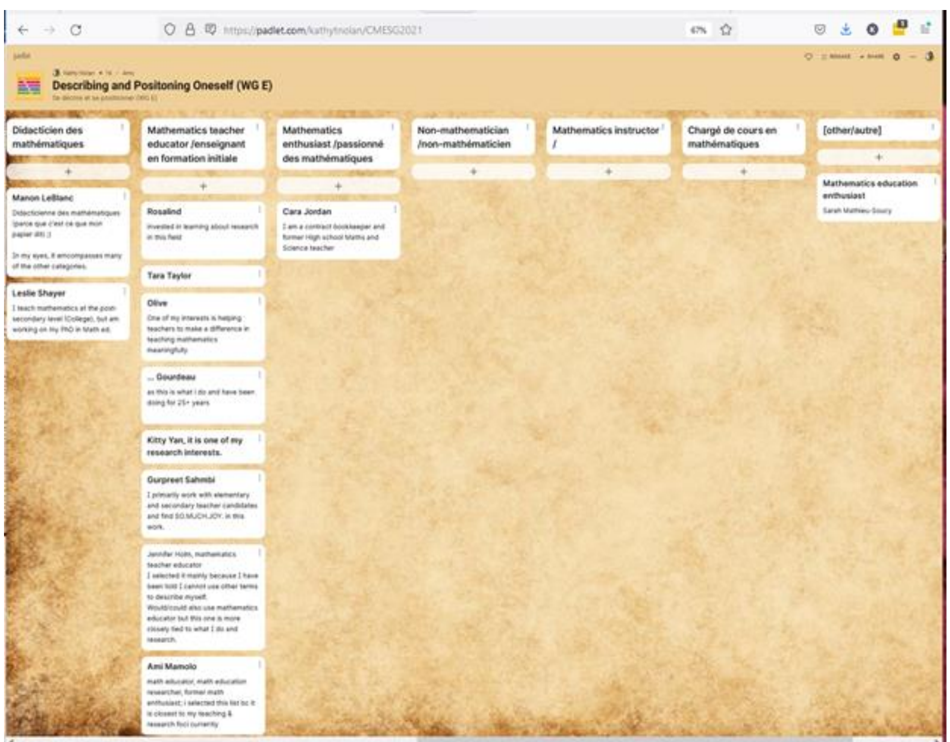

Figure 2. Describing and Positioning Oneself, Padlet Part (b)

Participants' reasons for their categorizations lead to rich discussions on how people perceived themselves based on life and work experiences. Looking closely at the entries in the padlet, it can be seen that most participants had at least dual or triple identities. At the same time, however, they acknowledged the tensions within themselves, and frequently with others, with regard to these identity labels. One participant, who positioned herself as a mathematics educator, shared a story of a recent encounter she had with a mathematician, where it was made clear that a mathematics educator is not a mathematician, leaving her feeling that the mathematician viewed her position as "less than." Others shared that such a hierarchy should not be perpetuated since there is so much to be learned on both sides, and that respect for what each brings to the collaboration needs to be present at all times. Without this mutual respect, participants noted, the creation of role 'silos' will continue, as well as very real fears of othering and of being judged. Some participants were surprised to hear that a mathematician could feel intimidated or fearful of mathematics education/educator, signaling that the fears could go both ways.

Following this informative discussion on participants' self-selected identities, we (the Working Group facilitators) then transitioned into the next activity by sharing a few details about a collaboration they each had with another person who, they decided, would be described using one of the other terms listed (a term not used to describe/position ourselves), and what made the collaboration successful (or not). We then placed participants in breakout room groups to share and discuss their own experience of collaborations, before returning to the larger room when we introduced the work of one key scholar (Michèle Artigue) who conducts research on relationships between mathematicians and mathematics educators.

With everyone back together in the large group, the WG facilitators shared a few key ideas and quotes from Artigue (1998). Artigue's goal in her research text was to address the relationships between mathematicians and research in mathematics education. After presenting an historical perspective on the teaching of mathematics, Artigue states:

> *Didactics is now considered as a legitimate speciality of research in applied mathematics and didacticians employed at tertiary level in mathematics departments are nationally elevated with the same criteria as other applied mathematicians. Nevertheless, these privileged links with the mathematics community remain fragile.* (p. 483)

Artigue continued to provide (what we referred to as) a creed; that is, a series of implicit I believe statements:

- "didactic research has to preserve strong and privileged links with the mathematical world";
- "mathematics as a discipline has to remain the fundamental root of didactics";
- "we need didacticians with a strong mathematical background";
- "didacticians from a mathematical origin have to try to preserve their present place within the world of mathematics production and mathematics education at the tertiary level";
- "the only way to guarantee that this strong connection [of didacticians] with the mathematical world remains effective is to foster it institutionally". (p.483)

In relation to the above statements, Artigue (1998) offered that "the only way to guarantee that this strong connection [of didacticians] with the mathematical world remains effective is to foster it institutionally" (p. 483). Artigue (1998) concluded her article by stating that positive action on the educational system is "different from the usual research work and has not to be left to the sole responsibility of didacticians" but instead it must "be carried out in a collaborative way, not only by didacticians, but also by mathematicians, by teachers at secondary and tertiary level, and also, at least at certain times, by educational researchers from other origins" (p. 487).

Following this general introduction to the work of Michèle Artigue, the group of participants moved into a small group activity designed to continue reflection on Artigue's work in connection with the relationship between mathematicians and mathematics didacticians, with specific attention to what participants thought about "fostering connections institutionally." Six quotes (A–F) from Artigue (1998) were prepared in advance (see Table 3), with one being randomly distributed to each small group for them to discuss in their breakout rooms.

A. "… the progressive institutionalization of the didactics community tends to withdraw the pressures of the educational world from the mathematicians. In fact, the existence of a community of specialists in the teaching and learning of mathematics inside the mathematics community, or at its border, in some sense allows mathematicians to stand back in a reflexive and critical position" (p. 482).	**B**. "… faced with the exponential increase of didactic productions, and faced with the strong development of more theoretical research, mathematicians can feel themselves more and more didactically incompetent: How to give advice on the pertinence of such and such a theoretical frame? How to judge what is really original research and what is not? How to evaluate the quality of didactic research?" (p. 483)	**C**."No doubt, [mathematicians] have been tempted in the past, and they continue to be tempted, to get rid of these embarrassing didacticians and to encourage them to join the community of educational science researchers. I have to confess, too, that a fairly large number of them live with a restrictive vision of science and are still wondering how the study of systems such as educational systems, which are so complex, so dependent on human beings, and so far from their mathematical world, could be the objects of real scientific work." (p. 483)
D. "mathematicians, most often, are not convinced of the legitimacy of those who do didactic research, and, as a consequence, have some a priori doubts on the pertinence of the results they obtain. More or less consciously, they tend to see the didactician as a kind of sub-mathematician who finds, in didactic research, a diversion from his or her lack of mathematical productivity" (p. 484)	**E**. "For a very long time, mathematicians have been protected from the problems induced by the democratization of teaching. They are no longer spared. They are more and more faced with students, less culturally-adapted, who need, in some sense, to learn what thinking mathematically is all about. Mathematicians are conscious of the increasing discrepancy between the lectures they give and the public they address. However, they do not find in the results of didactic research the means to remedy the problems that, in their opinion, this research should provide." (p. 484)	**F**. "… most often, [didactics] disturbs and destabilizes: It shows the failures of our usual teaching methods and tends to deprive us, as teachers, of the delusions which help us in our professional life. It shows us, as actors of the didactic system, we are involved with its malfunctioning" (p. 484)

Table 1. Selected Quotes for Discussion from Artigue (1998).

Even though Day 1 was winding down without much time remaining for small groups to report back, we closed the day by asking participants to reflect for a moment on their discussion and the quotes of Artigue through one or more of the following questions:

1. Based on some of what was shared earlier, discuss what makes a collaboration between research mathematicians different from collaborations between mathematics education researchers?
2. What do you think makes a collaboration between research mathematicians and mathematics education researchers 'successful'?
3. What are your thoughts on Artigue's claim that the connections and collaborations between the two communities must be fostered institutionally and involve others (K-12 teachers, researchers from disciplines other than mathematics, etc.)?

Finally, to effectively link Day 1's session to the session planned for Day 2, we asked participants to complete homework by writing a response to the following question: To what meaning of the word 'mathematics' do you subscribe to when you identify yourself (as a researcher) in your field?

DAY 2

Day 2 of our working group focused on the following questions:

- How can we initiate and/or sustain collaborations between two groups "somewhat suspicious" (Wheeler, 1977, p. 56) of each other?

- How might different research agendas and cultures, knowledge, interests, purposes, goals, languages, and 'success' cultures be coordinated?

Our first activity was a way to enter into an often tension-filled space between mathematicians and researchers in mathematics education. Participants separated into small groups to share and discuss their definition(s) of mathematics they had the opportunity to reflect on when this task was given as homework on Day 1. They were probed to discuss the following questions:

- In order to initiate and sustain collaboration between fields, is it important that the researchers adhere to the same definition of mathematics?
- Should their respective definition be made explicit at some point in the collaboration?
- What about personal definitions of learning, teaching, research, pedagogy?
- Does a potential disparity in definitions of mathematics matter? Is it a deal breaker for collaboration?
- How might the definition of mathematics impact a potential research result?

The group discussed how a definition of mathematics can emerge through respectful discussions on interests and connections. A number of different definitions were shared, some of them included

- Mathematical knowledge for teaching—curricular mathematics;
- Mathematical thinking;
- Definitions of mathematics as being contextual;
- Study of patterns, of relationships, and of ideas – shape, space, and numbers;
- Proof and argumentation;
- A language, a form of communication, that we can use to describe and understand the world around us, to relate ideas. Mathematics as a human way to understand things that can be put in a mathematical language;
- Processes;
- Truth seeking;
- Creating something new through deductive reasoning.

One participant noted that it was their first time being asked what their personal definition of mathematics was and started wondering how this might play a role in instigating and/or sustaining collaboration. The biggest takeaway of this conversation was that *relationships* form the foundation of any collaboration, that an interesting idea triggers the start of a collaboration, not the definitions related to mathematics, teaching and learning. Does one's worldview translate rigorously into the methodologies used in research? It is who the people are, the mutually negotiated purpose, and the claims they make depending upon their research methodology—contentious even amongst mathematics educators—that makes a collaboration.

As we asked participants to reflect on the idea that a collaboration can only happen if the participants have the same definitions of mathematics, participants noted that there may be not so much disagreement between researchers about the definition of mathematics so much as

- the worth of mathematics, and what mathematics is worthwhile to learn (deep understanding of foundational math vs advanced math);
- what it means to know and understand mathematics;
- who has/can access a strict definition of mathematics;
- the importance of having a definition;
- how to live out or enact the same definition.

The question of whether we really need to agree completely and always was raised. A common understanding seemed sufficient. Even among differences, the overlap in perspectives on mathematics, and on other topics that matter, is important for a collaboration (as compared to parallel play) and informs the research design. An analogy that was speaking to us was about musicians, how different musicians getting together to play brings their different skills, ideas of music, etc. but play together. Nevertheless, making understandings explicit and agreeing on the overlaps, while acknowledging places of difference, seem as important as the act of collaborating; the thinking and playing with ideas happen in these overlapping spaces. As researchers, asking ourselves what we are looking for, considering the lens with which we view mathematics, mathematics teaching, and mathematics learning as well as our goals for a project all contribute to a good place to start.

Our second activity was meant to explore possible tensions, with the ideas of building bridges and breaking down barriers. This was done through the study of excerpts of made-up dialogues from Nardi (2008), rooted in data, between RME (researcher in mathematics education) and M (a researcher in mathematics). M participated in a study RME was conducting, and it was the topic of a PME paper. We asked participants to read and share what comes to their mind as they read these tension-filled dialogues.

One group brought up a tension they noticed within mathematics education, a tension between the purpose of education research as one that improves education as opposed to understanding education. This tension within mathematics education research can come into play when thinking about collaboration between mathematicians and researchers in mathematics education, given that mathematicians are an important part of university education and therefore are prominent actors in questions of reform in university pedagogy. Who should make decisions about pedagogy in university mathematics? Who *should* teach, or who *can* teach, the mathematics classes in university? Whether these tensions are generalized or institution-dependent was at the forefront of our discussions. Participants emphasized how the relationships between mathematics and mathematics education research changes from one institution to another, depending on where the researchers in mathematics education are: either inside a mathematics department or in another department (building?) altogether.

Another group emphasized tensions in conversations between mathematicians and mathematics educators, where communications can be complicated by hierarchy and fear of losing status, where levels of mathematical knowledge are compared and used as precursors to respect and dialogue. Hierarchy in mathematics exists at all levels not just between mathematicians and researchers in mathematics education: upper elementary over lower elementary, high school over elementary, university over high school and then between mathematics fields in mathematics. Participants discussed that, from personal experiences, individuals who do not identify themselves on the 'mathematics' hierarchy are sometimes the ones willing to collaborate. Why could that be? Do they have no 'status' to lose? Are they open to improvement, change, growth and new opportunities? Are the ones on the top of the hierarchy in their own field more afraid to lose their status?

These ideas connect to tensions that occur as mathematics educators share their research using the words of the field, words that can be hard to understand to individuals that have not spent as much time in the field. It was discussed how conversations can be lost due to jargon or people not wanting to ask what a term or acronym means. If early in their career, maybe they do not want to be seen as not knowing; whereas, if late in their career, they may think of themselves as experts, and not want to be seen as less.

Many conversations across small groups addressed how the place a researcher is in regarding their career can play a role in the tensions of collaboration between the fields. Participants discussed how researchers who say they want to publish in the best mathematics education journals, those which could have the most impact on their career, need to first have some experience in the field; they cannot just 'jump in.' Ideas have to be wrapped and covered in layers of theory and methodology to meet expectations. The language of theory and methodology as well as the theoretical stances and the density of the language can be intimidating to newcomers. On the other hand, more accessible journals can be seen as having less impact on their research career.

Our group shared hopes of mathematics educators being incorporated into mathematics departments and in associated communities, such as the inclusion of mathematics education in Fields events.

DAY 3

The original plan for Day 3 was to go beyond the general discussions about collaborations and work towards more specific details for collaborations. The participants would brainstorm topics, form groups based on topics, then fill out a research proposal template as part of imagining an actual collaboration. However, we realized that many participants were still at the brainstorming stage and needed more time for discussions before settling on any specific topic and figuring out logistical details of a research program. Indeed, there was some resistance by several participants, but other participants were quick to decide on a topic.

We started the day with brainstorming of potential research topics, and used a padlet to share these. As with group discussions on the other days, the conversations were lively and there was a sense that people could keep talking for much longer. Figure 3 displays the Padlet results. Table 4 shows the four topics that were chosen for further discussions.

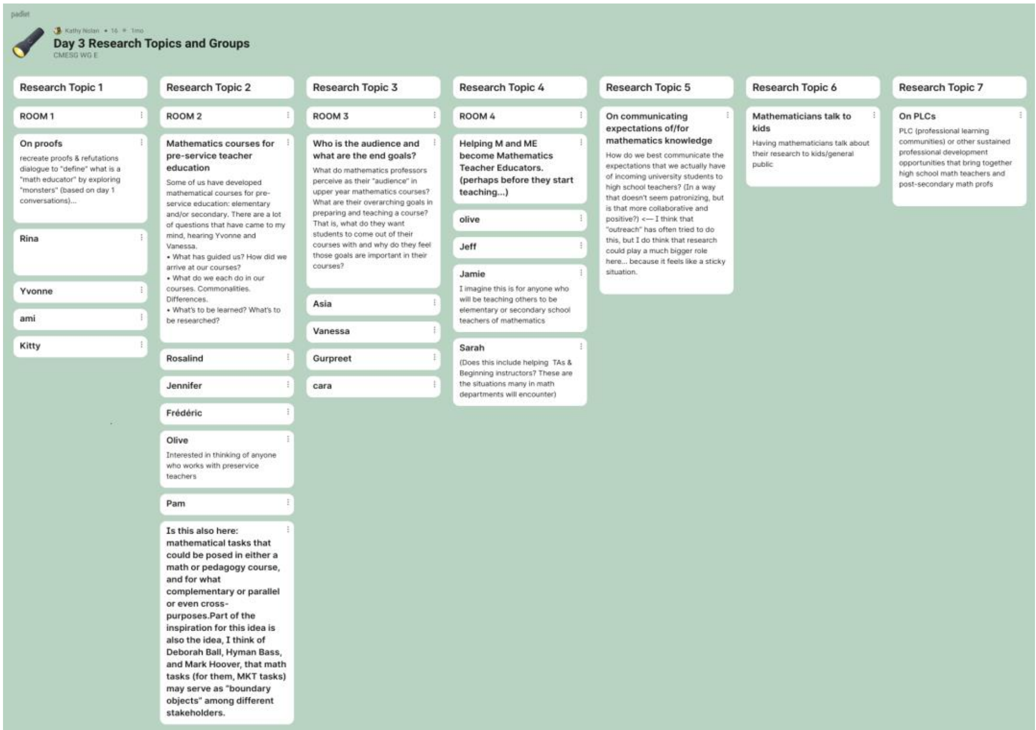

Figure 3. Day 3 research topics, Padlet part (c).

Other suggestions for topics:

- On communicating expectations of/for mathematics knowledge. How do we best communicate to high school teachers the expectations that we have of incoming university students? (In a way that does not seem patronizing but is that more collaborative and positive?). Outreach has often tried to do this, but research could play a much bigger role here.
- Mathematicians talk to kids / the general public.
- Professional Learning Communities (PLCs) or other sustained professional development opportunities that bring together high school mathematics teachers and post-secondary professors.

Once the four groups were formed, we encouraged participants to think about how they would do the research:

- What methodologies will be used?
- Do you need ethics approval?
- Do you have a budget? What are possible sources of funding? Internal / external?
- Who is involved in the collaboration? What are the roles of the collaborators?
- How will the collaboration take place?
- What is your proposed timeline?
- Other things to consider?

The discussions were about the topics in general without getting into more logistical details. While we may have felt surprised that the participants did not come away with actual plans for collaborations, the participants seemed to appreciate the discussions and could see more potential for collaborations.

Research Topic 1	Research Topic 2	Research Topic 3	Research Topic 4
Recreate proofs and refutations. Dialogue to define what is a ‘math educator’ by exploring ‘monsters’.	Some of us have developed mathematical courses for pre-service education: elementary and/or secondary. There are a lot of questions that have came to my mind, hearing Yvonne and Vanessa. • What has guided us? How did we arrive at our courses? • What do we each do in our courses? Commonalities. Differences. • What’s to be learned? What’s to be researched?	Who is the audience and what are the end goals? What do mathematics professors perceive as their ‘audience’ in upper year mathematics courses? What are their overarching goals in preparing and teaching a course? That is, what do they want students to come out of their courses with and why do they feel those goals are important in their courses?	Helping Mathematicians and Math Educators become Mathematics Teacher Educators. (perhaps before they start teaching...)
Rina, Yvonne, Ami, Kitty	Rosalind, Jennifer, Frédéric, Olive, Pam	Asia, Vanessa, Gurpreet, Cara	Olive, Jeff, Jamie, Sarah

Table 4. Day 3 research topics for groups.

CONCLUSIONS

The three sessions of this working group involved rich discussions, many of them revolving around the idea that, in a collaboration, we can all contribute, and we can all learn. It was noted throughout our working group’s time together that most collaborations begin with relationships, not with labels. Our working group modeled the way in which we can all work together and contribute, regardless of our individual positioning and roles, and regardless of whether our collaborative work is with respect to teaching, research…or CMESG working groups!

Les trois séances de ce groupe de travail ont été riches en discussions. Plusieurs de ces discussions ont fait ressortir que nous pouvons tous contribuer et nous pouvons tous apprendre dans une collaboration. Tout au long des échanges, il a été noté que la plupart des collaborations commencent par des connexions interpersonnelles. Notre groupe de travail a offert un modèle pour comprendre comment nous pouvons travailler ensemble et contribuer, quels que soient nos positions et nos rôles individuels, et ce, que notre travail collaboratif soit lié à l’enseignement, à la recherche…ou aux groupes de travail du GCEDM !

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Panel

Panel

CMS-CMESG JOINT PANEL: TACKLING DATA LITERACY IN THE CLASSROOM VIA REAL-WORLD DATA

INTRODUCTION

The panelists were

- Nat Banting, classroom mathematics teacher and mathematics education lecturer in Saskatoon, SK. Nat was unfortunately able to attend in person but created a video that was shared at the start of the panel.
- Kseniya Garaschuk, assistant professor of mathematics at University of the Fraser Valley, BC.
- Minnie Liu, classroom mathematics teacher in Vancouver, BC.
- Ryan Lukeman, associate professor of mathematics at St. Francis Xavier University, NS.

The panel was hosted by Patrick Reynolds, teaching professor, University of New Brunswick, NB.

Some of the discussion prompts were

- What are some skills, either practical/technical or conceptual, that students need to grapple with problems involving real-world data?
- If an instructor wishes to bring real-world data into the class, what are the implications for the overall course structure? Does the course need to be re-structured somewhat to allow for space/time to grapple with real-world problems?
- Have you experienced any socio-political discussion arising in the math classroom as a direct result of incorporating real-world data? If so, does bringing real-world data into the classroom require the instructor to prepare for this possibility?
- Broadly speaking, what are the pitfalls/challenges of incorporating real-world data?
- Broadly speaking, what are the benefits of incorporating real-world data?
- Is there a distinction between "using real-world data", and "fostering data literacy" in the math classroom? That is, if the goal is to improve data literacy, does the instructor need to take extra steps beyond bringing real-world data into the classroom?

Below are submissions from each of the panelists.

NAT BANTING

DATA LITERACY IN A 'REAL' WORLD

Society has never been more saturated with data. This assertion could have been safely made prior to March 2020 (a period I affectionally refer to as, "The before times"), but the persistence of the COVID-19 pandemic in the face of clear, data-driven recommendations for prevention erases any doubt. My work as a secondary mathematics teacher, therefore, is aimed not only at preparing students to engage with post-secondary courses in statistics (if they so choose to pursue them), but to prepare students to operate in such a society. It is against this backdrop that the question, "Is there a distinction between 'using real-world data' and 'fostering data literacy' in the math classroom?" struck me as a particularly important place to begin any panel discussion on data literacy. While I was unable to participate synchronously in the panel, my video contribution was primarily focused on affirming, defining, and describing this critical difference.

The difference jumps into relief when we look at classroom activities beyond the source of the data that students are interacting with and focus attention on the ways in which our practice asks students to act with that data. In doing so we begin to see data literacy in an active sense where the key to fostering data literacy is not necessarily about having students act with the *real world*; it is about having students act in *real ways*. It shifts emphasis away from *what* data is being interacted with and toward *how* that interaction unfolds. The difference is pedagogical in nature and aligns with the popular notion of statistical literacy as "the ability to interpret, critically evaluate, and communicate about statistical information and messages" (Gal, 2002, p. 1). Alongside Gal, I contend that the acts of interpreting, evaluating, and communicating take a prominent role in literacy while no particular mention is made to the source of the statistical information.

That is not to disparage the use of data sets from the natural or social sciences. It is only stating that there exists no difference, pedagogically speaking, if we simply replace sets of generated data with sets of data gathered from the real world and subsequently ask students to act in the exact same ways—as flesh-covered computers. Physics and sociology do not immediately imply the fostering of data literacy any more than the lack of such contexts preclude this development. To build data literacy, data cannot be interpreted *at* you; any discussions of how to foster data literacy need to be grounded in this spirit. A teacher's focus then shifts to how students are acting, and the job is to set the stage and the circumstance with potential for them to act in real, statistical ways.

A teacher could create scenarios where students might

- ask questions about the data—interrogate its source, its sample size, or its representation, etc.;
- select and justify an appropriate statistic rather than just calculating a prescribed statistic;
- expand the role of statistics beyond description and into prediction; and/or
- create statistics that they believe hold descriptive or predictive power.

These types of activities ask students to be both critical consumers and producers of data, and this active stance—these real ways of being statistical—is the cornerstone of becoming data literate. Never have these skills been more crucial as COVID-19 seems to have sorted those who interpret data from those who ignore it, and the fact that we find ourselves in the position we do, despite inundation of statistics from numerous agencies and influencers, suggests that *showing* people statistics from real-world contexts is not enough to sponsor data-literate decisions.

KSENIYA GARASCHUK

> *For most nonscientists, what's most important in science education is not the imparting of any particular set of facts (although I don't mean to denigrate factual knowledge), but the development of a scientific habit of mind: How would I test that? What's the evidence for it? How does this relate to other facts and principles? …Remembering this formula or that theorem is less important for most people than is the ability to look at a situation quantitatively, to note logical, probabilistic, and spatial relationships, and to muse mathematically.*— John Allen Paulos, *Beyond Numeracy*, p. 6

Data literacy means different things in different contexts. My context is (mostly) first-year calculus courses for non-majors, courses for students in life sciences and business. Students come into my classroom with a lot of preconceived ideas about what math is in general and what classroom math is in particular. I frequently observe such preconceptions when I teach applications, as most of the students have been desensitized to real-life examples in math. It is almost as if the life of the application and the realism of the data presented disappears the moment it ends up in a math classroom.

To me, data literacy means being able to muse mathematically about presented information: ask what simplifying assumptions we need to make to be able to model a phenomenon, what variables we decide to introduce and what quantities we decide to hold constant, what mathematical and statistical tools are suitable for analysis, how can we interpret the results. Data literacy is asking meaningful questions and deriving relevant information from the data. In my classes, we discuss mathematics of various authentic phenomena. The material is split between planned, such as prepared assignments and examples appearing in lectures, and just-in-time, which is motivated by current events or students' particular interests.

Just-in-time material requires drawing information from the news. Luckily, the current world is not short on data collection, so it is a matter of choosing something suitable out of the many available options. Let's start with graphs.

For example, in Calculus for Business in Fall 2020, my lectures and tests included discussions of pandemic effects on toilet paper supplies and Zoom (https://zoom.us/) popularity as well as importance of proper data reporting.

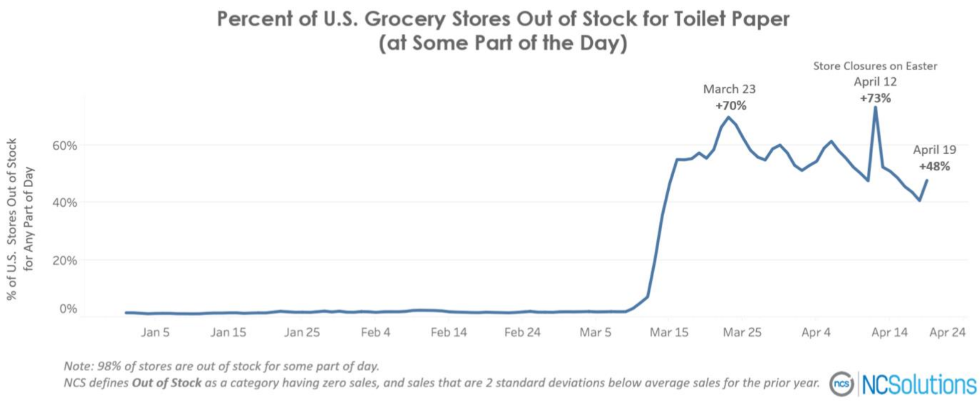


Figure 1. NCSolutions (2020).

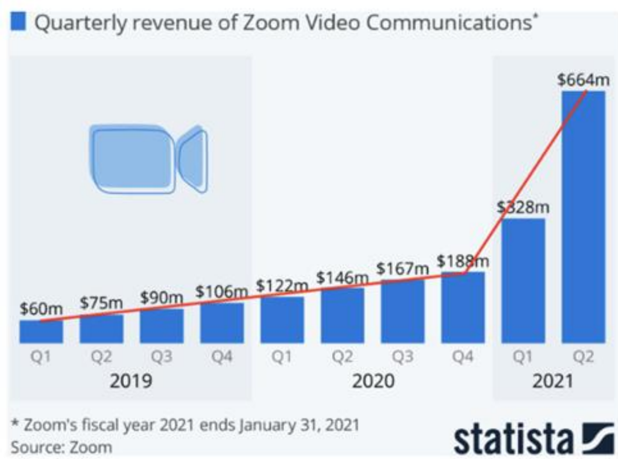


Figure 2. Richter (2021).

You can start asking data-related questions immediately; for example, what function can we model Zoom revenue with? Will only one function suffice, will only type of function (linear, exponential) suffice? Can we reliably predict future growth based on this data?

I also use mathematics to discuss social and economic aspects of society, such as unemployment rates and STEM gender gap. Do equal trends mean equal opportunities? Does increase in female STEM degrees mean the gap is narrowing? Arguing with data inevitably unveils logical misconceptions students have and hopefully dispels them.

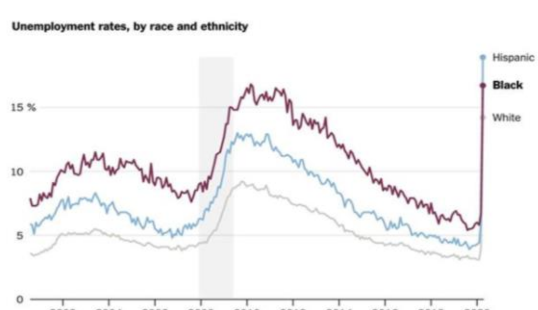


Figure 3. US Bureau of Labour Statistics as reported in Rattner and Higgins (2020).

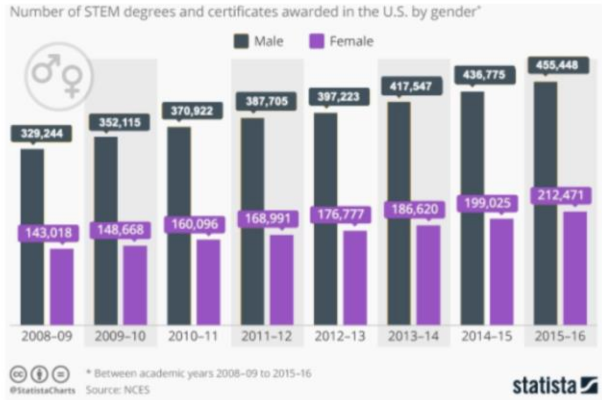


Figure 4. Feldman (2019).

Data literacy also means being able to read data, compute with data (in real-world numbers and units), and communicate with data to support the findings. Motivated by students' mention of a study of lake pollution in their ecology class, I created a project dealing with lake pollution in a local reservoir. Here is the set up, a few parts of the problem, and the sources used:

> *Lake Koocanusa is a reservoir in British Columbia and Montana, built as a joint Canada/USA project by damming of the Kootenay River in 1972. The lake is part of Elk River Valley, which hosts five open-pit coal mines, supplying a third of the world's steel-making coal [1]. For years, pollutants such as selenium have been flowing from the mines into the river. While selenium is a naturally occurring element in sedimentary rocks and coal, it is toxic to fish at elevated levels. The still waters of Lake Koocanusa allow even low concentrations of selenium to build up and have impacted the populations of sturgeon and trout that live in the lake [2, 3].*
>
> *Denote the amount of selenium pollution in Lake Koocanusa by P(t), measured in cubic meters of selenium, where t is in years. Your mission, should you choose to accept it, is to design and analyze the model for P(t).*
> *a) First, let us consider how selenium enters the lake. Annual discharge of a lake is the amount of water that enters and leaves the lake every year (so the volume of the lake stays constant). Annual discharge of Lake Koocanusa is* 9.46×10^9 *cubic meters per year [4]. Each cubic meter of water entering the lake contains 3 parts per billion of selenium. Find the rate at which selenium enters the lake.*
>
> *b) Now, consider how selenium leaves the lake. Volume of Lake Koocanusa is* 7×10^9 *cubic meters [5]. Supposing that selenium is evenly distributed throughout the lake, find its concentration. Recalling that* 9.46×10^9 *cubic meters of water leave the lake each year, all with selenium concentration found above, find the rate at which selenium leaves the lake.*
>
> *c) Use above information in parts (a) and (b) to find the total rate of change of selenium pollution P(t).*
>
> *d) …*
>
> *e) …*
>
> *f) …*
>
> *g) Scientists agree that the safe level of selenium is 0.8 parts per billion. If lake Koocanusa began with this level of pollutants, use Euler's method to approximate the level of selenium in the lake for the following 3 years with step size of 1 year (Note: you first need to find how much total selenium there is in the lake.)*
>
> *h) …*
>
> *i) …*
>
> *j) …*
>
> *k) …*
>
> *l) List and discuss at least three factors not taken into account by this model that would have an impact on the selenium level in lake Koocanusa.*
>
> *References:*

[1] Elk River, Wikipedia article: https://en.wikipedia.org/wiki/Elk_River_(British_Columbia)
[2] "Koocanusa Exceeds Selenium Standard" by Elk Valley Coal News, https://elkvalleycoal.com/koocanusa-exceeds-selenium-standard /
[3] "BC stalls on Koocanusa selenium pollution limit" by Wildsight, https://wildsight.ca/2020/10/15/b-c-stalls-on-koocanusa-selenium-pollution-limit/
[4] Koocanusa Reservoir by Ministry of Environment, https://www.for.gov.bc.ca/hfd/library/documents/bib61851.pdf
[5] Lake Koocanusa Stats from Lakepedia, https://www.lakepedia.com/lake/koocanusa.html

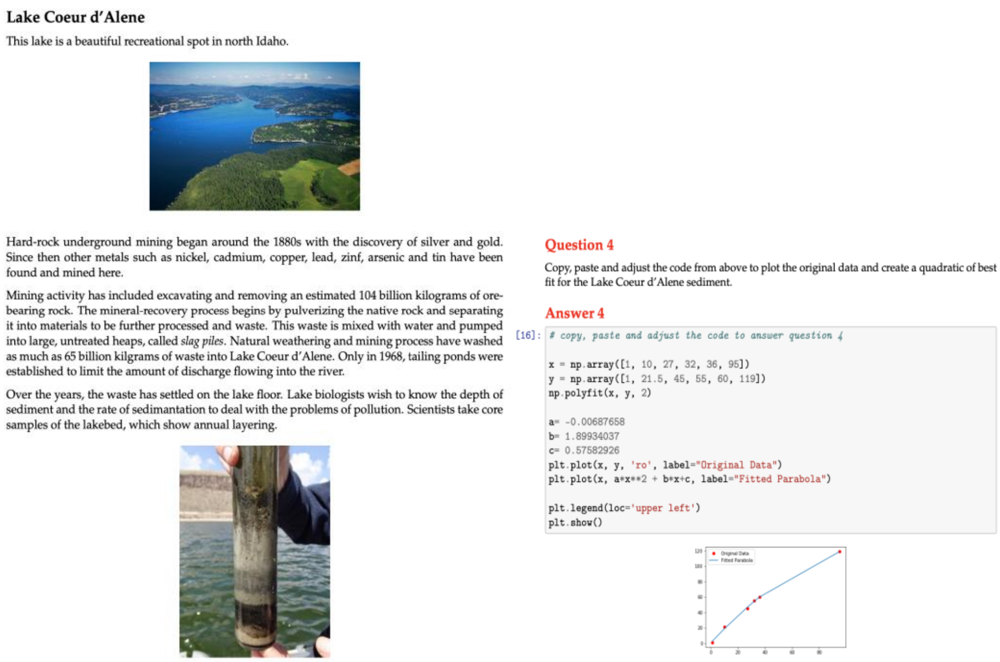
My pre-term prepared materials consist of various implementations of a problem-first approach. In particular, you can read about it in my piece "My problems are your problems: a problem-based approach to calculus" (Garaschuk, 2019). While some problems span the entire term, others make a briefer yet important appearance in several topics. As I want my students to be able to apply mathematical analysis tools in different contexts, I provide them with ample examples from various areas of study.

Data literacy can be further supported by developing data science skills, such as data analysis, coding and general computational thinking, all of which are important to have in the modern world. As such, it is not surprising that various initiatives across different levels of education have been sprouting to introduce students as young as age 10 to the basics of coding. For example, in 2018 Callysto project (https://www.callysto.ca) had a dedicated BC Ambassador, who has visited nearly 1000 Grade 5–12 students across the province to introduce them to coding basics. In 2018 and 2019, I have attended a few talks by my colleagues on using coding assignments in Python and Jupyter notebooks in their courses, both in first year and upper-level math. Jupyter notebook is a browser-based interactive environment ideally suited for producing computational narratives as it combines live code, explanatory text, plots and other visualizations, while the Python programming language is easy to read yet versatile. You can read more about it in "Jupyter notebooks for mathematics teaching and research" (Lamoureux, 2019). I consulted with my colleagues Andrijana Burazin and Miroslav Lovric to draw from their experiences introducing computational assignments into first year calculus courses. My goal was to create data-enabled computational assignments that prompted students to experiment with and explore mathematical ideas and algorithms in a highly dynamic and interactive environment. Together with two students, I designed and tested python assignments and implemented them into Calculus I for Life Sciences.

Assignments concentrated on computational techniques such as developing understanding of the definition of a derivative, user precision requirements of Newton's method, data transformation and data fitting. Each assignment consisted of two parts:

a) introductory part, in which students were required to work through examples and explore the code, and
b) question/answer part, where students had to modify the code from the introductory part to answer assignment questions about the mathematical concepts pertaining to the course.

Computational assignments were introduced as an alternative way for students to approach mathematical problems. They explored limits numerically, experienced limitations of such approaches, found out when Newton's method can fail, and they particularly enjoyed computer-assisted data fitting. As a side benefit, students were naturally introduced to basic coding elements, such as lists, 'for' loops and 'if' statements. The numerical explorations were once again rooted in authentic applications. Here is a set up and some parts of one such assignment:



Lake Coeur d'Alene

This lake is a beautiful recreational spot in north Idaho.

Hard-rock underground mining began around the 1880s with the discovery of silver and gold. Since then other metals such as nickel, cadmium, copper, lead, zinf, arsenic and tin have been found and mined here.

Mining activity has included excavating and removing an estimated 104 billion kilograms of ore-bearing rock. The mineral-recovery process begins by pulverizing the native rock and separating it into materials to be further processed and waste. This waste is mixed with water and pumped into large, untreated heaps, called *slag piles*. Natural weathering and mining process have washed as much as 65 billion kilgrams of waste into Lake Coeur d'Alene. Only in 1968, tailing ponds were established to limit the amount of discharge flowing into the river.

Over the years, the waste has settled on the lake floor. Lake biologists wish to know the depth of sediment and the rate of sedimantation to deal with the problems of pollution. Scientists take core samples of the lakebed, which show annual layering.

Question 4

Copy, paste and adjust the code from above to plot the original data and create a quadratic of best fit for the Lake Coeur d'Alene sediment.

Answer 4

[16]:
```
# copy, paste and adjust the code to answer question 4

x = np.array([1, 10, 27, 32, 36, 95])
y = np.array([1, 21.5, 45, 55, 60, 119])
np.polyfit(x, y, 2)

a= -0.00687658
b= 1.89934037
c= 0.57582926
plt.plot(x, y, 'ro', label="Original Data")
plt.plot(x, a*x**2 + b*x+c, label="Fitted Parabola")

plt.legend(loc='upper left')
plt.show()
```


Figure 5. Images from Century 21 Waterfront (2019).

In winter 2021, I asked my Calculus II for Life Sciences students to design a long answer final exam question that was rooted in an application not seen in class. Students did not disappoint: their submissions ranged from analysing tourism data from 1990 to 2020 to population fluctuations of Steller jays to growth patterns of baobabs—all supported by data, with sources cited. I will end with two students' comments from that term:

> *This class showed me that math is more about reason and logic than the numbers themselves; I never felt that getting a correct answer was more important than understanding core concepts.*
>
> *Overall this term sucked. My motivation has never been lower and I am not doing good in any of my other classes except for math. Math was definitely the highlight of my term as I felt excited to learn and I had motivation to do the work and study.*

MINNIE LIU

A REFLECTION ON DATA LITERACY

As I reflect on data literacy, I believe there are specific and important mathematical skillsets and knowledge surrounding data literacy which students need to successfully understand data and analyze reports, etc. However, I also believe that the lessons we provide to foster students' data literacy exceed beyond the mathematical skills and knowledge surrounding data literacy. As Schoenfeld (1992) explains,

> *The "lessons" students learn about mathematics in our current classrooms are broadly cultural, extending far beyond the scope of the mathematical facts and procedures (the explicit curriculum) that they study. …what one thinks mathematics is will shape the kinds of mathematical environments one creates—and thus the kinds of mathematical understandings that one's students will develop.* (p.341)

As such, I think of data literacy as something that goes beyond specific mathematical skills and content knowledge. Rather, I approach data literacy as a part of mathematical literacy or numeracy. In terms of skills, I think about data literacy from three major perspectives: an analyst's perspective, a participant's perspective, and a researcher's perspective.

From an analyst's perspective, I would like to see that students are capable of interpreting, evaluating, and factchecking the claims when reading a report; from a participant's perspective, to ask questions and have an understanding of what and how data is being collected and where the data is going and how it will be treated when participating in a study; and from a researcher's perspective, to collect, to accurately present, and protect data to their best ability. But above all, I wish for students to not only have these skillsets surrounding the treatment, analysis, and understanding of data. I wish for them to have something actionable—a willingness to apply these skills and their understanding, and to have a critical and inquisitive mindset whenever they deal with data.

Given that data literacy is more than a set of skills and knowledge and is closely related to a critical thinking mindset, I also believe that it is crucial to create a classroom culture where thinking is expected and required, and learning happens collaboratively as a community. Based on these ideas, I set up my classroom accordingly, where students learn and work in groups to analyze data and report their findings to develop their competencies in the areas of data literacy and much more.

From my experience in working with students, real-world examples are extremely different from the problems found in a textbook, and they can be complex and overwhelming for students. Also, moving from solving textbook problems to analyzing real-world data presents a huge learning curve. As such, in order to lead students to do something complex such as sorting through data or to find relevant data to represent or to discuss a specific phenomenon, I would start with small sets of *meaningful* and/or *realistic data*, which allows me to zoom in on what students find interesting and relevant, to build up specific skills, toolkits, along with students' confidence prior to the use of *real-world data* in class activities.

In my own journey of incorporating realistic and real-world data in my classes, I have the profound fear that I am not in complete control, because these activities involve much more than equations that lead to specific solutions. However, it comes into light very quickly that the mathematics we do on a daily basis and the mathematics that I want students to know is much more than mathematics in the classroom or those found in a textbook. This realization provides me the motivation to expand my teaching (and hopefully students' learning) beyond the mathematics found in a textbook to the mathematics and skills related to mathematics that students could apply on a daily basis.

At the end of the day, I believe that incorporating realistic tasks in my lessons is worth every minute of class time, as students are not learning just mathematical facts, they are also practicing mathematicians.

RYAN LUKEMAN

Most all mathematics students have received a problem in this, or similar form: "What is the largest rectangular area a farmer can enclose, given 100 feet of fencing?". Now, the motivations for such a problem are clear: you can introduce the process of optimization in calculus, the visualization is straightforward, and the requisite functions and algebra are uninvolved. But, if mathematics exists for students only in the realm of the toy problem, even when we tell our students that they are working on an *application*, it can be easy for the student to imagine that the application of mathematics to the 'real world' is either non-existent, or far beyond their capability. Both are, of course, untrue.

In my experience teaching mathematics at the undergraduate level, the incorporation of real data into the course elevates the pertinence and justification for their theoretical efforts, and leads to an increase in engagement. Here, I present some challenges and benefits, through example, in fostering data literacy in the classroom through real world data.

The expectation of sterilized mathematics problems can engender an unwillingness to combine the native challenge mathematics provides many students with the messy, complex issues that real data can bring to the fore. When bringing data into the class, instructors should mine data sources that incrementally explore the spectrum of complexity. In one class, I have used student home province data from our university registrar, together with COVID-19 prevalence rates, to make predictions about expected number of cases arriving at the start of the school year (Murphy, 2021). Here, the data is relatively clean, requires little processing, and the metrics used are clear and intuitive. Yet, the conclusions (especially when compared to actual arrivals) can be fascinating, displaying immediate utility of mathematics for informing policy decisions. On the other end of the spectrum, I sometimes use data on image-based collective animal tracking from my own research, in the classroom. Here, animals (ducks) can leave and re-enter the frame; sometimes the tracking fails; translating positions from camera frame to the real world is a challenge, and so on. When the source data has such inherent challenges, the use of well-chosen statistical measures to allow a signal to emerge from the noise is particularly striking for the student: a hidden story is revealed.

From a practical perspective, a primary consideration for developing data literacy in the classroom is the choice of the environment in which students will interact with the data. After some exploration, I have settled primarily on using MATLAB (2020), a numeric computing environment, for this purpose. In MATLAB, students can input data from a variety of formats, can perform statistical calculations, develop advanced visualizations, and program scripts to perform all of these operations efficiently, in one place. In contrast to single-purpose data processing software, MATLAB provides broad functionality and a skill-set that is highly transferrable. But, there is a cost: the breadth offered by the environment can simultaneously be intimidating for students to enter into, and the learning curve often does not comport with the restrictions of a one-semester course. The instructor therefore needs a clear plan of initiating students to the software, that usually involves a series of assignments or labs that involve sequentially less 'hand-holding' as the course progresses. Even better, if a mathematics department can collectively organize the curriculum around a single software environment, then students can build their skill-set over a number of years, and instructors of advanced courses can expect some basic familiarity of incoming students.

In a modelling course I teach, I start with an introductory assignment: determine the scaling relationship between the diameter of tree trunks, and the height of trees. The students must collect the data (involving direct measurement, and inferred measurement through similar triangles), input the data into MATLAB, visually assess any relationship, test models of best fit, and based on their conclusion, offer some biologically-inspired explanation for the relationship. Even in this simple project, students are developing data literacy: they grapple with measurement error (and how this arises in direct versus inferred measurement), obtaining data of a sufficient range to extract a trend, inputting data in a logical way, visualizing their data for making model hypotheses, and performing statistical analyses of their data. Such an exercise is helpful to start building the student's intuition around data, and the hands-on nature tends to support engagement through the initial technical challenges of orienting to new software and analytical techniques.

Starting simply, complexity can gradually be increased through the course of study. In another course, we build a model of the (currently under-construction) Grand Ethiopian Renaissance Dam, a hydro-power dam on the Nile River having downstream implications (both environmental, and for existing hydropower dams) on Nile flow for other countries (Keith, Ford, & Horton, 2017). Here, data is gathered (in summarized form) from published sources of information: academic papers, mostly, focusing on seasonal flow rates, evaporation rates, and physical parameters of the dam itself. After building a model for water flow, a captivating story emerges whereby damming water upstream in high altitude climates can reduce overall evaporation and offset flow losses downstream due to reservoir-filling. Students are able to test different rates and strategies for filling the reservoir behind the dam. In this course module, students become acquainted with sourcing data, and using data to motivate construction of, and parametrize, more complex models. And, by being true to the real-world system, resisting the urge to reduce it to a canned version of reality (perhaps at a cost of additional instructional time), students can become actual policy-makers. In this case, the budding policy-makers can directly interface with how water rights are determined in the case of resources shared across multiple countries, and put geopolitical arguments reported in the media by various countries to the test against their own model output.

Herein lies one of the strongest benefits of incorporating real world data in the classroom: the results of the student's mathematical work can have immediate interpretation and usefulness. Their computations result in milliseconds shaved off a sprinter's run, gigawatts of energy produced, litres of water saved, numbers of infections reduced, or lives saved—in the real context from which the problem emerges. That is not to say there is not benefit from making necessary mathematical simplifications—that is, after all, a cornerstone of mathematical modelling. But, we need not make those simplifications out of the gate: we ought to allow the data motivating the mathematics to be real, raw, and sometimes messy. And, we as instructors should guide the students in data literacy so that they can interface with the data with the proper skill set, work in the proper environment, and start asking the right questions.

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New PhD Reports

Présentations de thèses de doctorat

ENGAGING MULTIPLE REPRESENTATIONS IN GRADE EIGHT: EXPLORING MATHEMATICS TEACHERS' PERSPECTIVES AND INSTRUCTIONAL PRACTICES IN CANADA AND NIGERIA

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Lakehead University

Representation is one of the mathematical learning processes (National Council of Teachers of Mathematics, 2000), and is one of the reform-based instructional approaches to teaching and learning of patterning and algebra. Using concurrent mixed methods research, I investigated elementary in-service teachers' goals for, beliefs about and knowledge of representations, both in Ontario and Lagos. This research suggests that teachers generally, but particularly in Lagos, need a deeper understanding of representations and need to further develop the specialised mathematics content knowledge related to patterning and algebra.

INTRODUCTION

Recent research on teachers' engaging with representations in teaching mathematical concepts (Mitchell et al., 2014; Stylianou, 2010) offers a new approach to teaching which can help the teaching and learning of algebra. Student achievement may be improved if teachers are exposed to effective instructional practices in which use of representations may be encouraged. Hence, teachers are encouraged to incorporate a range of multiple representations in their teaching and solving of mathematical concepts. The use of multiple representations is expected to improve students' conceptual understanding in mathematics. Here I focus on patterning and algebra at the elementary level. However, improving teaching is only possible based upon the work of skilled teachers who understand the subject matter and make use of the appropriate mathematical learning processes such as representations

Representation is one of the mathematical learning processes (NCTM, 2000) and is one of the reform-based instructional approaches to teaching and learning algebra. The term representation also refers to models such as "concrete materials, pictures, diagrams, graphs, tables, numbers, words, and symbols" (Ontario Ministry of Education, 2005, p. 16). So, representation can be both a teaching process, as well as a teaching tool. The 2005 Ontario mathematics curriculum was used for my study.

In the Nigerian mathematics curriculum, representation is less clearly defined (not a specific learning process) as in Ontario. It is described only as written symbols, graphs, pictures, diagrams, and real-world situations applied to solve problems (Federal Ministry of Education, 2008). It should be noted that the Nigerian grade 8 algebra content focused on higher levels (such as factorisation, simultaneous equations) of procedural skills, and this in a way affects the teachers' teaching as well as student's understanding. This study contributes to this understanding based on a study that investigated grade 8 teachers use of representations in teaching patterning and algebra in Ontario and Lagos.

RELATED LITERATURE AND THEORETICAL PERSPECTIVE

Mathematics educators have suggested the use of representations to encourage students to develop a deeper, more conceptual knowledge of mathematics. Teachers' knowledge of mathematics, in particular specialised mathematical content knowledge, may be needed to make mathematics comprehensible to students, and use appropriate representations. Specialised content knowledge is to know more than just explaining the content; teachers must be able to explain why a concept works, why it is worth knowing, how the ideas are constructed, and how to relate it to other learning outcomes and other disciplines both in theory and in practice.

According to the literature (Ball et al. 2008; Kajander, 2010), SCK is referred to as 'other' mathematical understanding which could be seen "as facility with appropriate mathematical models, alternate approaches to concepts and ways of thinking and reasoning conducive to students" (Kajander, 2010, p. 50). Teachers may not use representation effectively when teaching because teachers may have gaps in their own ability to use mathematical representations when doing mathematics (e.g., Beatty & Bruce, 2012; Stylianou, 2010).

Several studies have highlighted issues with teachers' knowledge and beliefs about representations. For example, teachers tend to lack flexibility in switching between representations (Mitchell et al., 2014; Stylianou, 2010), have a lack of strategies allowing students to generate their own representations (NCTM, 2000) and use contextual learning tasks that encourage the use of multiple representations (Venkat, 2010). However, less attention has been given to teachers' goals of, knowledge and beliefs about the use of representations, which is the focus of the study being reported in this paper.

The study is framed in a theoretical perspective of three overarching aspects of teacher cognition, which include teachers' goals, knowledge and beliefs. **Goals**—are teachers' expectations relating to the intellectual, social, and emotional outcomes for students as a result of their classroom experiences. One of the goals of using representations is to model and interpret physical, social and mathematical phenomena. Some of these goals include how teachers use representation to facilitate learning, explain misconceptions, make connections and improve problem solving. **Beliefs**—are personalized assumptions of the teacher relating to the nature of the subject, the pupils, learning, and teaching. Beliefs relate to how representations are selected, how ideas are communicated, how students are able to restate problems in their words and learn ideas that are rooted in meaningful concrete models. **Knowledge**—teacher knowledge involves knowing central facts, concepts, and principles about the pupils, content, and pedagogy acquired over time. It is about teachers selecting, linking and recognizing what is involved in using particular representations as well as allowing students to explore their own representations.

METHODS

Mixed methods (quantitative and qualitative) was used concurrently to collect data.

Convenience sampling was used to recruit participants. There were 91 teachers who completed the online survey and 10 of these teachers (five teachers in Ontario and five in Lagos) voluntarily participated in the interviews. The instruments used during the study included a 24 item six-point Likert scale survey which included an open-ended question, and also semi-structured interviews in which I acted as a sole interviewer. The survey instrument was validated by my graduate student colleagues, and four faculty members (in educational psychology, educational foundations, and two mathematics education faculty members) who reviewed the instrument for face validity. Descriptive statistics and thematic analysis were used.

RESULTS

The Ontario teachers demonstrated a better knowledge of the modes of representations associated with the mathematics concepts, in particular, patterning and algebra. Symbols, Diagrams, graphs, concrete materials, algebraic expressions, manipulatives, pictures, models, number sentences, and words were the modes of representations the teachers mentioned in their responses to the open-ended question. These results were found to be consistent with the *Ontario curriculum, Grades 1–8: Mathematics* (Ontario Ministry of Education, 2005), "concrete materials, pictures, diagrams, graphs, tables, numbers, words, and symbols" (p. 16).

In contrast, the Lagos teachers focused hugely on symbols, graphs and diagrams. They demonstrated a narrower knowledge of the modes of representations, The majority of the Lagos teachers did not mention mathematical manipulatives in responses to the open-ended questions.

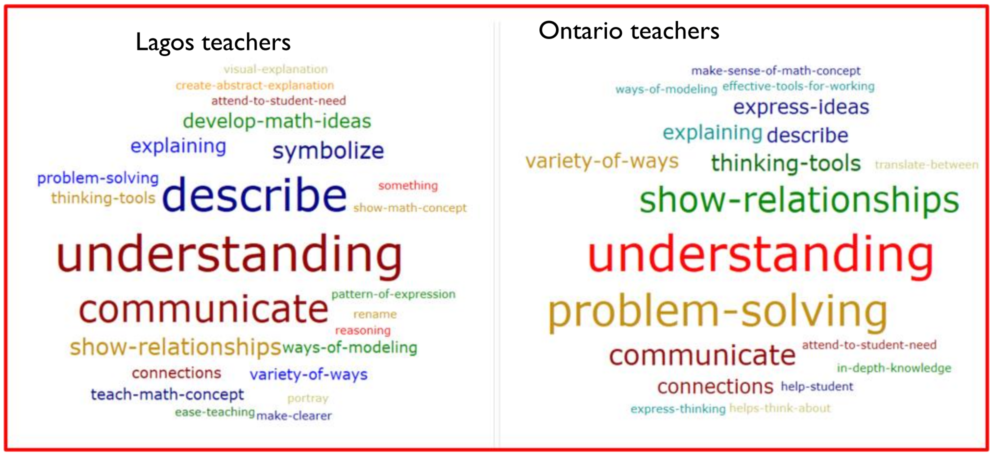

Figure 1.

The left half of Figure 1 reflects responses obtained from the Lagos teachers while the right side of Figure 1 reflects words used by the Ontario teachers. There were higher frequencies found in the Ontario sample on the use of representation to build *understanding* (15.4% ~ 29.6%), *relationships* (5.6% ~ 19.7%), and *problem solving* (2.8% ~ 25.4%). Other areas included *way of modeling* (2.8% ~ 4.2%), *variety of ways* (2.8% ~ 9.9%), *explaining* (4.2% ~ 9.9%), *connections* (2.8% ~ 9.9%), and *communicate* (11.3% ~ 15.4%).

Overall, Lagos participants were more comfortable with the use of symbols than other types of representations, while Ontario teachers tended to refer more to models, including concrete models. The survey results would be useful in the mathematics education field for anyone carrying out research in patterning and algebra.

Analysis of the key findings from all 10 teachers' interviews revealed their goals, beliefs and knowledge. Teachers from the Ontario subsample tended to describe more reform-oriented goals for and uses of representation than those in Nigeria, for example in these areas. In the interviews, all 10 teachers maintained that having SCK was critical. Perhaps more surprising was that only Sara from Ontario demonstrated strong SCK while all of the Lagos demonstrated weak SCK. This finding is consistent with what is found in the literature, namely that teachers struggle with SCK. Generally, the teachers believed that the curriculum must include attention to representations. And interestingly they are aware that they needed some support. Overall, the findings indicated the need for teachers' specialised content knowledge to be much better developed. All 10 teachers claimed that using contextual learning tasks should be related to students' experience.

DISCUSSION AND IMPLICATIONS

Students need opportunities to practise what they are learning and to experience performing the kinds of tasks in which they are expected to demonstrate competence. For example, if teachers want students to be proficient in problem solving, students must be given opportunities to practise problem solving. If strong deductive reasoning is a goal, student work must include tasks that require such reasoning.

Some of the Ontario teachers, and most of the Nigerian teachers, felt they need to understand more about the use and importance of representations in order to create the opportunity for students to effectively and appropriately use them. Research has also demonstrated that when students have opportunities to develop their own solution methods, they are better able to apply mathematical knowledge in new problem situations. Overall, the study suggests that teachers need to gain a deeper understanding of what representation is and what it looks like in the classroom. This study contributes to existing knowledge about how teachers perceive representations, and its role in teaching patterning and algebra in Ontario and Lagos.

CONCLUSION

Teachers need to be more aware of the role of representations and recognize their potential in student learning. Generating appropriate representations during problem solving, and reasoning and making connections related to a topic, are particularly important. Student achievement may be improved if teachers are exposed to effective instructional practices in which the use of representations is encouraged. The most significant overall finding and recommendation of my study is that enhanced mathematics teacher education is required to enhance teachers' understanding of the use of representation. Only then will the full potential of representations be realised in teaching and learning.

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BREACHING EXPECTATIONS: EXAMINING TWO NORMS OF U.S. HIGH SCHOOL GEOMETRY

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Prior research has shown that norms (i.e., often-implicit behavioural rules that people tacitly expect each other to follow in situations of a given type) have considerable influence on human behaviour (Legros & Cislaghi, 2020); in particular, the behaviour of teachers and students in mathematics classrooms (e.g., Stigler & Hiebert, 1998; Yackel & Cobb, 1996). It has also demonstrated that breaches of norms are often sanctioned, sometimes positively, but typically negatively (e.g., Garfinkel, 1963). Something that is not well-understood, however, is how two (or more) norms of a given situation may be related. In this paper (and my dissertation; Boileau, 2021), I discuss one possibility: that breaches of one norm may cause individuals to abandon their expectations that other norms will be followed[1].

To investigate this possibility, I used a specific case: two norms of a recurrent *instructional situation* (Herbst, 2006) in the canonical U.S. high school geometry course—the instructional situation of geometric calculation with algebra (GCA). In this situation, students are tasked with solving a mathematical problem that requires them to determine dimensions of a given geometric figure, represented through a diagram, both pictorially (by strokes and spaces) and by algebraic expressions. This can be accomplished by setting up and solving one or more equations, each based on a property of the type of figure represented by the diagram. An example is the typical GCA problem provided in Figure 1.

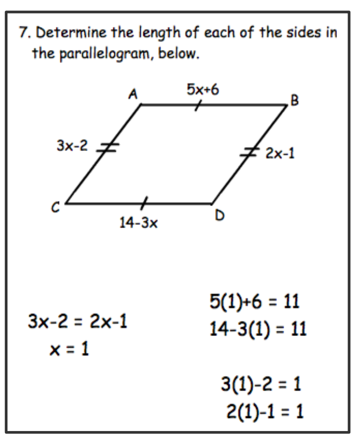

Figure 1. Normative GCA problem, with solution, from Boileau (2021, p.55). © 2020, The Regents of the University of Michigan, all rights reserved, used with permission.

Based on my observations of U.S. high school geometry classrooms and perusal of commonly used U.S. high school geometry textbooks, I conjectured that the instructional situation of GCA is regulated, in particular, by two norms. The first was that a figure of the type represented by the pictorial components of the diagram that also satisfies all of its algebraic expressions is expected to exist—hereafter, I refer to this hypothesized norm as the *GCA-Figure norm*.

[1] In the dissertation (Boileau, 2021), I provide reasons why it is reasonable to expect that this may occur.

The second hypothesized norm was that the geometric property that allows the student to set up any equation is expected to be stated verbally or left implied by the equation, but not written—hereafter, I refer to this as the *GCA-Theorem norm*.

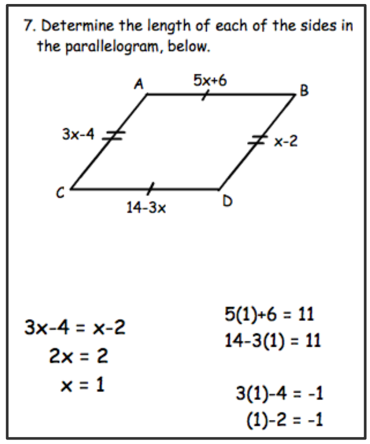

Figure 2. Non-normative GCA problem, with solution, from Boileau (2021, p.55). © 2020, The Regents of the University of Michigan, all rights reserved, used with permission.

The problem in Figure 1 is an example of a problem that complies with the GCA-Figure norm. In contrast, the problem in Figure 2 breaches it. The evidence of this is the calculations, which suggest that, if we assume that there is a parallelogram (in $\mathbb{R}^2_{\geq 0}$) whose side lengths could be represented by those expressions, then we must conclude that two of its side lengths are negative. Since we cannot, we must instead reject the premise that such a parallelogram exists.

Figure 3 represents a normative interaction between a teacher and student in which the GCA-Theorem norm is followed (as the teacher only asks the student to verbally state the geometric property that allowed them to write the equation that they are using to solve the problem). Figure 4 represents an interaction between a teacher and student in which the GCA-Theorem norm is breached by the teacher asking the student to write down the property, as if that was their expectation when they asked the student how they knew $5x + 6 = 14 - 3x$..

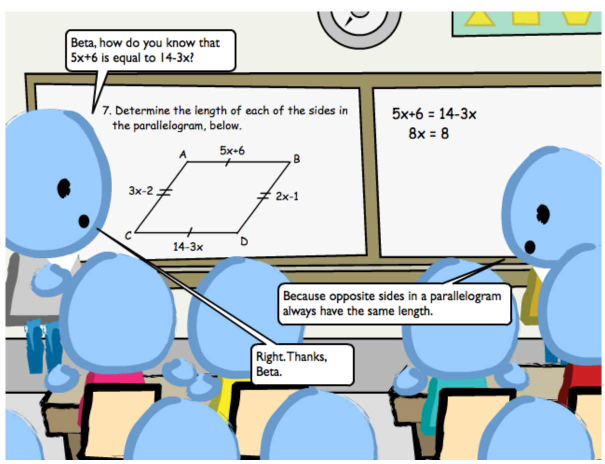

Figure 3. Storyboard in which the GCA-Theorem norm is followed, from Boileau (2021, p.57). © 2020, The Regents of the University of Michigan, all rights reserved, used with permission.

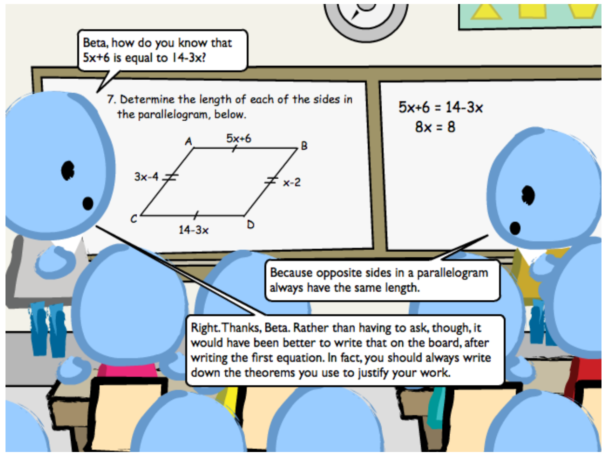

Figure 4. Storyboard in which the GCA-Theorem norm is breached, from Boileau (2021, p.57). © 2020, The Regents of the University of Michigan, all rights reserved, used with permission.

While the instructional situation of GCA was a context in which I could test my larger hypothesis (that breaches of one norm of a given type of situation may cause individuals to abandon their expectations that other norms will be followed), there were also several reasons to focus on it, as a mathematics educator. For example, if teachers were willing to assign GCA problems that I describe above as breaching the GCA-Figure norm, this situation would provide a unique opportunity in U.S. high school mathematics education to reason about the existence of a mathematical object. Such problems would allow a natural opportunity to learn to prove by contradiction—a reframing of the situation in which writing down one's geometric reasoning (i.e., breaching the GCA-Theorem norm) would likely be deemed acceptable.

As an attempt to confirm the existence of these two hypothesized norms and to determine whether breaches of the GCA-Figure norm would cause experienced geometry teachers to abandon their expectations that the GCA-Theorem norm would be followed, I conducted a *virtual breaching experiment with controls* (Dimmel, 2015). I focused on the expectations of experienced geometry teachers as teachers with little experience teaching a given course may not be aware of the norms of its instructional situations. While we do not know at what point in a teacher's career they become familiar with the norms of a given course (or when students do), in prior studies (e.g., Herbst et al., 2011), researchers have suggested that it is reasonable to expect that teachers would be familiar with the norms of a given course after three years of experience teaching it. As such, in the present study, I used this criterion to determine a sample of teachers that could help me answer my research questions.

Through this experiment, I also investigated experienced geometry teachers' attitudes towards decisions to follow or breach each norm. Given space constraints, I do not report my findings related to this question here but mention it as it influenced the design of the experiment.

METHODS

DATA COLLECTION

Broadly speaking, a *virtual breaching experiment* (Herbst & Chazan, 2011) consists of showing a virtual representation of a situation in which one or more hypothesized norms are breached to a sample of individuals who the researcher thinks are likely familiar with that type of situation, and therefore its norms. Participants are then asked what they notice and how likely and / or appropriate the behaviour they observed is. To date, these have consisted of showing U.S. mathematics teachers classroom videos and animations in the context of focus groups (e.g., Herbst et al., 2011), and storyboards embedded in online questionnaires (e.g., Dimmel, 2015), in order to investigate norms of other instructional situations. The goal of these studies has been to confirm the existence of hypothesized norms and elicit attitudes towards behaviours that violates them. In these studies, evidence that the hypothesized norms exist consists of a high percentage of participants remarking the behaviour that is hypothesized to breach it, as individuals tend to only notice and remark non-normative behaviour (Herbst et al., 2011).

However, one issue with this approach to studying norms is that the percentage of a group of people that are aware of a given norm (i.e., that expect it to be followed) depends on the norm in question (Bicchieri, 2006). Therefore, Herbst and Chazan (2011) suggested also showing participants representations of practice in which the hypothesized norm is followed (as an experimental control). This recommendation was first taken by Dimmel (2015) who conducted the *first virtual breaching experiment with controls* in order to investigate two norms of *doing proofs* in U.S. high school geometry (Herbst, 2006). This consisted of administering a set of questionnaires to a sample of U.S. high school mathematics teachers who varied in their years of experience teaching geometry. Each questionnaire contained a storyboard in which a student presented a proof at the board, and the teacher provided them feedback on their work. In each case, the proof either breached one of the hypothesized norms or followed both of them. The items included an open-response item that asked participants what they noticed about each storyboard, a closed-response item that asked them to rate the appropriateness of the teacher's actions, and an open-response item that asked them to explain their rating.

Inspired by this, I also designed a virtual breaching experiment with controls, although I had to adapt Dimmel's design to be able to investigate my hypothesized relationships between the GCA-Figure and GCA-Theorem norms: that breaches of the GCA-Figure norm would have experienced geometry teachers abandon their expectation that the GCA-Theorem norm would be followed and/or make them more open to breaches of that norm. I did this by creating three experimental conditions. In the storyboards in one condition, the *compliance-compliance* condition, both norms are followed. In the storyboards in a second condition, the *compliance-breach* condition, the GCA-Figure norm is followed, but the GCA-Theorem norm is breached. In the storyboards in the third condition, the *breach-breach* condition, both norms are breached.

I created four storyboards within each experimental condition, which differed in ways that I thought situations of GCA might naturally differ. Henceforth, I refer to these as the *storylines* of each storyboard. For example, the four storylines differed in terms of whether the students or the teacher chose the problem, and in terms of whether the student that presented their solution at the board was a volunteer or selected by the teacher. Storylines also differed in terms of the problem that students worked on, in particular the type of geometric figure represented by the diagram (and therefore the geometric knowledge required to solve the problem). This feature was used to name the storylines: similar triangles, trapezoid, isosceles triangle, and parallelogram. It was important to have multiple storyboards in each experimental condition to be able to argue that participants' reactions to breaches of either norm were characteristic of the instructional situation of GCA, rather than particular to a given GCA problem.

In total, there were twelve storyboards, each a unique combination of one of the three experimental conditions and one of the four storylines. Each storyboard consisted of three segments. In the first, the problem that the class would work on was chosen. During that segment, one frame provided an over-the-shoulder view of one student's complete, correct solution, which evidenced whether the problem breached or complied with the GCA-Figure norm. In the second segment, a student presents their solution at the board. The teacher interrupts them to ask them to justify the equation that they set up to solve the problem, in a way that either breaches or complies with the GCA-Theorem norm. The teacher also solicits feedback on the solution from the class. This feedback never includes mention of whether the GCA-Figure norm was followed or breached, although the determined side lengths provide evidence of this. This was intentional: by having the students accept the solution to each problem as normal, I expected that participants who did not expect the solution to imply that no figure of the type represented by the diagram exists would mention this when asked what they noticed about each breach-breach storyboard. In the third segment, the teacher proposes that they turn to another problem, making clear to the participant that the breach of the GCA-Figure norm would not be discussed.

After the first segment, participants were presented three items: an open-response item that asked "What did you see happening in this segment of the scenario?"; a Likert-style item that asked them to rate the appropriateness of the teacher's actions, with response options from "1–very inappropriate" to "6–very appropriate"; and an open response item that asked them to explain their rating. Then they were presented with a pair of GCA problems, one of which was the one being solved in the storyboard and another that either breached or followed the GCA-Figure norm but was otherwise equivalent. Participants were asked, "Which of the two problems below is more appropriate for a Geometry teacher to present to students?", and provided six response options, ranging from "1–Option A is much more appropriate than option B" to "6–Option B is much more appropriate than option A". They were then asked to explain their rating, in an open-response field. This itemset was included both to learn whether participants preferred GCA problems that comply with the GCA-Figure norm over ones that breach it, as well as to increase the probability that participants notice when the GCA-Figure norm is breached before viewing the rest of the storyboard, so I could learn whether the breach of the GCA-Figure norm would influence their reaction to the breach of the GCA-Theorem norm.

Participants were then presented the second segment of the storyboard, followed by the same three items that they were presented after viewing the first segment. Last, after viewing the third segment of the storyboard, they were then asked to rate the appropriateness of the teacher's actions throughout the scenario (using the same six response options) and to explain their rating.

The set of questionnaires—hereafter, the INR GCA instrument—was administered along with a number of other instruments and a background questionnaire to a national sample of over 700 U.S. high school mathematics teachers. Each participant was randomly assigned to one of three groups. Each group was strategically assigned four INR GCA questionnaires. The storyboard in each followed a different storyline and belonged to two experimental conditions. One group received two questionnaires that contained a compliance-compliance storyboard and two questionnaires that contained a compliance-breach storyboard. Another group received two questionnaires that contained a compliance-breach storyboard and two questionnaires that contained a breach-breach storyboard. The third group received two questionnaires that contained a breach-breach storyboard and two questionnaires that contained a compliance-compliance storyboard. The background questionnaire contained an item that asked participants how many years they had taught geometry, which was used to determine the subset of participants that I would consider familiar enough with the U.S. high school geometry course to be familiar with its norms (those with three or more years of experience teaching it).

In total, 480 teachers from this sample completed the four INR GCA questionnaires that they were assigned, 303 of whom had three or more years of experience teaching U.S. high school geometry. This resulted in approximately 11,700 open responses.

ANALYSIS

A set of dichotomous codes were applied to each of the open responses that indicated whether the response contained evidence that the participant recognized that the GCA-Figure and / or GCA-Theorem norm was followed or breached, as well as whether they positively or negatively appraised those decisions.

I coded a random subset of 2,160 of the approximately 11,700 open responses with another mathematics education researcher. I then coded the rest of them independently. However, given the number of responses and time constraints, I did so using a set of search terms that identified responses to which one or more of the codes would be applied, which I developed and tested using the 2,160 responses that I coded with the second coder. Specifically, I used Microsoft Excel's FIND function to locate responses that contained one or more of those terms, then coded those responses manually, then coded the rest of the responses as not containing evidence of recognition or appraisal of the decision to breach or follow either norm.

To answer the question of whether the GCA-Figure norm exists (i.e., whether experienced geometry teachers expect that it will be followed), I compared responses to the first item associated with the first segment of the compliance-breach and breach-breach storyboards that followed the same storyline. I compared those pairs of storyboards because they only differ in terms of whether the GCA-Figure norm is followed. I only considered responses to that item as it was the participant's first opportunity to express whether they were surprised by the choice of problem, making it the best evidence of their expectations of GCA problems. Similarly, to answer the question of whether the GCA-Theorem norm exists, I compared responses to the first item associated with the second segment of the compliance-compliance and compliance-breach storyboards that followed the same storyline. Again, I compared those pairs of storyboards because they only differ in terms of whether the GCA-Theorem norm is followed. And I only considered responses to that item as it was the participant's first opportunity to express whether they were surprised by the first interaction between the teacher and the student at the board, making it the best evidence of their expectations about whether and how students will share the geometric reasoning behind their choice of equation.

To answer the question of whether experienced geometry teachers prefer GCA problems that follow the GCA-Figure norm over GCA problems that violate it, I compared responses to the compliance-breach and breach-breach storyboards that followed the same storyline, again, as those pairs of storyboards only differ in terms of whether the GCA-Figure norm is followed. Similarly, to answer the question of whether experienced geometry teachers prefer when the GCA-Theorem norm is followed over when it is breached, I compared responses to the compliance-compliance and compliance-breach storyboards that followed the same storyline, as those pairs of storyboards only differ in terms of whether the GCA-Theorem norm is followed. To answer these research questions, however, I considered all responses to each storyboard, as evaluations primed by the requests to evaluate the teacher's actions were just as relevant as ones that were unprimed. It was similarly important to consider evaluations of decisions to follow or breach either norm that were only noticed after the participant had viewed the entire storyboard, which were captured by their responses to the final pair of items in each questionnaire.

Last, to answer the question of whether breaches of the GCA-Figure norm have experienced geometry teachers abandon their expectations that the GCA-Theorem norm will be followed and/or make them more open to breaches of the GCA-Theorem norm, I compared responses to the second segment of compliance-breach and breach-breach storyboards that followed the same storyline. I compared responses to those pairs of storyboards because the GCA-Theorem norm was breached in both of them, but the GCA-Figure norm was only breached in the breach-breach storyboards.

To make these comparisons, I created mixed-effects regression models in STATA, using the *melogit* command to compare dichotomous scores (representing whether the participant recognized or appraised the decision to breach or follow either norm) and the *mixed* command to compare responses to Likert items. Each model regressed a given participant score or rating on fixed effects indicating the experimental condition and storyline of the storyboard to which the score or rating was associated, as well as a random intercept indicating which participant contributed it. This random effect was included because each participant responded to multiple storyboards. Each model also included a third fixed effect indicating whether the participant was an experienced geometry teacher. Last, each model also includes interactions between all of the fixed effects (condition, storyline, and experience), as this was needed to compare how only the experienced teachers responded to the three storyboards that follow each storyline but belong to different experimental conditions.

RESULTS

As mentioned above, due to space constraints, in this paper, I only report the results related to the questions of whether the two hypothesized norms exist and whether breaching the GCA-Figure norm is likely to have an experienced geometry teacher abandon their expectation that the GCA-Theorem norm will be followed.

In terms of whether the GCA-Figure norm exists, the predicted probabilities that an experienced geometry teacher would remark that the GCA-Figure norm is breached when responding to the breach-breach version of the similar triangles and parallelogram storyboards were 0.07 and 0.13, respectively. While small, these were significantly higher than the predicted probabilities that an experienced geometry teacher would remark that the GCA-Figure norm is followed when responding to the compliance-breach version of those storyboards: 0.00 (meaning that no one remarked this decision). In contrast, these predicted probabilities were not significantly different in the case of the other two pairs of storyboards. This analysis therefore provided some evidence in support of the hypothesis that the GCA-Figure norm exists, but also some evidence to the contrary[2].

In terms of whether the GCA-Theorem norm exists, the predicted probabilities that an experienced geometry teacher would remark that the student breaches the GCA-Theorem norm (and / or that the teacher asks them to do so) when responding to the compliance-breach version of the trapezoid and parallelogram storyboards were 0.32 and 0.28, respectively. These were significantly higher than the predicted probabilities that an experienced geometry teacher would remark that the student follows the GCA-Theorem norm (and/or interpret the teacher's request for justification as a request for that justification to be stated verbally) when responding to the compliance-compliance version of those storyboards: 0.10 and 0.05, respectively[3]. In contrast, these predicted probabilities were not significantly different in

[2] That said, in the dissertation (Boileau, 2021), I provide more evidence of its existence, based on other analyses.

[3] In the case of the trapezoid questionnaire, the predicted difference is 0.23, SE= 0.06 (z=3.84, $p<0.001$). In the case of the parallelogram questionnaire, the predicted difference is 0.23, SE= 0.05 (z=4.20, $p<0.001$).

the case of the other two pairs of storyboards. This analysis therefore also provided some evidence in support of the hypothesis that the GCA-Theorem norm exists, but also some evidence to the contrary[4].

Last, in terms of whether breaches of the GCA-Figure norm are likely to have a teacher abandon their expectation that the GCA-Theorem norm will be followed, the predicted probabilities that an experienced geometry teacher would remark that the GCA-Theorem norm is breached when responding to the compliance-breach version of the trapezoid and parallelogram storyboards (again, 0.32 and 0.28, respectively) were significantly higher than the predicted probabilities that they would remark that it is breached when responding to a breach-breach version of those storyboards (0.15, in both cases)[5]. Again, these predicted probabilities were not significantly different in the case of the other two pairs of storyboards. This analysis therefore also provided some evidence in support of the hypothesis that breaches of the GCA-Figure norm may have such teachers abandon their expectation the GCA-Theorem norm will be followed, but also some evidence to the contrary.

DISCUSSION

The present study makes several important contributions to the literature on mathematics instruction. First, it adds to a growing body of literature that demonstrates that mathematics instruction is regulated by subject-specific norms: norms that only apply to a particular mathematical activity. It also suggests one way that these norms may be related. Last, it demonstrates how one might investigate whether two norms are related in this way.

Research like this is important because it deepens our understanding of the complexity of mathematics teaching and the decision making behind it. As such, it also helps us imagine different approaches to instructional improvement. For example, the research presented in this paper helps us understand how teachers' actions are not simply expressions of their individual beliefs, knowledge, and intentions, and that instructional improvement might therefore require changing some of the norms that bound them. Directions for future research should therefore not only include more studies of how the various (subject-specific and general social) norms that influence teachers' decisions are related, and how they influence mathematics teachers' decisions, but also research on which norms should be changed, and how this might be done. Fortunately, there is general guidance on the latter in the literature on norms outside of education (e.g., Bicchieri, 2017), although our field will need to consider whether and how such theories of change will need to be adapted to improve mathematics teaching and learning.

ACKNOWLEDGEMENTS

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[4] That said, in the dissertation (Boileau, 2021), I report on additional analyses that suggest that the norm may instead be that students are only expected to share their algebraic work, and that either verbally stating or writing down the geometric property that justifies each equation that they set up would therefore be unexpected.

[5] In the case of the trapezoid questionnaire, the predicted difference is 0.18, SE= 0.07 (z=2.73, $p<0.001$). In the case of the parallelogram questionnaire, the predicted difference is 0.13, SE= 0.06 (z=2.06, $p<0.05$).

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USING A MIXED METHODS APPROACH TO STUDY THE RELATIONSHIP BETWEEN MATHEMATICS ANXIETY, MATHEMATICS TEACHER EFFICACY, AND MATHEMATICS TEACHING ANXIETY IN PRE-SERVICE ELEMENTARY SCHOOL TEACHERS IN ONTARIO

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INTRODUCTION

Think of the individual who is highly proficient in *solving* a math problem but has a hard time explaining their solution. Or, what about the gifted teacher of elementary mathematics who tenses up at the thought of integrals, differentiation, and quadratics (all concepts that are unrelated to teaching mathematics in an elementary school). This disconnect between an individual's relationship with mathematics as a student and their relationship with mathematics as a teacher is often ignored. When measuring mathematics anxiety, the individual is placed in the student role of mathematics, not in the teacher role of mathematics. Brown et al. (2011) note this disconnect. They found that "preservice teachers with low or no mathematics anxiety in their prior experiences can still possess mathematics teaching anxiety when teaching mathematics to students, and vice versa for preservice teachers with high levels of mathematics anxiety in their backgrounds" (p. 11).

The goal of this research was to explore this disconnect, examine mathematics teaching anxiety in pre-service elementary school teachers, and investigate the relationship with mathematics anxiety and mathematics teacher efficacy. Specifically, this research aimed to answer the following question and sub questions:

1. What roles do mathematics teaching anxiety, mathematics anxiety, and mathematics teacher efficacy have on pre-service elementary school teachers?
   a) What is the relationship between mathematics teaching anxiety, mathematics anxiety, and mathematics teacher efficacy?
   b) How are varying levels of mathematics teaching anxiety, mathematics anxiety, and mathematics teacher efficacy experienced in elementary school pre-service teachers?

LITERATURE REVIEW

This literature review will serve as a brief overview for each of the three constructs.

MATHEMATICS ANXIETY

It has been highlighted before that pre-service teachers are an at-risk group for mathematics anxiety (Bursal & Paznokas, 2006; Gresham, 2009; Vinson, 2001). It has been suggested that high levels of mathematics anxiety in pre-service teachers can lead to increased levels of mathematics anxiety in their future students (Vinson, 2001), poor mathematical performance from their students (Gresham, 2009), and a higher probability of using traditional teaching methods such as lecturing, devoting more time to seatwork, and avoidance of using engaging unstructured teaching methods such as implementing mathematical manipulatives and asking open-ended questions (Bursal & Paznokas, 2006; Gresham, 2004; Vinson, 2001).

Gresham (2007) and Sloan (2010) have shown that a methods course can reduce mathematics anxiety in pre-service elementary school teachers, but if this reduction is permanent is not clear. For example, Gresham (2017) surveyed and interviewed 10 in-service teachers who were a part of a study five years prior that involved the levels of mathematics anxiety in pre-service teachers before and after a methods course. It was discovered that although there was an initial reduction in mathematics anxiety, all 10 of the in-service teachers had their mathematics anxiety return during their teaching. This is concerning as it shows the reduction in mathematics anxiety as only temporary.

MATHEMATICS TEACHING ANXIETY

Mathematics teaching anxiety is often conflated with mathematics anxiety. Researchers often use instruments focused on measuring mathematics anxiety where mathematics teaching anxiety appears to be closer aligned with their research questions. Brown et al. (2011) put forward an initial effort to distinguish mathematics teaching anxiety from mathematics anxiety. Fifty-three pre-service elementary school teachers were asked to self-report after teaching a minimum of three elementary mathematics lessons at local elementary schools. The results from coding these self-reports showed that over one-third of the pre-service teachers reported having high mathematics anxiety but did not experience mathematics teaching anxiety and / or identified as having no mathematics anxiety, but high mathematics teaching anxiety. This evidence shows the need to separate and compare mathematics teaching anxiety and mathematics anxiety.

MATHEMATICS TEACHER EFFICACY

Mathematics teacher efficacy is housed in Bandura's (1986) self-efficacy and social cognitive theory. Bandura (1977) believed that efficacy beliefs were dependent on the context that the individual was in. In the context of teaching mathematics, mathematics teacher efficacy is the extent to which a teacher believes they have the ability to affect a student's mathematical performance when specifically *teaching* mathematics.

Gavora (2010) shows that when challenges occur while teaching, teachers with low levels of teacher efficacy put forward less effort and persevere less compared to their highly efficacious counterparts. Mathematics methods courses have shown to increase pre-service elementary school teachers' levels of mathematics teacher efficacy (Charalambous et al., 2008; Swars et al., 2006; Utley et al., 2005) as well as upon completion of a mathematics content course (Alsup, 2004).

Mathematics teacher efficacy is often connected to pre-service teachers' relations with past mathematics teachers and/or to their parent's relationship with mathematics (Brown, 2012; Swars, 2005). This is concerning as it hints towards mathematics teacher efficacy being cyclical in nature—a teacher with poor mathematics teacher efficacy, negatively impacts their students and creates a potential future teacher with low levels of mathematics teacher efficacy.

METHODS

In this section the research design will be described, followed by a description of the participants and their context, the instruments used, and finally the methods of data analysis.

RESEARCH DESIGN

A modified version of an explanatory sequential mixed methods design (Creswell & Plano Clark, 2007) with three phases was used. The first phase involved the distribution of a questionnaire which was comprised of demographic questions, the Revised Mathematics Anxiety Rating Scale (Alexander & Martray, 1989), a modified version of the Teaching Anxiety Scale (Parsons, 1973), a modified version of the Teachers' Sense of Efficacy Scale (Tschannen-Moran & Woolfold Hoy, 2001), and the open-ended question: "Tell me about an experience you had during your teacher education program learning (in class or on a placement) that might have come to mind while completing this survey."

The second phase of this study was a set of semi-structured interviews with pre-service elementary school teachers from the survey who agreed to be interviewed. The interview was comprised of 10 questions and was conducted in person or via a video chat program such as Skype (https://www.skype.com/en/) or Zoom (https://zoom.us/). This interview was used to further develop an understanding of the pre-service teachers' relationships with mathematics and teaching mathematics.

For the third phase, pre-service teachers who were interviewed from phase two were contacted for an additional interview. The same interview protocol from phase two was used for phase three with the addition of individual notes for each interviewee based on the results from the initial analysis of their first interview data to delve deeper and gather any information potentially missed.

PARTICIPANTS

One hundred and eighty-five pre-service elementary school teachers across six major universities in Ontario, Canada responded to the questionnaire with 87% identifying as female. For phase two, 16 participants were interviewed with six of these participants interviewed again in phase three.

INSTRUMENTATION

The questionnaire contained demographic questions, three separate scales used to measure mathematics anxiety, mathematics teaching anxiety, and mathematics teacher efficacy, and one open-ended question. The Revised Mathematics Anxiety Rating Scale (RMARS; Alexander & Martray, 1989). The RMARS is made up of 25-items on a five-point Likert scale and has three subconstructs: mathematics test anxiety, numerical task anxiety, and mathematical course anxiety.

To measure mathematics teaching anxiety, a modified version of the Teaching Anxiety Scale (TCHAS; Parsons, 1973) was used. The questions in this instrument were modified to fit the context of a mathematics classroom. For example, the question "I feel uncertain about my ability to improvise in the classroom setting," was changed to; "I feel uncertain about my ability to improvise in a *mathematics* classroom setting." The TCHAS is comprised of 25 items on a five-point Likert scale.

Similarly, a modified version of the short form Teachers' Sense of Efficacy Scale (TSES; Tschannen-Moran & Woolfolk Hoy's, 2001) was used for measuring pre-service teachers' mathematics teacher efficacy. The TSES was modified by specifying the context of the questions to a mathematics classroom. The TSES consists of 12 items on a nine-point Likert scale and has three underlying constructs: efficacy for instructional strategies, efficacy for classroom management, and efficacy for student engagement.

For the interviews, ten open-ended questions were developed and structured to be approximately an hour in length. The development of the interview questions was guided by the collection of literature surrounding mathematics anxiety, mathematics teaching anxiety, and mathematics teacher efficacy. The purpose of the interview was to gain a better understanding of pre-service teachers' relationship between these three constructs.

DATA ANALYSIS

Data collected from the online questionnaire was entered into the data analysis software *SPSS Statistics Version 25*, with reverse coding performed as required. Before any analyses was done, factor analyses and reliability checks were completed to ensure the instruments were behaving as expected. Afterwards, the means for the RMARS, TCHAS, and TSES scales were computed for each individual and used to place participants into categories based on mathematics anxiety, mathematics teaching anxiety, and mathematics teacher efficacy (high, moderate, or low anxiety/teaching anxiety / teacher efficacy).

Analyses of variances (ANOVAs) were computed for the groupings for each category. This analysis served to analyse differences between the high, moderate, and low groups for each construct and contrasts were used to see which groups differed (Field, 2018). Multiple separate ANOVAs were computed instead of a MANOVA because of the structure of the research questions.

For the qualitative data analysis interviews from phase two and phase three were recorded, transcribed, and imported into the qualitative analysis program *NVivo* Version 12 and coded using emergent coding (Creswell, 2007). This research set out to examine the relationship between mathematics anxiety, mathematics teaching anxiety, and mathematics teacher efficacy; therefore, the qualitative analysis procedure was designed to isolate one construct and examine it through the lens of another. For example, all individuals with high mathematics teacher efficacy were grouped together and then these interviews were examined by looking for instances of mathematics anxiety and then coded again separately for mathematics teaching anxiety. This coding procedure allowed for a perspective of how an individual with high mathematics teacher efficacy experiences the other two constructs. Triangulation, member-checking, and inter-rater reliability were used to validate these findings.

RESULTS

The following section will outline the results found from the quantitative data followed by the qualitative data and then mixed in the discussion and conclusion sections.

QUANTITATIVE RESULTS

To begin, each of the three instruments were verified for their validity and reliability. Due to the size constraints, it can only be reported that all three instruments were reliable and valid.

A two-tailed bivariate correlational analysis was done with the RMARS, TCHAS, and TSES to determine the relationship between the three constructs. It was found that mathematics teaching anxiety was positively correlated with mathematics anxiety and negatively correlated with mathematics teacher efficacy. There was no correlation found between mathematics anxiety and mathematics teacher efficacy.

For a better interpretation of the differences between groups of high, medium, and low for each construct, multiple ANOVAs were computed with the Gabriel procedure as a post-hoc test. It was found that

- individuals with low mathematics anxiety had significantly lower mathematics teaching anxiety than those with average mathematics anxiety [$F(2,182)=57.99$, $p<.01$] and high mathematics anxiety [$F(2,182)=57.99$, $p<.01$],
- individuals with average mathematics anxiety had significantly lower levels of mathematics teaching anxiety than those with high levels of mathematics anxiety [$F(2,182)=57.99$, $p<.01$],
- individuals with low mathematics teacher efficacy had significantly higher mathematics teaching anxiety than those with average mathematics teacher efficacy [$F(2,182)=15.275$, $p<.01$] and high mathematics teacher efficacy [$F(2,182)=15.275$, $p<.01$],
- individuals with average mathematics teacher efficacy had significantly higher levels of mathematics teaching anxiety than those with high mathematics teacher efficacy [$F(2,182)=15.275$, $p<.01$],
- there was a significant difference in mathematics teacher efficacy scores amongst individuals with low levels of mathematics anxiety and those with average levels of mathematics anxiety [$F(2,182)=3.441$, $p<.05$], and
- individuals who were considered to have high mathematics anxiety had no significant difference in terms of their mathematics teacher efficacy than individuals with average or low mathematics anxiety.

QUALITATIVE RESULTS

The qualitative results were gathered from the short answer question and the interviews. For the category of a ‘typical’ student, the data was coded once using emergent coding without any lens of a construct. The results can be seen in Table 1.

DISCUSSION AND CONCLUSION

Mathematics teaching anxiety and mathematics anxiety have been shown in past research to be positively correlated (Adeyemi, 2015; Haciomeroglu, 2014; Peker & Ertekin, 2011; Unlu et al., 2017). The findings from this research are in alignment with this conclusion. The quantitative results show a positive correlation between the two constructs and the qualitative results bolster that conclusion.

In this research, mathematics teaching anxiety and mathematics teacher efficacy was found to be negatively correlated—in alignment with the current literature (Peker, 2016, Unlu et al., 2017). We see a moderately negative correlation from the quantitative findings, and this is echoed in the qualitative findings as well.

	Mathematics Anxiety Lens	Mathematics Teaching Anxiety Lens	Mathematics Teacher Efficacy Lens
High Mathematics Anxiety Bin		Developing Strategies to Cope	Pushing Through Anxiety
Low Mathematics Anxiety Bin		Relation to Peers Bringing the Passion	No Themes Found
High Mathematics Teaching Anxiety Bin	Good Student, Poor Student Experience		Reliance on Resources
Low Mathematics Teaching Anxiety Bin	Mathematics Content Knowledge		Overcoming Early Teaching Obstacles
High Mathematics Teacher Efficacy Bin	Overcoming Early Teaching Obstacles	Enjoyment of Mathematics	
Low Mathematics Teacher Efficacy Bin	Small Scale Confident, Large Scale Worried Taught One Way, Asked to Teach Another	Comfortability with Mathematics	
Average Bin	Optimistic Hesitance		

Table 1. Themes generated from qualitative analysis.

Interestingly, the evidence regarding the relationship between mathematics anxiety and mathematics teacher efficacy misaligns with the current literature—that mathematics anxiety is negative correlated to mathematics teacher efficacy (Gresham, 2008; Swars et al., 2006; Unlu et al., 2017). The quantitative and qualitative evidence from this research shows little to no correlation between the two constructs. In the quantitative data we see the total score for the RMARS not correlated to the total for TSES and even the subconstructs between the two show little to no correlation. In the qualitative data we see instances of low mathematics anxiety actually causing poorer teaching practices as the pre-service teachers cannot relate to their struggling students.

The research presented here introduced mathematics teaching anxiety and examined the interactions between all three constructs. As stated earlier, mathematics teaching anxiety is often conflated with mathematics anxiety. It is believed that the correlation between mathematics anxiety and mathematics teacher efficacy observed in prior research was influenced by the overlooked construct of mathematics teaching anxiety. When mathematics teaching anxiety is introduced as a distinct and separate construct from mathematics anxiety, a better representation of what is happening with pre-service elementary school teachers is given. This does not completely discredit mathematics anxiety and its impact on mathematics teacher efficacy. High or low levels of mathematics anxiety could impact mathematics teaching anxiety therefore impacting mathematics teacher efficacy, but the primary connection influencing teaching mathematics is between mathematics teaching anxiety and mathematics teacher efficacy.

It would be a bold conclusion to assume that mathematics teaching anxiety has a greater impact on teaching than mathematics anxiety but given the amount of research showing the impact mathematics teacher efficacy has on teacher effectiveness and student learning, it is powerful to see that mathematics teaching anxiety has a greater impact on mathematics teacher efficacy than mathematics anxiety. With this knowledge, I argue that teacher educators should focus their attention on mathematics *teaching* anxiety not mathematics anxiety. Pre-service teachers are resilient, resourceful, and have found ways to overcome their mathematics anxiety. Instead, we should be looking at those who are nervous to teach mathematics, not to study it. I believe this shift in perception would greatly improve pre-service teaching and ultimately benefit elementary school students.

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THE DEVELOPMENT OF (NON-)MATHEMATICAL PRACTICES THROUGH PATHS OF ACTIVITIES AND STUDENTS' POSITIONING: THE CASE OF REAL ANALYSIS

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In 2012, I took my first steps into the world of mathematics education research as an undergraduate student at the annual meeting of the CMESG; nine years later, I participated as a New PhD! This report synthesizes my presentation, in which I aimed to portray the essence of my doctoral thesis (Broley, 2020): the motivation behind it, the study on which it is based, and some of the main contributions it makes.

MOTIVATION

My journey as a doctoral student began from a general interest in the process of becoming a mathematician, given the considerable gap that seems to exist between what students do in math courses and what mathematicians do to solve new mathematics problems. My own experiences, as a mathematics student and in research in mathematics education, led me to wonder if students learn to behave mathematically (i.e., like mathematicians) in their progression of mathematics coursework. From a research point of view, one issue with this wondering is that the term *mathematical behaviour* is ill-defined. Scholars have proposed numerous definitions over the years; and I would guess that each reader could have a different way of explaining what this means to them. Hence, I needed to decide what it would mean to me in the context of my thesis.

For this, I turned to the notion of praxeology offered within the Anthropological Theory of the Didactic (ATD; Chevallard, 1999) and I posed the following definition: For a specified area of mathematics, an individual has developed a *mathematical practice* if they can (1) identify a given task t as belonging to a general mathematical type of task T; (2) choose and implement an appropriate mathematical technique τ for solving t; (3) describe in a mathematical discourse θ how and why τ works; and (4) acknowledge the existence and importance of the mathematical theory Θ that supports θ, where 'mathematical' means 'would be considered acceptable among the community of mathematicians working in the specified area.' In other words, to have developed a mathematical practice, one must have both a mathematical know-how, $\Pi = (T,\tau)$, and a mathematical know-why, $\Lambda = (\theta,\Theta)$. My original wondering was thus specified: Do students develop mathematical practices in their progression of mathematics coursework?

Much research has documented how the activities offered in Calculus courses can enable and encourage the development of *non-mathematical practices* (e.g., Bergqvist, 2007; Brandes, 2017; Cox, 1994; Hardy, 2009; Lithner, 2004; Orton, 1983; Selden et al., 1999; Tallman et al., 2016). These studies have shown that Calculus students can obtain good passing grades by learning to solve a limited set of task types through highly algorithmic techniques, with little to no understanding of the mathematical theory that can be used to describe and explain those techniques. But what happens as students progress to more advanced mathematics courses?

Theoretically speaking, it has been proposed that students' practices should become more mathematical in Analysis courses (Winsløw, 2006): if Calculus students focus on mastering know-how (e.g., techniques for calculating limits, derivatives, integrals, and so on), then Analysis students are typically expected to learn the corresponding know-why (e.g., definitions, theorems, and proofs that describe and explain limit-finding techniques); they are also typically expected to develop more abstract proof practices that may be more likely to require an understanding and use of underlying mathematical theory. This said, there is also a lot of work that has pointed to obstacles (cognitive, epistemological, didactic, …) that could impede students from developing mathematical practices in Analysis courses (e.g., Bergé, 2008; Kondratieva & Winsløw, 2018; Maciejewski & Merchant, 2016; Przenioslo, 2004; Raman, 2004; Tall, 1992). Some studies highlight the possibility of students successfully completing Analysis courses having developed limited, algorithmic, non-mathematical practices (e.g., Weber, 2005).

MY STUDY

Motivated by this literature, I wanted to carry out an exploratory study to gain an understanding of (1) the nature (mathematical or otherwise) of the practices developed by students in a first Analysis course; and (2) the factors that may be shaping the development of those practices.

Given the impossibility of considering every possible factor, I had to decide which ones I would consider in my thesis. I formulated my choice of factors as three interconnected layers of influence on the practices developed by students in any mathematics course: the broad *institutional context* of the course, the *activities offered to students* by the teacher of the course, and the *positions adopted by students* in the course, which dictate how they interact with the activities offered to them. The broadest layer (institutional context) represents my choice to work from an institutional perspective (Chevallard, 1991) whereby any mathematics course operates within a relatively stable social structure called a didactic institution, which differs from the professional institutions on which it is based. For example, in university mathematics, curricula, textbooks, and evaluation procedures define practices to be taught (by teachers) and practices to be learned (by students), which can already be significantly different from the mathematical practices produced and used by mathematicians. The two other factors (activities and positions) were inspired by literature I read. On the one hand, when a student takes a mathematics course, they are offered numerous activities (in lectures, tutorials, textbook exercises, …). What past research (e.g., Hardy, 2009) suggests is that the activities that arise in assessment situations can play a special role: they can communicate to students a minimal core of practices to be learned to receive a good passing grade, which can shape the practices that students develop. On the other hand, past research (e.g., Liljedahl & Allen, 2013; Sierpinska et al., 2008) also suggests that students can take different approaches towards the activities offered to them, which can also shape their practices. The literature I read emphasized a distinction between acting in a position of Student (a subject of the didactic institution, for whom mathematics is the course to be passed, and whose actions are shaped by features of the course) and acting in a position of Learner (a cognitive subject, for whom mathematics is a mental endeavour to be shared with the teacher, and whose actions are shaped by principles and standards of the discipline). With this choice of factors, I specified my second research objective to gain an understanding of how students' practices may be shaped by the positions they adopt and the activities they are offered.

The study I carried out to address my objectives focused on one first Analysis course (A1) at a large North American university. A1 is a core course in mathematics programs leading to graduate work (e.g., in statistics or pure mathematics). Students taking A1 will have usually completed courses in single variable and multivariable Calculus. The topics covered in A1 are the same as those typically covered in single variable Calculus: sequences of real numbers, and limits, continuity, and differentiability of single variable real-valued functions. The difference is that A1 also aims to introduce students to mathematical rigour and proofs.

When designing my study, I knew that I wanted to conduct *task-based interviews* (Goldin, 2000) with students after they passed A1. In such an interview, tasks are presented by an interviewer to an interviewee in a pre-planned way, and the latter is expected to think aloud while engaging with the tasks. By asking a successful A1 student to think out loud while solving mathematics tasks, I knew that I could observe how they identify the tasks, choose and implement techniques, and describe and justify techniques; ultimately, I would be able to infer the practices they had developed. Moreover, by spending a significant amount of time planning for the interview, I knew that I could carefully design the tasks and an interview protocol not only to elicit students' practices, but also to make it possible to examine their nature (objective 1) and to infer possible links with the positions students had adopted and the activities they had been offered (objective 2). But to do this, I first needed to conduct an *analysis of activities* typically offered in A1. Since I was operating under the assumption that assessment activities can play an important role in shaping students' practices, I chose to analyze the kinds of activities students received on their weekly assignments, midterm, and final exam, as well as the past midterms and final exams they could use to guide their studying. I also analyzed any solutions to those activities that were made available by the teachers of A1, the course textbook, and the course outline.

After analyzing over 200 activities, I designed six interview tasks and a protocol, which I implemented with 15 students (S1 to S15) after they successfully completed (with passing grades) A1. Interviews lasted between two and three hours, and after combining audio recordings with participants' written work, I ended up with over 600 pages of transcripts to analyze.

TWO THEORETICAL CONTRIBUTIONS

As I carried out my analyses, two theoretical tools emerged for thinking about how activities and positions may shape the practices developed by students in a mathematics course: the notion of a path to a practice and a positioning framework containing five positions. To describe these tools, I will provide some examples related to my second interview task (T2): *Show that the function* $f(x) = e^x - 100(x-1)(2-x)$ *has two zeros*. Before proceeding, readers may want to think about how they would go about solving T2: Do you have a (mathematical) practice for doing so?

PATH TO A PRACTICE

The notion of a path to a practice arose during my analysis of assessment activities offered in A1. As I worked on constructing models of practices students may have been expected to develop in completing those activities, I started to notice that I could separate the activities into two main groups. First, there were activities that were relatively *isolated* in that they related to practices addressed in only one or two assignment activities. Second, there were activities that were connected to each other in that they related to the same practice, which was addressed many times, on both assignments and (past) exams (they formed what we called a *path to a practice*). This observation led to several key ideas. I think the first time a student encounters an activity, their actions are relatively isolated in the sense that they are primarily focused on solving the activity at hand; however, as the student encounters other activities linked to the same type of task, I think they are encouraged to develop a practice. Moreover, there may be different aspects influencing the developed practice: not only the activities themselves (e.g., how they are stated, the kinds of objects they involve), but also what is made explicit about the activities (e.g., in teachers' solutions or textbooks) or the context in which the activities take place (e.g., those on past exams may have a stronger influence than those on assignments). Based on these ideas, I decided to focus my analysis of assessment activities on identifying paths to practices and developing models of practices students may develop in following those paths. Then I selected some paths on which to base my interview, and I created tasks that would help me see both the nature of students' practices and the potential influence of the activities. More precisely, I aimed for tasks that were *recognizable* (so students would apply a practice they had developed in A1) and *deceptive* (so any non-mathematical nature of their practices would be revealed).

<u>An Example: The Creation of T2</u>

Figure 1 shows a collection of assessment activities that I identified as belonging to a path to a practice and my model of practices students may have developed in following the path. For instance, it seemed that students were expected to learn a practice for solving the type of task: prove that a function $f(x)$ has at least n zeros on a specified domain D. The most commonly illustrated technique in the solutions I analyzed was to show that there are n sign changes in f and to cite the Intermediate Value Theorem (IVT). Solutions were inconsistent in including an explanation for how or why the IVT supported the technique (e.g., the required continuity condition on f), and there were no methods explicitly illustrated for finding the sign changes. In fact, D was typically an interval for which the sign changes could be found easily (by plugging in the endpoints and possibly a few simple points in between). In conducting this analysis, I began to wonder if students would have developed a specified (perhaps non-mathematical) practice and I created T2 to try to see if that was the case: I did not specify a domain, and I constructed f so that plugging in integer values at random would not be enough to locate sign changes since it leads only to positive values (these were the deceptive features of the task).

Show that the polynomial
$P(x) = x^5 - 3x + 1$
has a zero in the interval (0,1).

Show that $f(x) = e^{-x^2}$ has a fixed point in the interval (0,1).

Prove that the function $f(x) = e^x - 100x$ has exactly one zero in the interval [0,1].

Prove that the equation $\cos(x) = 5x(1-x)$ has exactly two solutions in [0,1].

Let $f: [0,1] \to [0,1]$ be continuous. Prove that the equation $f(x) = x^2$ has a solution in [0,1] (you may use the Intermediate Value Theorem).

Show that the equation $e^x = 3x^2$ has at least two positive solutions.

How many zeros does
$f(x) = 4x^3 - 32x^2 + 79x - 60$
have in the interval [0,5]?

…

$\dot{T}_2$: Prove that $g(x)$ has exactly n fixed points on a domain D. ⇒ (let $h(x) = x$)
$\widetilde{T_2}$: Prove that $g(x) = h(x)$ has exactly n solutions on a domain D. ⇔
$\widehat{T_2}$: Prove that g and h intersect exactly n times on a domain D ⇒ (let $f(x) = g(x) - h(x)$)

T_2: Prove that a function $f(x)$ has exactly n zeros on a domain D.
Typically: $n \in \mathbb{N}$ is small (e.g., 1, 2, 3, or 4) and D is a specified interval $[a, b]$ with $a, b \in \mathbb{Z}$.
$T_2 = T_{2_a} \wedge T_{2_b}$

T_{2_a}: Prove that $f(x)$ has at least n zeros on D.

τ_{2_a}: Find n sign changes of $f(x)$ on D.
Typically: calculate $f(a)$ and $f(b)$, and maybe $f(c)$ for c equal to integers in (a, b) or midpoints between integers.

θ_{2_a}: "By the Intermediate Value Theorem."

T_{2_b}: Prove that $f(x)$ has at most n zeros on D.

$\tau_{2_{b_1}}$: Show that f' is strictly positive (or negative) on n intervals I_i that form a partition of D. Illustrated for $n = 1$: Show that f' is strictly positive (or negative) on D.	$\tau_{2_{b_2}}$: Assume that f has $n + 1$ zeros and derive a contradiction. More specifically, argue that f' has n zeros, f'' has $n - 1$ zeros, …, and f^n has 1 zero; and show f^n has no zeros. Illustrated for $n = 2$: Assume that f has 3 zeros, whereby f'' has 1. Show that f'' has no zeros.	$\tau_{2_{b_3}}$: Illustrated for f a polynomial: Note that the degree (or order) of f is n.
$\theta_{2_{b_1}}$: "If $f' > 0$ (or < 0) on an interval I, then f is strictly increasing (or decreasing) on I and can cross the line $y = 0$ at at most one point."	$\theta_{2_{b_2}}$: "By Rolle's Theorem and by contradiction."	$\theta_{2_{b_3}}$: "If f is a polynomial of order n, then f has at most n zeros."

Figure 1. Examples of activities forming a path to a practice (left) and the corresponding model of practices students may develop (right) that led to the creation of T2.

In Broley et al. (submitted), we describe the behaviour of the 15 interviewed students when solving T2, highlighting three ways in which their practices revealed themselves to be non-mathematical (or mathematical). These ways relate to expected patterns of behaviour aligning with the above model: e.g., 12 of the 15 participants quickly identified T2 with the use of a special "theorem," "property," or "method"; and 11 of the 12 participants who sought sign changes in *f* faced the struggle of finding only positive values. This said, there were also some interesting differences in the nature of the practices participants seemed to be using.

POSITIONING FRAMEWORK

To try to make sense of how different students may develop different kinds of practices when faced with the same collection of activities, I introduced a positioning framework (Figure 2), which builds on the positions of Student and Learner emphasized in past work. The framework emerged from an in-depth analysis of five participants who seemed to embody five different positions in both the nature of their practices and the way they described their actions in A1.

Definition: The *position* one adopts in a given mathematics course is an idealized relationship with what one perceives to be the practices to be learned in the course.*

Student	Skeptic	Mathematician in Training	Enthusiast	Learner
does what is necessary to obtain a good grade in the course; that is, seeks to identify the minimal subset of practices to be learned in order to succeed in assessment activities.	questions what they perceive to be the practices to be learned in the course in the sense of critically reflecting on whether they should be learning them or not.	incorporates the practices to be learned in the course into a larger collection of practices they perceive as important for participating in a community of mathematicians.	dedicates themselves to what they perceive to be the practices to be learned in the course.	seeks their own understanding of the practices to be learned in the course.

** It is not necessarily the result of a conscious choice; it is possible to adopt more than one, to different degrees, in different circumstances; and it is a theoretical tool for understanding (not perfectly representing) reality.*

Figure 2. A positioning framework.

An Example: A Student Response to T2

S11 seemed to embody the Student position. During his interview, he spoke about several strategies he used in A1, including "analyzing the teacher," "taking crazy notes," and "selective studying." His task solving behaviour for T2 provides one example of how his positioning as a Student seemed to lead to practices that reflect an attempt at identifying the minimal core of practices to be learned to succeed in the course.

When S11 received T2, the first thing he spoke about was using the IVT. Then he used a calculator to find *f(x)* for different values of *x* (0,1,-1,…), explaining: "Usually, what we saw in [A1], was that the interval in which the function alternates between negative and positive is close to zero." As S11 continued to make calculations, he found all positive responses. He eventually seemed stuck: the problem, he said, was that he was not sure which interval to consider. He then interacted with the interviewer (I) as follows:

> I: *"Would it help if this problem was phrased differently and it had an interval?" […]*
> S11: *"Yeah, but then, if you just tell me an interval, it's kind of like, I can just plug in values at this point. […] Like if you tell me, ok, like it's not this function, it's another function, and you tell me, ok, it's zero to five [writing [0,5]]. Then at that point you can just plug in the values and just see how many zeros you have. No?"*
> I: *"Ok. Which values do you mean? Do you mean like plug in zero and five?"*
> S11: *"Yeah. Exactly. Zero, one, two, three, four, five. In this function. And you'll see which one alternates between negative and positive. And you'll figure out how many zeros you have. But… Yeah, the hard part is not having the interval." […]*
> I: *"How do you know that you can apply the Intermediate Value Theorem in this case?"*
> S11: *"I don't. […] My logic with finding zeros is finding a value before and finding a value after that point at which it's equal zero that are alternating signs. And the only theorem that we have that talks about that is… Well that I know, is the Intermediate Value Theorem. So that's why I instantly thought of that."*

In my analysis, I found that S11 had developed a non-mathematical practice for "finding zeros": his know-how included a technique composed of steps that just seemed to work (at least for the assessment activities in A1), and his know-why included an acknowledgement of theory (the IVT), which he did not know how to use to mathematically describe and justify a technique.

<u>A Contrasting Example: A Learner Response to T2</u>

S3 seemed to embody the Learner position. His way of describing his actions in A1 stood in contrast with S11: for instance, while S11 said he used lectures to analyze the teacher and take crazy notes, S3 said his goal during lectures was to understand the mathematics and, if he succeeded, he did not need to take notes. S3's solving of T2 provides one example of how his positioning as a Learner seemed to lead to practices that reflect a personal understanding of the practices to be learned, which seemed to be weakly shaped by the assessment activities.

S3 did not identify T2 with "the IVT." When he received T2, the first thing he spoke about was that the question says that $e^x = 100(-x^2 + 3x - 2)$ has two solutions. Then he explained:

> *What I am trying to prove is that when I graph it, ok? Ok, so this is e^x [starting to draw Figure 3]. Ok, so I have no idea yet how this works, how this one* [$g(x) = 11(-x^2 + 3x - 2)$] *looks. But what I'm trying to prove is that it is going to cross twice.*

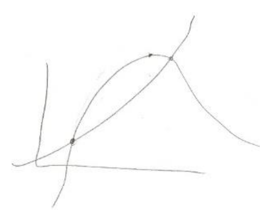

Figure 3. S3's reinterpretation of T2 as a task about intersections of graphs.

To do his proof, S3 analyzed *g*, confirming the pertinent elements depicted in his drawing. He noted that the negative sign in front of x^2 means that *g* is negative for ±1 000 000 000. Then he found (by solving $g'(x)=0$) that the maximum point of g occurs at $x = 1.5$, where the value of *g* (i.e., $g(1.5)=25$) is greater than the value of $e^{1.5}$. In comparison with S11, S3 seemed to progress through his solution, not by recalling mathematically superficial features of assessment activities and solutions from A1, but by identifying and using mathematically relevant concepts (e.g., *x* is a zero ⇔ $f(x)=0$). In this sense, his practices appeared to be more mathematical.

CONCLUSIONS

In my thesis (Broley, 2020), the above examples can be found among numerous others, which led me to three general conclusions: (1) students' practices can be (non-)mathematical in different ways and to varying degrees; (2) the activities offered to students can form paths that enable and encourage the development or the maintenance of practices that are not exclusively mathematical; and (3) the nature of students' practices can also be shaped by the positions they adopt; each position (Student, Skeptic, Mathematician in Training, Enthusiast, and Learner) may be productive towards (non-)mathematical practices in different ways; moreover, students may adopt different positions, to different degrees, in different circumstances.

When I think about my own experiences in mathematics courses, I can certainly relate to the Student position. I can relate to the participants in my study who spoke about trying to manage the massive amount of course material and intense exams by focussing on what was necessary to get the grade. I can relate to finishing a course with a superficial understanding of how to solve certain types of tasks. As alluded to above, I think the Student position can be productive towards mathematical practices: for example, in skillfully following the paths laid out for them, the Student develops robust practices that can be examined, questioned, and transformed. This said, I also conjecture that to become a mathematician (or an expert in any field), acting in a position of Student is not enough: one must somehow, at some point, develop an ability to flexibly shift out of studenting (to learn, train, be skeptical, be enthusiastic, …). Perhaps it is to be expected that most students will Student to some degree, and most students will develop practices that are non-mathematical in some senses. The question then becomes: How can students be given opportunities to go beyond studenting, to acquire an ability to shift their positioning and develop practices that are more mathematical in nature?

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THE FINANCIAL NUMERACY AFFORDED IN SECONDARY MATHEMATICS: A STUDY ON THE TEXTBOOKS, PERCEPTIONS AND PRACTICES OF TEACHERS IN QUÉBEC, CANADA

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INTRODUCTION

In recent years, financial education has become more important in academia and society (Arthur, 2012). However, this matter has yet to find mainstream attention in the field of mathematics education. With increasing efforts to incorporate financial numeracy in curricula around the world, the question of how to teach it still lacks an appropriate answer. I addressed this issue by focusing on financial numeracy from the perspective of one key stakeholder: mathematics teachers.

A mathematics education perspective is necessary because, despite the role of mathematics in financial education being recognized by researchers and institutions, we still lack a conceptual framework to understand how mathematics and financial education interact and contribute to each other. The current literature shows that integrating financial education seems to increase student engagement (Attard, 2018) and enhance their understanding of both mathematical and financial concepts (Garcia-Santillán et al., 2016). It can also help tackle social justice and develop a sense of empathy (Lucey & Tanase, 2012).

In this research, I used the concept of financial numeracy to refer to the knowledge, confidence, and ability to use numerical/quantitative information in financial situations. I constructed the analytical framework of this research based on two theories: the Theory of Measurement (Hand, 2004) and the Anthropological Theory of the Didactic (Chevallard, 2006). The epistemological contribution of the Theory of Measurement informed my understanding of money as a measurement practice. This theory involves looking at money as a unit of measurement but also unpacking financial concepts and practices to understand how they are defined and how they work. The didactic contribution of the Anthropological Theory of the Didactic informed my understanding of social practices and has important implications to the teaching and learning of financial numeracy. This theory helps us unpack financial situations by looking at their practice block (procedures and practical knowledge) and theory block (justifications to the practice).

The combination of these frameworks helped me conceptualize a framework of three dimensions of financial numeracy which provide a lens to analyze the textbooks, perceptions and practices of secondary mathematics teachers. The contextual dimension refers to the study of mathematics in financial contexts. Consequently, learning about financial concepts and practices is not necessarily part of the goals of the contextual dimension of financial numeracy. Because of that, financial measures (such as price) are only explored through representational measurement (as units of measurement). The conceptual dimension refers to the teaching and learning of financial concepts which requires multiple mathematical processes such as modelling, representing, estimating, measuring, comparing, counting, predicting, rounding, etc. In this sense, we start to unpack financial measures and how they are defined, therefore mobilizing pragmatic measurement. Many ideas, such as profit and investments, gain deeper layers of complexity once we incorporate the conceptual dimension. The systemic dimension refers to the justification, or unpacking, of financial practices and concepts in relation to other epistemological systems: ethics, values, beliefs, politics, economy, etc. The systemic dimension entails investigating certain financial measures and how they are calculated, defined, modelled, and portrayed in society. Figure 1 presents a visual diagram of the two theories and the analytical framework of this research.

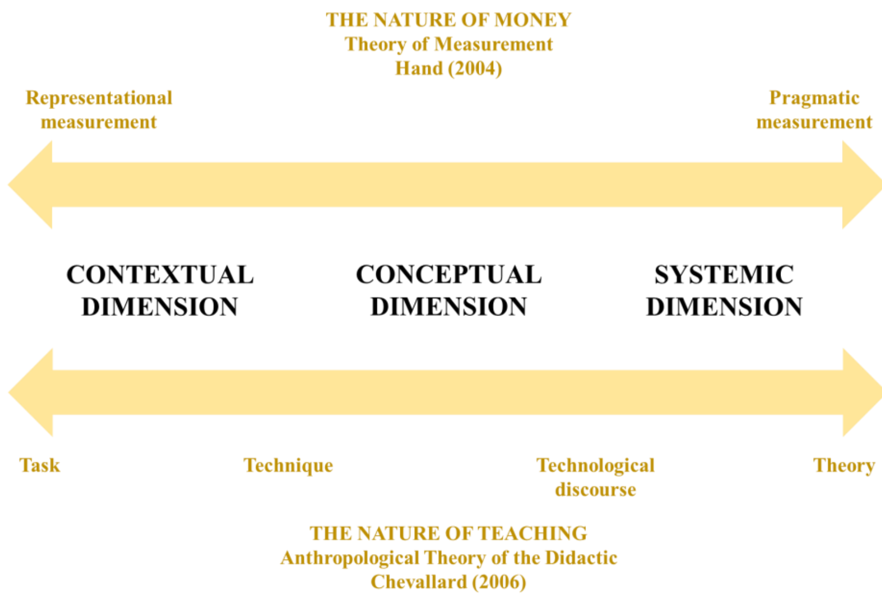

Figure 1. Analytical framework of the research.

These three dimensions engender a spectrum of financial numeracy; they provided me with the conceptual tools to investigate how financial numeracy emerges among secondary teachers. The question that remains to be answered is how these dimensions come together in moments of teaching and learning financial numeracy. To understand their context and inform future professional development to teachers, I asked three research questions:

1. What are the representations of financial numeracy in the didactical materials available to secondary mathematics teachers in Québec?
2. What are their perceptions of financial numeracy in the context of a professional development session?
3. What aspects of financial numeracy do they emphasize in their teaching?

CONTEXT AND METHODS

To answer the above-stated research questions, I used quantitative and qualitative methods to analyze financial numeracy tasks from 40 textbooks, the perceptions shared by 35 teachers in six professional development focus groups, and the teaching practices enacted by six teachers when they implemented financial numeracy lessons. All three strands of data were collected in the context of research projects in partnership with different mathematics educators from Québec.

I gathered data from the three textbook collections approved and recommended by the Québec Ministry of Education. In each textbook, I identified every task with a financial reference. This included tasks that talked about money, represented numbers through currency or referred to financial practices (selling, purchasing, investing, etc.).

Thirty-five secondary mathematics teachers participated in six focus groups in five different regions of Québec. The focus groups were designed for participants to discuss their experiences with financial education and mathematics. First, the teachers filled out a multiple-question assessment of financial concepts developed by the Canadian government which helped to initiate a discussion about everyday financial situations that they might have experienced. The second part was dedicated to their professional development and discussions of their professional needs. We used situational problems that tackled financial and mathematical concepts from the grade 11 curriculum in order to stimulate the teachers' thinking while also providing them with teaching resources.

Finally, during the recruitment process for the focus groups, I also invited teachers to plan a lesson of their choice to be video recorded for analysis. Six teachers volunteered to participate and planned a lesson using the following prompt: "Can you teach a lesson, on the topic of your choice, using a financial situation?"

All three strands of data (tasks, focus group and classroom videos) were coded individually and analyzed using the three dimensions of financial numeracy and broad themes. This approach ensured consistency in the analysis and data integration. In the next section, I present a summary of the findings for each data strand.

FINDINGS

PROFESSIONAL DEVELOPMENT

Results from the focus groups show that teachers seemed to merge both professional and personal stances when expressing their ideas about teaching financial numeracy. Their professional stance referred to perceptions on their students and their own teaching practices. Their personal stance referred to perceptions related to their personal lives, particularly financial practices of consumption, investment, and debt.

The contextual dimension of financial numeracy was apparent whenever teachers talked about what they currently do in their classes. It became apparent that they perceived the provincial curriculum documents as constraints to the content they can afford to teach in their classes. In that sense, the participants envisioned financial numeracy as an instrument to reach their goal of teaching mathematical concepts such as percentage, proportions, exponentials, exponential functions, etc. This heavy focus on mathematics concepts (as opposed to broader competencies, for instance) was evident among the participants who shared concerns about some of the situational problems we presented. In their perception, the mathematics in these tasks was not explicit enough for them to see the benefits to use in class. These participants showed little regard for the potential benefits to developing financial or even mathematical competencies.

The contextual dimension of their perceptions of financial numeracy was also related to student thinking. Since the participants felt more comfortable discussing the mathematics aspect of the tasks, particularly the consumption task, they also mentioned how students usually go about solving similar tasks. They believed the consumption task was closer to their current practices and therefore they were able to speak about how students would probably solve such a situation. Whenever this sort of conversation happened, the teachers did not think the financial nature of the tasks imposed any relevant obstacles to student understanding and performance.

The conceptual dimension emerged whenever participants expressed discomfort with their own understanding of a range of financial concepts and products related to the situational problems. During the discussions, they seemed to value financial numeracy for its potential to their own personal finances. They perceived the researchers as resources to learn more about these matters (particularly investment), but the implications to teaching financial numeracy were not debated in depth. The vast majority of the teachers acknowledged that they do not go beyond explaining some vocabulary in word problems that describe financial situations. They acknowledged the value of learning financial concepts through mathematics by discussing these matters in their own lives.

Finally, the teachers also discussed financial numeracy through a systemic dimension perspective. They did so whenever they shared their own values regarding financial practices and products, but also when they explained their visions for students' financial competencies. When doing so, they made references to a wider range of epistemologies that went above and beyond the boundaries of school mathematics.

Overall, there seemed to be a tension between professional practice and personal lives in the teachers' perceptions of financial numeracy. To address such a tension, I proposed two orientations for the professional development of teachers. The first orientation is toward teachers who have concerns about the appropriateness of financial numeracy in mathematics. These teachers attend to structural barriers such as time and curriculum to avoid integrating financial numeracy. Appropriate support must help them identify the mathematics underlying financial situations more explicitly. The second orientation is toward teachers who feel strongly about financial numeracy but need support in their own content knowledge. These are teachers who were excited to share their individual stances about financial situations. Appropriate support for these teachers includes providing sufficient content and pedagogical content knowledge so they can realize their existing interest in financial numeracy.

TEXTBOOK TASKS

I collected a total of 1362 financial tasks that appeared in the Québec textbook collections, for an average of 30.3 tasks in each grade of secondary school per textbook collection. The contextual dimension constitutes an average of 11.13 tasks in each grade of secondary school, or 37% of the total. For the conceptual and systemic dimensions, the averages are 14.02 (46%) and 5.38 (18%) respectively. Financial numeracy tasks spammed from short word problems that emphasized the explicit mathematical content to open-ended tasks that incorporated students' personal perspectives into the justification of the problems.

The contextual dimension reflected tasks that highlight how mathematics could be used in daily life, which in turn could help students see the value of learning certain procedures that would otherwise be demotivating for them. Accordingly, the financial situation did not play a role in the solution of the word problems, and money was treated as a unit of measurement like time or length. This is why there would not be any significant loss if we changed the financial context for another one. This dimension was also reflected in how the literature refers to financial numeracy as a context to teach mathematics (Bansilal et al., 2012; Pournara, 2013) that makes mathematics more authentic (Althauser & Harter, 2016; Sawatski, 2013).

The conceptual dimension contemplated tasks that required students to engage in multiple mathematical processes. In doing so, mathematics served as an instrument to help students make sense of each financial situation. The financial measures presented in these tasks were questioned and explored through mathematical processes. Consequently, they provided an opportunity to be unpacked, therefore allowing students to learn more about the underlying financial concepts depicted in these situations. In this dimension, the relationship between mathematics and the financial situation were symbiotic: while mathematics helped us understand the situation in depth, the financial situation provided meaning to the equations, graphs and other representations that students needed to construct.

In the systemic dimension, tasks mobilized the same elements described in the previous two dimensions and added another layer of complexity by asking students their own ideas and opinions related to each financial situation. Students would then use mathematics coupled with personal ideas (informed by a variety of epistemologies) to construct an argument to justify or challenge the premises established in the financial situation. Explaining why certain financial situations work as they do (such as the price of gas or the design of a survey) is at the core of the systemic dimension. In addition to that, these questions were more conceptual in the sense that they asked for explanations on top of mathematical procedures.

Overall, there is a noticeable shift in the transition from grade 7, 8 and 9 (early secondary) to grades 10 and 11 (late secondary). Financial numeracy seems to shift from the contextual dimension to the conceptual dimension. The trend holds for all three streams of mathematics in late secondary. In all cases, the systemic dimension was the least frequent.

TEACHING PRACTICES

The teaching practices enacted in class had important similarities that characterized the teachers' approaches to financial numeracy. With the exception of one, all teachers framed their lessons and presented them to students as useful for their future lives. These teachers emphasized the lack of financial knowledge among the general population and how important it was for their students to start developing good practices from a young age. Consistent with this goal, the teachers often built on their personal experiences to either illustrate the content or delve deeper into the implications of their classes.

Overall, four teaching practices emerged in the financial numeracy lessons: emphasizing procedural fluency in financial situations; using technology to focus on the interpretation of results; sharing personal experiences when discussing financial situations; and providing practical advice regarding financial matters.

The contextual dimension of financial numeracy emerged in the teaching practices whenever the teachers emphasized procedural fluency in financial situations. In those lessons, students explored more complex mathematics content when compared to others. Topics included statistical methods for regression, logarithms and exponential functions. These were moments in which both teacher and students focused on the procedural aspect of these topics. A focus on procedures not only revealed the contextual dimension, but also the specific approach to this dimension. It is important to notice here that a lesson would still have been categorized as contextual had the teacher emphasized conceptual aspects from the mathematical perspective. For instance, when students were conducting linear regression in statistics class, the teacher focused on the procedures for doing it, but did not discuss as much what linear regressions represented (either mathematically or financially).

In these lessons, students generally had to perform certain calculations using techniques being taught by the teacher. In fact, at some points the financial nature of the lesson was lost or forgotten. At the same time, it must be noted that the classes mostly situated in the contextual dimension were also those which had more explicit connections to the mathematics curriculum in Québec. The teachers were clear about what topics they were teaching, and it was noticeable how the tasks proposed by them addressed such topics.

The conceptual dimension emerged in classes whenever teachers explained concepts related to their tasks. Using technology to focus on interpretations was a common teaching practice in this dimension. In most cases, teachers did not constrain their explanations to a single concept, but rather unpacked a series of ideas coming from the financial situations in their lesson. These concepts varied in breadth and depth, covering a spectrum of financial numeracy practices associated with credit cards, consumer debt, mortgages, interest rates, income taxes, etc. The mathematics implicated in this dimension was generally less complex than those in the previous one. The use of technology in these tasks facilitated the transition toward the conceptual dimension as the teachers were able to focus the discussions on the interpretation of results. For instance, in one class the use of spreadsheets helped students make calculations that would otherwise take too long. Because of that, they had enough time to focus their energy into getting more information about the progression of debt and making interpretations of the results (how long it would take to pay off different items). They also provided meaning to these results and connected them back to the financial situations described in their tasks. This circular aspect made these lessons more consistent with their goals of developing foundational knowledge of certain financial concepts.

The systemic dimension of financial numeracy also emerged in several classes. It did so mostly when teachers shared their personal experiences with the financial situations depicted in their tasks and provided practical advice to students. Upon making those connections, various teachers provided practical advice on financial matters to students. They often highlighted the multitude of factors that impact one's personal finance decisions, therefore offering nuanced perspectives into the concepts being explored in the lessons. However, it is yet inconclusive whether the systemic dimension was mobilized by teachers intentionally. It might have emerged as a natural consequence of teachers' connecting to students on a personal level, but it might also have been part of each teachers' intention of making the mathematics class more authentic and useful for students' lives.

Overall, the interpretations I presented here should be understood with caution: these classes are not necessarily representative of a typical math class, nor do they represent teachers' common practices. Instead of looking for generalizations, my interpretations were rather an attempt to understand what financial numeracy practices are afforded (possible) in mathematics classes. Through these interpretations, I hope to develop theoretical distinctions about the ways that financial numeracy can emerge in classroom practices.

DISCUSSION AND CONCLUSION

The results of this research contribute to advancing this emerging field of financial education by constructing one coherent framework of financial numeracy. It provides insights to what teachers can afford to do based on their own perspectives and the institutional affordances of the school system. Besides shedding light into the financial numeracy teaching context of secondary mathematics in Québec, this research also contributes to advancing our understanding of the concept of financial numeracy. I now propose a new, detailed framework of financial numeracy education grounded in the existing literature but expanded through my analysis. Figure 2 summarizes the results of this research in relation to the framework.

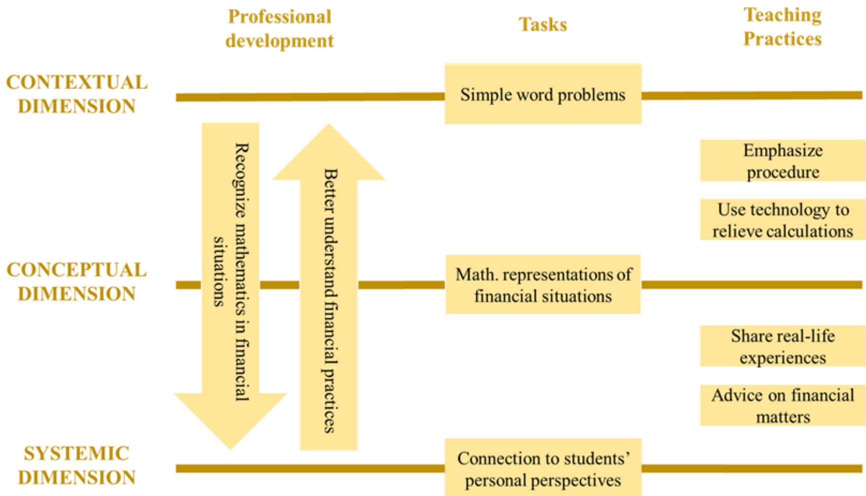

Figure 2: Summary of the findings in relation to the analytical framework

Due to the very nature of the data presented here, textbook tasks revealed the most static aspect of financial numeracy. They were typically situated within distinct dimensions and were composed of one of the following three categories: simple word problems that promote repetitive applications of mathematical procedures, mathematical representations of financial situations in multi-step problems which involve various processes, and connections to students' personal perspectives of such financial situations in open-ended questions.

Teaching practices offered more nuanced approaches in class. They were typically situated in between the dimensions as teachers most frequently touched on more than one dimension in their lessons. Two practices emerged between the contextual and conceptual dimensions: emphasizing procedures in financial tasks and using technology to alleviate some of the calculation burden on students. Two other practices emerged between the conceptual and systemic dimensions: sharing real-life experiences with financial matters and providing practical advice on these issues.

The professional development sessions revealed a dynamic relationship between the three dimensions. Teachers most often transitioned through each of them in their discussions. The trends found in the data provided evidence of what needs to be done in teacher preparation to support teachers to navigate the different dimensions of financial numeracy seamlessly. On the one hand, they need to recognize the mathematics that exists in financial situations (which can be explicit or implicit depending on the context), the financial practice being represented and their own epistemologies of mathematics. On the other hand, they also need to develop sufficient understanding of financial situations (concepts, practices, products, instruments, etc.) in order to be able to implement them in mathematics classes.

From the beginning, the Theory of Measurement and the Anthropological Theory of the Didactic were used to inform the conceptualization of the financial numeracy dimensions. My model initially constructed the three dimensions as distinct entities to capture the complexities of the concept they envelop. Despite this attempt, the research findings pointed to financial numeracy tasks, perceptions and practices in between these dimensions. Instead of static categories, they seemed to construct a spectrum of characteristics that go from formal mathematics (contextual dimension) to complex systems in which mathematics acts as one component (systemic dimension). In other words, the lines have become blurred, and each dimension of financial numeracy can be interpreted as zones that concentrate certain characteristics of this concept.

For instance, when teachers discussed in their focus groups the introduction of financial contexts but restricting their practices to the mathematical aspect, I situated their discussion in the contextual dimension. I acknowledge, though, that these contexts can have some potential to develop students' understanding of financial concepts to a certain degree. The same applies to the tasks that have been coded in the contextual dimension. Even the most procedural task can contribute to students' financial skills depending on the context and the procedure (whether it is used in everyday life or not). Conversely, tasks that were situated in the systemic dimension can also be implemented with the intention of teaching strict financial concepts (without expanding to other factors) depending on the teaching practices.

In addition to blurring the lines that distinguish each of the financial numeracy dimensions, this research contributes to our understanding of an equitable relationship between these dimensions. I had originally constructed the analytical framework with a vision of the systemic dimension as the higher order aspect of such a concept. Yet, after analyzing the teaching practices as well as the mathematics teachers' perspectives on the situational problems, it seems that the dimensions possess a rather complementary relationship. Although the systemic dimension can be perceived as more authentic in its connections to real life, the other two also contribute in important ways to the development of financial numeracy. The contextual dimension is better aligned with the mathematics curriculum and can be mobilized more frequently than the others. It can also serve as a starting point for both students and teachers to have conversations about financial matters. The conceptual dimension strengthens the role of mathematics in modelling financial concepts effectively. It sheds light on the structure and dynamics of these concepts, hence developing better understandings of financial products and practices. Finally, the systemic dimension contributes to a critical appreciation of the world around us. Not only does it serve to analyze the complexities of financial practices in real life, but it also highlights the usefulness as well as the limitations of mathematics in describing the world.

Overall, the concept of financial numeracy as I have mobilized in this research is a powerful conceptual tool to think about the ways of introducing and integrating financial situations in mathematics classrooms. It provides insights to what teachers can afford to do and justify their choices based on their own perspectives and the institutional affordances of the school system. Financial numeracy fundamentally concerns mathematics educators, and this research has shown possible paths they can build to have their voices heard in this matter.

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COMPARING THE EFFECTS OF TWO INQUIRY-BASED TEACHING STRATEGIES ON SECONDARY STUDENTS' CONCEPTUAL UNDERSTANDING AND ACHIEVEMENT IN MATHEMATICS: A MIXED-METHODS APPROACH.

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ABSTRACT

This study investigated and compared the effects of Investigation and Exemplification on secondary students' achievement and conceptual understanding of the three primary trigonometric ratios using a pre-test–post-test, randomized, experimental design. Thirty-five fourth-form (grade 10) students from one secondary school in Dominica were randomly assigned to two groups. The researcher taught both groups for three weeks, one using Investigation and the other using Exemplification. Mixed methods were used to analyze students' responses on the pre-test and post-test to give measures of their achievement and conceptual understanding. Both groups had a significant increase in achievement and conceptual understanding. However, the achievement and conceptual understanding of the group taught by Exemplification was higher than that of the group taught by Investigation.

INTRODUCTION

BACKGROUND OF THE PROBLEM

For many years, Dominican secondary school leavers have been performing poorly on the Caribbean Secondary Education Certificate (CSEC) mathematics examinations, a high-stake test for Dominican youths. This consistent run of poor performances raised how these students are being prepared to take this test and the strategies they are exposed to in the teaching and learning process. Several studies have reported that inquiry-based approaches to teaching mathematics have improved students' mathematics performances at various educational levels. In a recent study, Charles (2015) found that, while most Dominican secondary mathematics teachers were aware of inquiry-based teaching approaches, such as the investigative approach, these teachers seldom used inquiry-based approaches in mathematics teaching at secondary schools.

Secondary schools in Dominica have five grade levels, called forms, with the first-form being the lowest level and the fifth-form being the highest. At the end of the fifth-form, students take the CSEC examinations in the subject areas they study at the fourth-form and fifth-form levels. CSEC mathematics is usually taken by all students leaving the fifth-forms. Approximately 1500 Dominican secondary school leavers take the CSEC mathematics examination every year.

PURPOSE OF THE STUDY

Education officials, principals and teachers, parents, and the wider Dominican society want more students to obtain a passing grade (grades I, II, or III) on CSEC mathematics, given that it is a high-stakes examination. A more significant number of CSEC passes in mathematics would help more Dominican youths qualify for entry-level jobs in the public and private sectors, meet the matriculation criteria for university programs, and further studies in STEM-related areas. In response to this problem, this study investigated the effects of two inquiry-based approaches to teaching mathematics on students' achievement and conceptual understanding of mathematics. I wanted to find out if they would help students increase their achievement in and conceptual understanding of mathematics and if they are equally effective in doing so. These approaches are Investigation (Jaworski, 1986) and Exemplification (Watson & Mason, 2005).

Studies such as Jaworski (1986), Sangster (2012), and Staples (2011) concluded that Investigation improves students' understanding of and achievement in mathematics. Moreover, the CSEC mathematics syllabus, which drives the curriculum in Dominican secondary schools, advocates for using Investigation to get students to solve mathematics problems (Caribbean Examination Council [CXC], 2003). Other studies such as Dahlberg and Housman (1997), Rawson and Dunlosky (2016), and Watson and Mason (2005) concluded that the act of learners generating examples—Exemplification—have improved students' understanding of and achievement in mathematics. The mathematics concepts used in this study were the three primary trigonometric ratios of Sine, Cosine, and Tangent.

Research Questions

Driving this study is the primary research question: *How are students' achievement and conceptual understanding of the three primary trigonometric ratios affected when taught using Investigation compared with Exemplification?* Answering the following four sub-questions helped to formulate plausible answers to this primary research question.

1. How was the level of students' achievement affected after being taught by Investigation?
2. How was the level of students' achievement affected after being taught by Exemplification?
3. How do the levels of students' achievement differ after being taught using Investigation compared with Exemplification?
4. How does students' conceptual understanding differ after being taught using Investigation compared with Exemplification?

RESEARCH DESIGN

This study used a two-group, pre-test–post-test, independent measures (between-group) experimental design. A convenient sample of 35 fourth-form students (17 females and 18 males) from one secondary school in Dominica, considered average mathematics learners, were randomly assigned to two groups. One group was taught using Investigation as proposed by Jaworski (1986), and the other group was taught using Exemplification as proposed by Watson and Mason (2005). I taught the three primary trigonometric ratios to both groups of students; each group received instructions for an equal amount of time. Figure 1 shows the sequence of activities during the teaching and data collection periods.

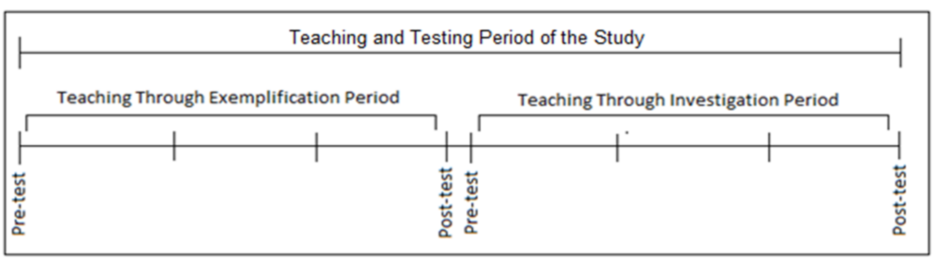

Figure 1: Sequence of activities during teaching and data collection.

Students' responses on the pre-test and post-test were the only sources of data used in this study to assess participants' achievement and conceptual understanding. Both tests assessed the same mathematics content and constructs and were similarly constructed. They comprised both multiple-choice items and question prompts to mimic the CSEC mathematics examination. All test items were marked, and raw scores (achievement), one for each test, were assigned to each student. These achievement scores were tabulated and analyzed using a mixed ANOVA along with other quantitative methods.

Students' written responses were analyzed using pre-determined themes. A separate analysis was done for each group, and group profiles were created. The profiles highlighted students' correct and incorrect representations of the Sine, Cosine, and Tangent ratios. Each profile pointed to the related group level of conceptual understanding, and they were compared to determine how conceptual understanding differed between groups.

RESULTS

Three of the 35 students in the sample did not complete the post-test; hence, the study's results compared how teaching affected 32 participants' achievement and conceptual understanding of the three primary trigonometric ratios. Of these 32 participants, approximately 53% (n = 17) were female and approximately 47% (n = 15) were male. Each group comprised 16 students. The Investigation group comprised of 62.5% (n = 10) male and 37.5% (n = 6) female, and the Exemplification group had 31% (n = 5) male and 69% (n = 11) female.

ACHIEVEMENT

A participant's achievement (A) was taken as the raw score (RS) obtained from adding his score on the written response (WR) section to his score on the multiple-choice (MC) section. There were 12 multiple-choice items, each contributing one mark for a correct answer and three written response prompts, each contributing a maximum of four marks. Thus, a participants' raw score ranged from zero to 24, with 24 being the maximum attainable raw score; see Table 1. Both groups' marks for the written responses (WR) on the pre-test are all zeros because no student attempted that section.

Investigation						Exemplification					
Pre-test			Post-test			Pre-test			Post-test		
MC	WR	**RS**	MC	WR	**RS(A)**	MC	WR	**RS(A)**	MC	WR	**RS(A)**
2	0	**2**	10	8.75	**18.75**	4	0	**4**	9	10.00	**19.00**
3	0	**3**	8	10.25	**18.25**	3	0	**3**	7	11.50	**18.50**
1	0	**1**	8	8.50	**16.50**	2	0	**2**	9	9.25	**18.25**
2	0	**2**	7	7.75	**14.75**	2	0	**2**	9	9.00	**18.00**
4	0	**4**	7	5.75	**12.75**	1	0	**1**	8	8.75	**16.75**
1	0	**1**	6	6.00	**12.00**	4	0	**4**	8	8.75	**16.75**
3	0	**3**	6	5.75	**11.75**	2	0	**2**	9	7.75	**16.75**
2	0	**2**	5	6.25	**11.25**	5	0	**5**	8	7.75	**15.75**
1	0	**1**	7	3.50	**10.50**	2	0	**2**	8	7.13	**15.13**
9	0	**9**	6	4.00	**10.00**	1	0	**1**	5	9.50	**14.50**
2	0	**2**	5	4.50	**9.50**	6	0	**6**	6	7.63	**13.63**
1	0	**1**	5	3.75	**8.75**	1	0	**1**	5	7.38	**12.38**
1	0	**1**	3	4.00	**7.00**	5	0	**5**	8	3.25	**11.25**
2	0	**2**	3	2.50	**5.50**	2	0	**2**	5	4.25	**9.25**
7	0	**7**	2	1.25	**3.25**	1	0	**1**	4	4.00	**8.00**
2	0	**2**	1	1.25	**2.25**	4	0	**4**	4	3.75	**7.75**

Table 1. Participants raw scores obtained in the pre-test and post-test.

Analysis Of Variance

A two-way, 2×2 mixed ANOVA with repeated measures on the Test-type variable was conducted using version 25 of the IBM *SPSS* software. The Omnibus test was performed using a significance level of 0.05. Neither the sphericity assumption nor the homogeneity assumption was violated.

Descriptive Statistics

A breakdown of the pre-test scores showed that students taught by Investigation had a mean score of 2.69 (11.21%) with a standard deviation of 2.27. Those taught by Exemplification had a mean score of 2.81 (11.71%) with a standard deviation of 1.64. These statistics indicated that the difference in achievement between the two groups in the pre-test was minimal (0.5%). A breakdown of the post-test scores showed that students taught by Investigation had a mean score of 10.80 (45%) with a standard deviation of 4.85. Those taught by Exemplification had a mean score of 14.48 (60%) with a standard deviation of 3.75. These statistics indicate that the difference in achievement between the two groups in the post-test was 3.68 (15%) in favour of the group taught by Exemplification with students taught by Investigation obtaining a wider range of scores.

Within-subjects' Effects

The ANOVA results for the within-subject effect for both groups of students showed a significant main effect on Test-type, $F(1, 30)=126.404$, $p<.001$, $\eta p^2=0. 808$. This main effect indicated that the post-test scores for at least one group are significantly higher than that group's pre-test scores. Also, the teaching accounted for approximately 81% of the variance between the post-test scores and pre-tests scores. It also, shows that Test-type did not interact significantly with Method, Test-type * Methods, $F(1, 30)=4.085$, $p=.052$, $\eta p^2=0.120$. This non-significant interaction indicates that the post-test scores were significantly higher than the pre-test scores for both levels of Methods (Exemplification and Investigation).

Between-subjects' Effects

The ANOVA result for the between-subject effect showed that there was a significant main effect on Methods, $F(1, 30)=5.603$, $p=.025$, $\eta p^2=0.157$. This effect indicates a significant difference between the two levels of Method, Investigation and Exemplification, for at least one level of the Test-type. The difference in achievement between the two groups in the pre-test was minimal (0.5%); thus, the significant difference between the groups came in the post-test.

Summary Of Findings

- Sub-question 1: How was students' level of achievement affected after being taught by Investigation? Result: *Students taught by Investigation had a statistically significant increase in their level of achievement.*
- Sub-question 2: How was students' level of achievement affected after being taught by Exemplification? Result: *Students taught by Exemplification had a statistically significant increase in their level of achievement.*
- Sub-question 3: How did students' levels of achievements differ after being taught by Investigation compared with Exemplification? Result: *Students taught by Exemplification level of achievement was statistically significantly higher than students taught by Investigation.*

CONCEPTUAL UNDERSTANDING

Participants' level of conceptual understanding was determined by a mixture of quantitative and qualitative methods. However, priority was given to the qualitative method, the thematic analysis. Three themes were pre-determined based on the work of Kilpatrick et al. (2001). They were: representing a contextual problem, multiple representations of a single concept, and comparing related concepts. A profile for each group was developed based on participants' correct responses–strengths–and errors–weaknesses–within each theme. Table 2 shows a comparison of the groups' profiles. Texts that are written in italics highlight the differences between the groups.

Themes		Exemplification	Investigation
Representing a contextual problem	Strengths	Presented: an appropriate diagram, appropriate trigonometrical ratio and justification, and *an appropriate algorithm.*	Presented: an appropriate diagram, appropriate trigonometrical ratio and justification.
	Weaknesses	Presented: a flawed justification, an inappropriate algorithm, and *an inappropriate diagram.*	Presented: a flawed justification, and an inappropriate algorithm.
Multiple representations of the cosine ratio	Strengths	Presented: the correct formula, an appropriate graph, an appropriate table of values, an appropriate diagram, an appropriate justification.	Presented: the correct formula, an appropriate graph, an appropriate table of values, an appropriate diagram, and an appropriate justification.
	Weaknesses	Presented: an incomplete formula, and an inappropriate diagram.	Presented: an incomplete formula, an inappropriate diagram, *an inappropriate graph, an incorrect formula, an inappropriate table of values, and an inappropriate justification*.
Comparing the three primary trigonometric ratios	Strengths	Presented: correct same form representation, appropriate comparisons based on sides, and appropriate comparisons based on values.	Presented: correct same form representation, appropriate comparisons based on sides, and appropriate comparisons based on values.
	Weaknesses	Presented: incorrect comparisons based on values.	Presented: incorrect comparisons based on values, *incorrect same form representation, and inappropriate comparisons based on their sides.*

Table 2. Comparison of groups' profiles.

Between Group Differences

Drawing from their profiles, both groups of students appeared to have attained a limited understanding of the concepts. They produced similar responses in many instances, with both groups giving appropriate and inappropriate responses. However, there are differences in the groups' profiles, thus, suggesting differences in the group's conceptual understanding. These differences suggest that the group taught by Exemplification attained a higher conceptual understanding than the group taught by Investigation.

For the theme representing a contextual problem, the group taught by Exemplification presented a correct algorithm to solve the problem while the group taught by Investigation did not. On the other hand, the group taught by Exemplification presented an incorrect diagram representation of the problem, while all diagram representations of the problem presented by the group taught by Investigation were adequate.

The most significant difference came from the number and type of errors produced by each group. For the theme, multiple representations of a single concept, the group taught by Investigation presented several errors that were not presented by the group taught by Exemplification: inappropriate graph, incorrect formula, inappropriate table, and inappropriate justification for their choice of representations. For the theme, comparing related concepts, the group taught by Investigation also presented comparison errors that were not made by the group taught by Exemplification: the same form of representation that did not correctly depict all three ratios and inappropriate comparisons of the three primary trigonometric ratios based on their sides.

DISCUSSION

The primary purpose of this study was to determine the effects that Exemplification compared to Investigation had on students' achievement and conceptual understanding of mathematical concepts at the secondary level. Moreover, I wanted to find out if the effects of Exemplification, which is not yet widely investigated at the secondary school level, were different from the effects of Investigation, which is widely investigated at all levels of education. The study attempted to answer these through four research sub-questions.

For sub-question one, the result shows that students taught through Investigation performed significantly better in the post-test than in the pre-test. This finding is consistent with several recent studies (e.g., Budak, 2015; Ekwueme et al., 2015; Mainali & Heck, 2017). For sub-question two, the result shows that students taught through Exemplification performed significantly better in the post-test than in the pre-test. This finding is also consistent with other studies (Dahlberg & Housman, 1997; Dinkelman & Cavey, 2015; Iannone et al., 2011; Rawson & Dunlosky, 2016; Sandefur et al., 2012).

For sub-questions three and four, the result showed that students taught through Exemplification performed significantly better on the post-test than the group taught by Investigation both in terms of their achievement and conceptual understanding. These results suggest that Exemplification might be a more effective approach to teaching aspects of mathematics to secondary students than Investigation. Unfortunately, no study was found in the literature that compares the effects of Investigation and Exemplification on students' mathematics learning.

The researcher argues that this potential difference in the effectiveness of Investigation and Exemplification could be because the three essential factors (learners as active participants, the teacher as facilitator, and an enabling environment) in a constructivist approach to teaching might be better attended to during teaching through Exemplification compared to teaching through Investigation. During Exemplification, both the learners and the teacher are genuinely actively engaged cognitively by generating examples and socially through meaningful discussions with their peers and a more knowledgeable other (Vygotsky, 1978)—their teacher.

On the other hand, students' cognitive and social activities might not always be 'meaningfully' enacted during Investigation. During an Investigation, students usually work collaboratively in groups doing hands-on activities. During such activities, students are often left to initiate and maintain cognitive and social activities such as conjecturing, testing and modifying, and discussing within groups. These activities can easily be neglected or abused because the teacher cannot attend to every group simultaneously. Also, helping students make, test, and modify conjectures while teaching by Investigation, which is sometimes necessary, might reduce the tasks' cognitive demands

(Stein et al., 2009). According to Stein et al. (2009), reducing the cognitive demands of a task leads to lower levels of understanding.

I would be remiss not to mention at least one possible confounding factor: the difference in the ratios of boys to girls between groups. The better performance of the group taught by Exemplification could be because of its higher percentage of female students (69%) compared to that of the group taught by Investigation (37.5%). In Dominica, girls perform better than boys in mathematics at elementary and secondary school levels. Therefore, the better performance of the group taught by Exemplification over the group taught by Investigation could have reflected the gender performance differences in Dominica and not the effects of the different strategies since these gender differences were not accounted for in the study.

CONCLUSION

Students need to be exposed to a multitude of teaching approaches when learning mathematics. Strategies that adhere to the constructivist theory of learning appear to be most effective. Both Investigation and Exemplification are aligned with the constructivist theory of learning and have been shown in this study to increase students' achievement and conceptual understanding of mathematics. Moreover, this study has added to the literature showing that Exemplification may be more effective than Investigation in raising students' achievement and conceptual understanding of mathematical ideas at the secondary level.

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UTILISATION DE LA GÉOMÉTRIE DYNAMIQUE AVEC DE FUTURS ENSEIGNANTS DE MATHÉMATIQUES AU SECONDAIRE POUR REPENSER LE DÉVELOPPEMENT DU RAISONNEMENT

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PROBLÉMATIQUE

Dans les recherches portant sur l'enseignement des mathématiques, deux types d'apports sont attribués à l'intégration de la technologie : les *apports pragmatiques* et les *apports épistémiques* (Artigue, 2013). La production de résultats de façon rapide (apport pragmatique) doit se réaliser en considérant aussi le travail sur la compréhension des concepts (apport épistémique). La conception de tâches et d'activités mathématiques intégrant la technologie doit tenir compte de ces deux types d'apports, tout en permettant aux élèves d'exercer un jugement critique (Caron, 2003).

Dans le programme de formation de l'école québécoise (PFEQ), le raisonnement mathématique est valorisé, mais peu de place est laissée à la preuve. La place accordée à la preuve et l'articulation entre exploration et preuve dans les cours de mathématiques pourraient dépendre du rapport à la validation qu'entretient l'enseignant dans sa classe de mathématiques. Selon une recherche de Mary (1999), les futurs enseignants ont tendance à utiliser la répétition d'expériences à laquelle ils accordent un potentiel de preuve dans la généralisation de leurs observations. L'utilisation de la technologie tend à amplifier ce phénomène, car elle permet la vérification expérimentale d'un grand nombre de cas. Plusieurs chercheurs mentionnent que les futurs enseignants utilisent, par exemple, la visualisation en géométrie sans chercher une preuve plus élaborée, car cela est perçu comme un substitut acceptable à la preuve (Boileau et Garançon, 2009 ; DeVilliers, 2007).

OBJECTIFS DE RECHERCHE

Dans ma recherche doctorale, je me suis intéressée à ce sujet en élaborant d'abord une séquence d'activités misant sur un travail de réflexion et de preuve mathématique à partir de constructions et d'explorations dans un environnement technologique impliquant l'outil GeoGebra (https://www.geogebra.org/). Dans cette séquence, un premier objectif était d'enrichir les apprentissages des futurs enseignants relativement à des concepts du secondaire ainsi que sur l'outil utilisé. Le second objectif de cette séquence était de faire émerger, chez les futurs enseignants, un besoin de prouver les régularités observées et d'établir des liens entre différents objets géométriques.

Suite à une analyse *a priori* de cette séquence, j'ai expérimenté le tout auprès d'un groupe de futurs enseignants de mathématiques au secondaire. L'objectif de l'analyse *a posteriori* était d'évaluer l'apport de la séquence développée chez les futurs enseignants, au niveau (1) de leur instrumentation avec l'outil et de son utilisation projetée avec des élèves du secondaire, (2) du raisonnement mathématique mobilisé et (3) de leur vision de l'enseignement des mathématiques.

CADRE THÉORIQUE

Dans la conception et l'analyse de la séquence, deux approches ont été utilisées : l'approche anthropologique du didactique (Chevallard et Cirade, 2010) ainsi que la genèse instrumentale (Trouche, 2007). Ces approches ont permis de rédiger les différentes tâches des activités tout en considérant l'environnement technologique utilisé dans ces tâches. Nous avons également utilisé quelques idées en lien avec les constructions robustes et molles (Soury-Lavergne, 2011) ainsi que le réseau déductif (Tanguay, 2006) afin de bien articuler les observations faites dans GeoGebra et les questions menant à une explication, voire même une preuve. Suite à l'expérimentation de la séquence, nous nous sommes appuyés sur deux construits théoriques pour analyser les productions des étudiants : les niveaux de preuve de Balacheff (1988, 1998) et l'espace de travail mathématique (Kuzniak, 2011).

NIVEAUX DE PREUVE

Balacheff (1988, 1998) distingue deux catégories de preuves : les *preuves pragmatiques* et les *preuves intellectuelles*. Dans la première catégorie, on retrouve l'empirisme naïf, l'expérience cruciale et l'exemple générique. Dans l'empirisme naïf, étant souvent observé chez les futurs enseignants selon Mary (1999), une simple étude de quelques cas permet de vérifier la validité d'un énoncé. Au niveau de l'expérience cruciale, l'élève va généraliser en utilisant un cas plus général que les autres, alors que dans le niveau de l'exemple générique, un début d'explication du processus derrière la généralisation apparaît. Dans les *preuves intellectuelles*, les élèves vont tenter de prouver leurs observations en élaborant des démonstrations plus ou moins complètes. Trois sous-niveaux sont définis : l'expérience mentale, le calcul sur les énoncés et la démonstration. Dans un environnement de géométrie dynamique, il arrive souvent que les deux derniers sous-niveaux des preuves pragmatiques et le premier sous-niveau des preuves intellectuelles soient difficiles à distinguer. Ainsi, nous avons regroupé le tout en un niveau que nous avons nommé intermédiaire comme illustré à la Figure 1 ci-dessous.

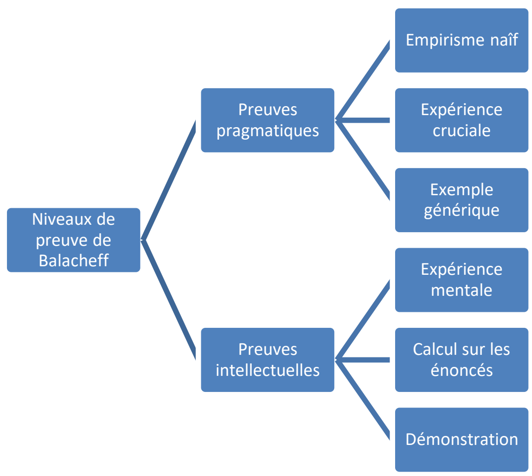

Figure 1. Niveaux de preuve dans un environnement technologique.

ESPACE DE TRAVAIL MATHÉMATIQUE

Dans l'*espace de travail mathématique* (Kuzniak, 2011), différents processus au niveau épistémologique et cognitif peuvent expliquer le travail mathématique des élèves dans une séquence intégrant la technologie. Au *niveau épistémologique*, il convient de considérer les interactions entre les trois composantes que sont l'espace réel et local, l'ensemble d'artefacts et le système référentiel. Les contenus mathématiques ciblés pour une activité doivent donc être structurés dans un but précis. Au *niveau cognitif*, les processus de visualisation, de construction et de preuve sont importants pour comprendre et analyser le travail des élèves. Ainsi, si les élèves utilisent davantage les preuves pragmatiques, leur travail mathématique pourra être lié à la *genèse figurale* ou *sémiotique* alors que si les preuves intellectuelles sont utilisées, un travail au niveau de la *genèse discursive* est noté.

MÉTHODOLOGIE

La méthodologie de cette étude repose sur l'*ingénierie didactique* développée par Artigue (1988, 1996, 2002) comprenant des analyses préalables, une conception et une analyse *a priori* de la séquence ainsi qu'une expérimentation et une analyse *a posteriori* des différentes données recueillies.

DESCRIPTION DE LA SÉQUENCE

La séquence développée comporte 7 activités (notées de A à G) et recoupe trois grands axes, comme illustré dans la Figure 2 ci-dessous. Les premières activités de la séquence servent à outiller les étudiants dans leur utilisation de GeoGebra et comportent des notions mathématiques du premier cycle du secondaire (les quadrilatères). Par la suite, les notions mathématiques s'orientent vers les lieux géométriques et les coniques pour aboutir à la définition de l'excentricité. Tout au long de la séquence, les étudiants ont l'occasion de faire des apprentissages techniques, informatiques et mathématiques ainsi que de les réinvestir dans des activités ultérieures. Ils peuvent ainsi développer une pratique mathématique instrumentée.

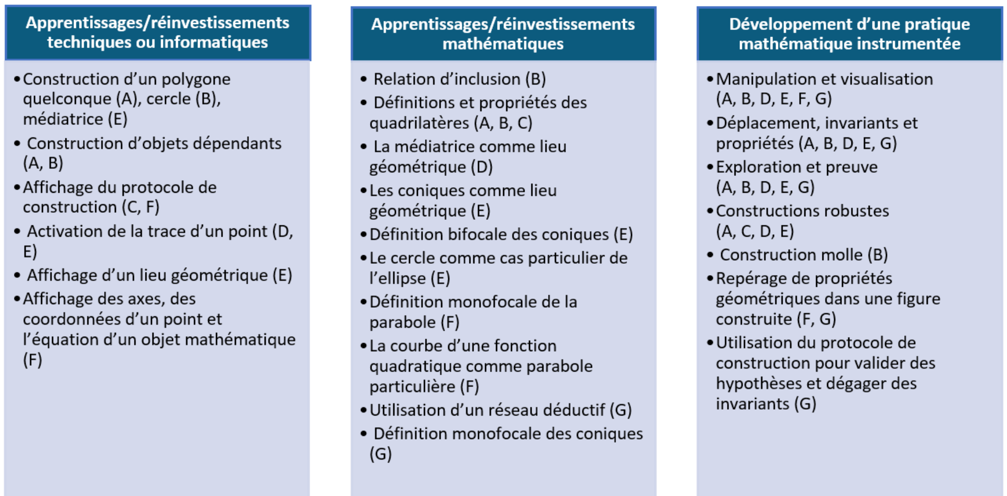

Figure 2. Organisation synthèse de la séquence selon trois axes.

CONTEXTE DE L'EXPÉRIMENTATION

L'expérimentation de la séquence a eu lieu avec de futurs enseignants en mathématiques au secondaire dans une université québécoise. Avant de débuter la séquence, les étudiants ont eu à répondre à un questionnaire préexpérimentation afin de dégager certaines caractérisations relatives à leur vision des mathématiques, leur enseignement souhaité et anticipé avec des élèves ainsi qu'à leur vision de l'utilisation de la technologie. Dans chacune des séances, les étudiants avaient une clé USB ainsi que la feuille d'activité et laissaient des traces de leurs constructions et de leurs réflexions tant sur du papier que sur un support informatique. La séance débutait par une courte présentation de l'activité, puis les étudiants commençaient à réfléchir à leurs constructions. Après chacune des activités, une discussion suivait afin d'échanger sur les observations réalisées, les constructions demandées et leurs découvertes. Une institutionnalisation de certains éléments techniques, mathématiques ou théoriques pouvait être prise en charge par la chercheure durant cette discussion si le besoin s'en faisait sentir. À la fin de la séquence, la passation d'un questionnaire reprenant certaines questions du questionnaire préexpérimentation permettait de voir si les caractérisations des étudiants avaient évolué après leur expérience des activités proposées dans la séquence.

TROIS PROFILS D'ÉTUDIANTS

En analysant ces différentes données, nous avons vu que trois profils d'étudiants semblaient émerger : les *inspirés*, les *sceptiques* et les *encadrants*. Ces profils ont été définis en tenant compte non seulement des niveaux de preuve mobilisés par les étudiants, mais aussi des idées que ceux-ci valorisent relativement à leur utilisation (ou non) des activités de la séquence avec leurs futurs élèves et la façon dont ils le feraient le cas échéant.

Figure 3. Synthèse des trois profils d'étudiants.

LES INSPIRÉS

Dans cette catégorie, nous avons regroupé les étudiants qui avaient au départ une orientation plus technique ou appliquée dans leur enseignement souhaité ou anticipé et qui, après l'expérimentation, manifestaient une plus grande ouverture vers le développement du raisonnement et le sens de la démonstration. Ces étudiants ont mobilisé dans leurs réponses des éléments de preuve pouvant être associés au niveau intellectuel. Lors de la séquence, ils semblaient avoir cherché à expliquer les phénomènes observés et ont valorisé le raisonnement déductif.

Du côté didactique, ces futurs enseignants mentionnent qu'ils utiliseraient des activités de la séquence en favorisant l'exploration par les élèves avec GeoGebra et voient même la possibilité de se servir de cet outil pour introduire de nouveaux concepts, faire formuler des conjectures, faire émerger le questionnement des élèves et favoriser l'élaboration de preuves. Dans les discussions, ils mentionnent des idées pour expérimenter avec les élèves ou les faire manipuler dans le but de faire émerger les relations existantes entre les différents objets géométriques.

Brigitte mentionne que « c'est intéressant de voir la formation de l'ellipse. Cela amène à se questionner sur les propriétés des lieux géométriques ». On note ici que le questionnement sur ce que l'on observe semble être important pour elle et pourrait mener à un travail permettant de faire émerger les relations entre les objets géométriques. Alexandre précise que l'activité « permet d'aller plus loin qu'avec les manuels scolaires. L'élève peut découvrir certains concepts par lui-même ». Cette ouverture relativement à l'introduction de nouveaux concepts catégorise bien les inspirés qui pensent à utiliser les activités pour amener la réflexion et le raisonnement déductif chez leurs futurs élèves. Alexandre ajoute que GeoGebra peut être « utile pour réaliser des preuves et des demonstrations ».

LES SCEPTIQUES

Dans cette catégorie, on retrouve des étudiants qui liaient au départ leur enseignement visé au développement du raisonnement, et qui semblent avoir opté, après expérimentation, pour des visées plus procédurales, tout en accordant plus de place aux technologies et aux applications dans leur enseignement futur. Ainsi, la séquence semble les avoir déstabilisés au point de les décourager à développer le raisonnement chez les élèves ou encore, les avoir confortés dans la vision que « l'on n'a plus besoin de prouver » grâce à la visualisation permise par la technologie. Ces étudiants ont mobilisé des éléments de preuve qui restaient au niveau pragmatique. Ils semblent ne retenir de GeoGebra que la visualisation que cet outil permet. Ils ont utilisé un raisonnement inductif en généralisant rapidement leurs observations, mais n'ont pas cherché à les expliquer ou à les prouver, semblant s'appuyer sur l'idée que ce qu'on voit est une preuve en soi. Ils ont même réinterprété en ce sens le but de certaines activités. Une certaine méfiance à l'égard de l'utilisation de la technologie semble aussi partagée par ces étudiants.

Du côté didactique, ces futurs enseignants mentionnent qu'ils pourraient utiliser cette séquence avec des élèves, mais qu'ils se limiteraient à la manipulation des constructions et à l'observation des phénomènes en misant ainsi sur un raisonnement inductif. Une étudiante, Valérie, mentionne que GeoGebra est « un bon support visuel pour la géométrie » et qu'il peut être utilisé « pour manipuler les figures ». La méfiance de Valérie envers la technologie s'illustre par les précautions qu'elle prend et qui l'amène à vouloir « prévoir un plan B ».

Une autre étudiante classée dans ce groupe, Ophélie, mentionne que « certaines constructions sont difficiles à faire pour des élèves du secondaire. [L'utilisation de la technologie] demande la connaissance des constructions préalablement ». Ces propos indiquent qu'elle n'utiliserait probablement pas certaines des activités de la séquence avec des élèves à cause de la complexité de certaines constructions. De son côté, Valérie laisse entrevoir la possibilité d'utiliser la séquence avec des élèves, mais mentionne qu'il faudrait « faire un retour sur les notions avant ».

LES ENCADRANTS

Les étudiants de cette catégorie caractérisaient leur enseignement anticipé et souhaité, avant l'expérimentation, avec autant d'éléments liés au raisonnement qu'à l'application de procédures. Après avoir réalisé la séquence, ils mentionnent des idées d'utilisation des activités avec un rôle accru de l'enseignant pour encadrer la démarche des élèves dans les activités ou même pour garder un contrôle de l'exploration ou de la manipulation avec le logiciel. Dans leurs réponses aux activités de la séquence, ces étudiants montrent des niveaux de preuve plus variés allant souvent du côté pragmatique et quelquefois, du côté intellectuel.

D'un point de vue didactique, ces étudiants expriment des idées d'utilisation des activités de la séquence en donnant un rôle accru à l'enseignant pour simplifier l'activité. Cette réponse donnée par Geneviève en témoigne : « Si l'enseignant le fait, je crois que ce serait plus simple ». Pour ces étudiants, la technologie est un support à l'observation de phénomènes et l'idée de visualisation y est importante. Geneviève mentionne que GeoGebra est « très pratique, surtout en géométrie [et] permet de produire une bonne visualisation aux élèves. [Il] permet de manipuler ». Cependant, les étudiants de cette catégorie limiteraient la manipulation par les élèves, car l'enseignant devrait occuper ce rôle devant la classe et montrer ce que l'on peut voir dans la construction. Ophélie nous donne un exemple d'utilisation en disant : « Je ferais probablement l'activité en groupe tout en donnant plus de précisions sur la matière utilisée dans l'activité. » Il est aussi possible de noter que certains étudiants perçoivent la construction réalisée comme une preuve en soi, comme en témoigne l'appréciation de Mylène qui explique que celle-ci « permet la preuve par construction ».

CONCLUSION

L'objectif premier de notre recherche était de développer une séquence d'activités dans GeoGebra permettant d'enrichir les apprentissages des futurs enseignants en mathématiques tout en faisant émerger le besoin de prouver leurs conjectures sur les relations observées entre les objets géométriques issus des différentes constructions. Cet objectif a été atteint, tout au moins en partie, et nous avons constaté que la séquence élaborée dans notre recherche a eu un impact variable, mais généralement positif, chez les étudiants. Suite à l'analyse des différentes données recueillies lors de l'expérimentation, trois profils se sont dégagés en lien avec les différents rôles d'un enseignant ainsi que leur vision des mathématiques. Certains étudiants semblent avoir été inspirés, d'autres s'en sont distancés et d'autres encore encadreraient davantage les élèves dans leurs manipulations. Il importe aussi de préciser que notre étude s'est réalisée auprès d'un groupe restreint de 7 étudiants, ce qui limite la généralisation que l'on peut en tirer. Néanmoins, nous sommes d'avis qu'il peut exister, dans la pratique enseignante, des profils d'enseignants ayant des idées similaires aux trois profils que nous avons dégagés dans notre étude.

D'un côté plus technique, certaines difficultés ont été soulevées dans les discussions avec les étudiants tant sur le plan de la séquence que sur les apprentissages mathématiques et techniques qu'ils y ont faits. D'abord, la séquence élaborée est longue et certaines activités, dont les constructions étaient déjà réalisées, ont pu agir, malgré l'accès au protocole de construction, comme une sorte de boîte noire et empêcher certains étudiants de prouver leurs observations. La dernière activité, particulièrement ambitieuse, combinait un réseau déductif partiellement rempli pour faire comprendre le concept d'excentricité des coniques. Comme ce concept n'est pas typiquement enseigné au secondaire, il s'agit de l'activité ayant généré le plus de productions incomplètes chez les étudiants. Ceux-ci ont d'ailleurs mentionné cette activité dans leur appréciation de la séquence en la caractérisant de longue et complexe. Au fur et à mesure que la séquence avançait, les étudiants cherchaient à prouver les phénomènes observés dans les constructions, même si parfois, les niveaux de preuve mobilisés ne dépassaient pas celui des preuves pragmatiques. Nous supposons que cela pourrait être lié à la pratique en géométrie au secondaire qui, en dehors des calculs de mesure, se limite souvent à ce type de preuve.

Certaines améliorations peuvent sans aucun doute être apportées à notre séquence tant au niveau de l'*orchestration instrumentale* (Trouche, 2007) qu'au regard de la structure même de la dernière activité. Au niveau de l'orchestration instrumentale, il aurait pu être intéressant d'utiliser un *élève sherpa* qui vienne à l'avant de la classe pour expliquer ses constructions et ses découvertes dans les différentes activités avant d'en faire une institutionnalisation par la chercheure. Le facteur temps dans notre recherche a motivé le choix de ne pas y avoir recours. Dans la dernière activité, le réseau déductif nous semble un milieu didactique intéressant pour formuler et structurer une preuve, mais cette représentation devrait faire l'objet d'un enseignement, voire même une activité en soi. De plus, un retour théorique sur les notions liées aux coniques, qui, n'étant qu'un vague souvenir chez les étudiants, ont paru constituer un frein à leur raisonnement sur les constructions dans GeoGebra, pourrait être un atout dans la séquence d'activités.

Finalement, la séquence a permis de faire ressortir le potentiel de GeoGebra comme instrument d'exploration et de visualisation permettant de formuler des conjectures et de favoriser un travail mathématique au niveau du raisonnement déductif. Ce travail est permis en faisant varier les types de constructions robustes et molles (Soury-Lavergne, 2011) et en questionnant les étudiants relativement aux invariants d'une construction et aux propriétés géométriques observées dans cette construction. Les réponses données par les étudiants au cours des activités et dans les discussions montrent que pour plusieurs d'entre eux, quelques exemples observés peuvent être suffisants pour prouver. L'expérience qu'ils ont vécue comme élèves au secondaire, ce qu'ils perçoivent dans leurs stages, ou ce qu'ils anticipent comme futurs enseignants peuvent expliquer ce constat. Notre étude a montré que l'on peut jouer sur cette vision, mais qu'une seule séquence n'y suffira pas. Il nous semble néanmoins important d'envisager une meilleure articulation des raisonnements inductif et déductif dans les activités géométriques proposées aux élèves du secondaire ; cela permettrait aux élèves d'approfondir et d'organiser logiquement leurs connaissances, d'en éviter l'oubli prématuré et peut-être même ainsi d'apprécier la cohérence de la géométrie et des preuves.

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MY DOCTORAL JOURNEY: (RE)IMAGINING POSSIBILITIES FOR MATHEMATICS EDUCATION THROUGH/WITH INDIGENOUS KNOWLEDGES AND UNDERSTANDINGS OF INTERFACING

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INTRODUCTION: WHY THIS RESEARCH?

After completing my Master of Education degree, I stepped away from learning / teaching for a period of four years. At that time, I was uncertain I would ever return to the realm of education because I wanted to be a more present parent for my two daughters. Consequently, I was rather surprised when one of my master's thesis supervisors, Dr. Kathy Nolan, contacted me to ask if I would be interested in pursuing doctoral studies with her, along with support from a Graduate Research Fellowship. Although I had no idea what I wanted to study, I knew that such opportunities for personal growth were rarely granted in this way, and I happened to be ready to embark on another educational journey after my break. Besides, I had not forgotten that when I defended my master's thesis, I suppressed lurking feelings of dissatisfaction about my theoretical framework. While complexity theory vocabulary and understandings helped me gain a sense of freedom, introducing me to multiple ways of knowing and being, the theory was criticized for lacking ethical intent (Kuhn, 2008), unable to provide moral / ethical guidance to educational philosophers (Alhadeff-Jones, 2008; Morrison, 2008). As a result, I turned to Situated Learning Theory (Lave & Wenger, 1991) as complement to the perceived moral / ethical limits of complexity theory, however, I was never fully satisfied with the merger.

During my first semester of doctoral studies, my first assignment required that I delve into one of thirteen areas of research pertinent to mathematics education and present findings to the class. Initially, I avoided the area of 'Indigenization of mathematics education' and Sterenberg's (2013) pre-selected article, partly because I knew nothing about Indigenous knowledges but mainly because I thought mathematics was a neutral / universal subject. I wanted to avoid anything remotely politically charged (having to do with culture, race or power and so on). Although this topic and article were listed second, I explored them second to last. Unexpectedly, I was immediately drawn to the area because Indigenous ways of knowing, as represented through the vocabulary that I was reading, strongly resonated with some of my prior understandings of complexity theory. Additional readings led me to realize that Indigenous worldviews might even offer complexity theorists the moral / ethical guidance that was perceived to be lacking. Needless to say, I was hooked on learning more about Indigenous ways of knowing and being from the onset of my doctoral journey. I had enjoyed my Master of Education experiences because they afforded me an onto-epistemological leap from positivism to complexity and it did not take me long to recognize that my Doctor of Philosophy in Education experiences could afford me another onto-epistemological leap, from complexity to relationality.

This background is important, for as Aluli-Meyer (2008) reminds, "your relationship to your research topic is your own. It springs from a lifetime of distinctness and uniqueness only you have history with" (p. 220). Thus, as my doctoral journey progressed, I became aware of the multiple experiences and related subjectivities (identity labels or positionalities) that influenced my research. As a Settler Canadian I wanted my work to "signal to others that [I am] ready and committed to honestly addressing settler colonialism in Canada" (Battell Lowman & Barker, 2015, p. 123). Therefore, Chapter 1 of my thesis outlined aspects of who I am and how I came to my research foci: to disrupt my Euro-Western onto-epistemologies, my ways of knowing and being in relations to mathematics education; to (re)imagine / decolonize my mathematics education learning / teaching practices; and to explore non-reductionist possibilities for and / or interfacings of Indigenous knowledges and mathematics education, especially after noticing curricular tokenisms and Indigenous absences.

LITERATURE REVIEW

Many terms or phrases within mathematics education literature conveyed to me, in some manner, attempted interfacings of Indigenous onto-epistemologies and mathematics or mathematics education, including *teaching mathematics for social justice* (TMfSJ) (Bartell, 2012; Ortiz-Franco, 2006), *equity in mathematics education* (Gonzalez, 2012; Tutak et al., 2011), *multicultural mathematics* (Hall, 2007; Weist, 2006; Zaslavsky, 2006), *intercultural mathematics education* (Greer & Mukhopadhyay, 2015), *culturally relevant mathematics* (Belczewski, 2009; Ezeife, 2003; Nichol & Robinson, 2000; Rosa & Orey, 2015), *culturally responsive mathematics education* (Gay, 2009; Greer et al., 2009; Moses et al., 2009; Nicol et al., 2013; Nolan & Graham, 2017; Ukpokodu, 2011), *two-ways or both-ways approaches* (Jorgensen, 2015), *cross-cultural teaching* (Aikenhead, 2017), *culturally based mathematics* (Bishop, 2002; Cajete, 2012a; Nutti, 2013), *place-based school mathematics* (Aikenhead, 2017), Mathematics in Indigenous Contexts (MIC) (Perry & Howard, 2008), *Math in a Cultural Context (MCC)* (Lipka et al., 2012; Lipka et al., 2009), *Indigenous mathematics education* (Parra & Trinick, 2018; Stavrou & Miller, 2017), *ethnomathematics* (D'Ambrósio, 2016; Gavarrete, 2015; Mukhopadhyay et al., 2009), *ethnomathematical barters* (Parra-Sanchez, 2017) and *innovative approaches in ethnomathematics* including *ethnocomputing, ethnomodelling and trivium curriculum* (Rosa & Orey, 2016).

I found it difficult to select a term/phrase that reflected my intentions with respect to exploring how Indigenous knowledges might open possibilities for (re)imagining mathematics education. This difficulty came from knowing Grande's (2011) wariness of "miscegenation between Indigenous and Western constructions of knowledge whereby Indigenous formations are recast within Eurocentric frames" (p. 40). She argues that terms including "*integration, accommodation, reconciliation, incorporation, and amalgamation*—only seem to domesticate the violent collision of competing moral visions; of epistemicide" (emphasis in original, p. 40).

Eventually, I turned to Doolittle's (2007) concept of interfacing because it reflected the possibility of two independent systems simply meeting to interact, being open to communications, collaborations and the possibility of (re)imaginations through complex conversations. The idea of an interface was also mentioned by Kanu (2011), Mukhopadhyay and Roth (2012), as well as Armour et al. (2016) who describe "the interface" (p. 423) as being "where two cultures meet" (p. 423) when considering a two-way learning approach. This approach involves "equal partnerships between Indigenous and non-Indigenous people" so that "meaningful exchanges can occur" and "everyone is learning together" (Armour et al., 2016, p. 422). Nakata (2002) notes a Cultural Interface can be "a place of constant tension and negotiation of different interests and systems of Knowledge" (p. 286); "a space that abounds with contradictions, ambiguities, conflict and contestation of meanings" (Nakata, 2007, p. 323). He argues "for embedding the underlying principles of reform in this space" (Nakata, 2002, p. 285) because such principles allow for other possibilities that shape our future by "accept[ing] that the intersections of different knowledges and discourses produce tensions and condition what is possible but do not directly produce certainty of outcomes" (p. 286). Thus, it was with the intention of exploring Indigenous and mathematical interfaces that I began my thesis research.

THEORETICAL FRAMEWORK

In understanding that theory can be "a *powerful* force" (emphasis in original, Holman Jones, 2016, p. 232) for thinking through and with research experiences, I was drawn to the idea of a decolonizing theoretical framework, but it proved difficult to pinpoint and articulate. I chose a form of artistic assemblage during the development process, gathering pre-existing research to foreground / background concepts and embrace a heteroglossia. All the while, maintaining the idea of a fluid composition, knowing that my framework development and use would continue well beyond the completion of my doctoral journey. I felt supported in my decision to take up a decolonizing theoretical framework because decolonizing processes and the notion of decolonization have been emphasized by many Indigenous scholars as a means to challenge and resist present structures and systems (Andreotti et al., 2015; Battiste, 2013; Cajete, 2012b; Iglioliorte et al., 2012; Laenui, 2009; Monchalin, 2016; Pirbhai-Illich et al., 2017; Swadener & Mutua, 2008; Waziyatawin & Yellow Bird, 2012). To inform my framework, I turned to understandings of postcolonialism; radical hope (Lear, 2006; Wiseman et al., 2015); a beyond reform interpretation of decolonization (Andreotti et al., 2015); a theory of social justice for abnormal times (Fraser, 2005, 2008); and dangerous coagulations (Baker & Heyning, 2004).

In reflecting upon how postcolonial aspects informed my decolonizing theoretical framework, I brought a number of ideas to the fore as a way of increasing my awareness of colonial manifestations. Such awareness worked to improve my capacity to identify and possibly resist and/or challenge postcolonial aspects—including: nationalism, universality / standardisation, Eurocentrism, hierarchy, capitalism, distinction / difference, racialization, categorizations or classifications, objectism / objectification, totality, hybridity and sacredness—as I worked at (re)imaginings of mathematics education.

The idea of radical hope, "directed toward a future goodness that transcends the current ability to understand what it is" (Lear, 2006, as cited in Wiseman et al., 2015, p. 237) resonated with Settler Canadian and Indigenous relations not yet thought (Battell Lowman & Barker, 2015). Thus, by way of understandings, curricula and subjectivities not yet thought, I found the concept of radical hope encouraged a (re)imagining of mathematics curricula and educator subjectivities. The idea of transcending or going beyond was also taken up specifically in relation to decolonization by Andreotti et al. (2015) who developed three conceptualizations of reform (soft, radical and beyond) by mapping spaces rather than offering definitions. When considering how to describe spaces, beyond an *incorporation into* (soft) or an *expansion of* (radical) existing power structures / systems, a beyond reform space allows for "recognition of ontological dominance" whereby "the modern system itself is perceived as inherently violent, exploitative, and unsustainable" (Andreotti et al., 2015, p. 27). Key to a beyond reform space is the understanding that "alternatives articulated from within modernity's frames will tend to reproduce it" (p. 28).

From Fraser's (2008) work, I was most drawn to the idea of misframing. I related the idea of misframing to mathematics education, due to the associated drawing of fabricated and arbitrary mathematical boundaries. During my doctoral research journey, I realized that mathematics and therefore mathematics education are framed or constructed in ways that exclude many mathematical forms. Capacity to identify such misframings seem important to a decolonizing theoretical framework because, in my experience, identification brings awareness which can then be followed by possibilities that lead to (re)framings and (re)imaginings. Finally, I turned to dangerous coagulations as explanation for my constant use of the prefix (re-) within my writing. Coagulation expresses how something fluid begins to form a semi-solid state; and, in linking this semi-solid state understanding with the idea that stagnating forms could be dangerous or have "the potential to be harmful" (Toll & Crumpler, 2004, p. 385), I am reminded that imaginings of mathematics education could cease without emphasis of the prefix (re-). Such stagnation could become yet another form of dogma, akin to the idea of *an* imagining of mathematics education being mistaken for *the* mathematics education solution, just as "near-universal, conventional mathematics…(NUC-mathematics)" (Barton, 2008, p. 10) has been mistaken for mathematics.

METHODOLOGY

Story / autoethnography as methodology encouraged me to explore interfacings of Indigenous knowledges and mathematics education through my own doctoral research journey experiences and participations yet also in relation to broader social, cultural and political phenomenon. I was not particularly distressed that "the meanings and applications of autoethnography have evolved in a manner that makes precise definition and application difficult (Ellis & Bochner, 2000/2013, p. 134). I accepted that "there is no consensus on what the term [autoethnography] means or what the approach involves" (Sikes, 2013, p. xxii). However, through prolonged reading for the purposes of trying to better understand autoethnography as methodology, I came across a variety of autoethnographic types, including collaborative, organizational, performative, evocative, interpretive, deconstructive, postcolonial, and critical. Although concepts and understandings from each autoethnographic type were useful to me in conceptualizing my methodology, especially when repetitions occurred, I found myself most drawn to vocabulary from deconstructive, postcolonial and critical autoethnographies (Dutta & Basu, 2013; Jackson & Mazzei, 2008; Holman Jones, 2016; Pathak, 2013). These forms encourage the complicating, troubling, problematizing, challenging, deconstructing and disrupting of dominant discourses; decentering and dismantling of authorial certainties; straining and provoking of interpretations, stories and experiences; and engaging in processes of becoming.

Additionally, during my doctoral comprehensive examination, I was challenged to think about how autoethnography as methodology was sufficient for pursuing research reflective of learning through / with Indigenous knowledges.

Consequently, after further reading, I identified my methodology as story/autoethnography to emphasize that I drew from Indigenous understandings of story (Archibald, 2008; Ermine, 1995; Kovach, 2009; McLeod, 2007) as well. For instance, one understanding is that once something is learned, there is a responsibility to share one's understandings so that others might use them. Archibald (2008) explains that "sharing what one has learned is an important Indigenous tradition" (p. 2). Here, sharing is meant as a means of "giving back" (p. x) and "is done with a compassionate mind and love for others" (p. 2). This form of sharing can be about humbling oneself and revealing one's own learnings from experiences, much like my intent and purpose became as I wrote my thesis.

METHODS

In returning to the idea of assemblage, I found this approach useful not only for developing a decolonizing theoretical framework but also for engaging in data collection. My interests in exploring disruptions to my Euro-Western onto-epistemologies and (re)imagining possibilities for mathematics education through / with Indigenous knowledges and complex conversations did not waver since the first semester of my doctoral journey. Since that 2014 Fall term, when I was so greatly impacted by Sterenberg's (2013) article, my research had been underway. Consequently, I counted and valued each memorable experience that challenged and informed my thinking about interfacing Indigenous knowledges and mathematics education. By way of assuming that my research could not lead to any one particular reimagining of mathematics education or definitive and generalizable interfacing of mathematics education with Indigenous knowledges, the research journey / exploration itself instead stood as my focus. However, in thinking about isolating a few components of the data collection process so as to create a research methods section through assemblage, I experimented with journaling / marginalia; extending coursework; recording complex conversations with research participants; and reflecting on my role as a sessional instructor of an introductory finite mathematics course for pre-service teachers.

DATA ANALYSIS AND DISCUSSION

Initially, I proposed to keep a journal of my explorations / experiences because I wanted to track and possibly share experiences that I perceived led to contradictions and disruptions of my Euro-Western worldview, especially moments that seemed pivotal and related to mathematics education issues and concerns. However, I quickly learned that I am not a journal writer, even when well-intentioned and under pressure to collect data for a thesis. Extension of course work was more valuable. My main take-away was that my research and work, if I want to practice decolonizing and reconciliatory processes, requires my continued commitment to participate in relationship building and learning from Indigenous peoples. I also learned that personal complacency / contentedness should be resisted for decolonization requires restructuring of power imbalances, rejecting dominations, disrupting dominant discourses and breaking cycles of mindless reproduction. Such disruptive / decolonizing ideas were regularly brought to the fore through class discussions so as to encourage students to be agents of change.

When I began exploring / researching the idea of interfacing Indigenous knowledges and mathematics education, I worked with the assumption that interfacing these realms could be beneficial for all learners of mathematics. However, as my research progressed, I began to doubt this assumption. I feared that, as a Settler Canadian, I might not be adequately positioned to explore interfacings or that such exploration, no matter how well meaning, could cause harm to Indigenous peoples in some way. Therefore, through guiding questions for discussion, I presented my concerns to research participants (who were anonymized using names of known mathematicians). Our complex conversations were recorded and transcribed by me, as a way of dwelling in the data. Then, selected excerpts were formatted and analyzed by means of "transcription as data analysis" (Ravitch & Riggan, 2012, p. 98), to represent a plurality and diversity of voices and perspectives.

The first guiding question was a question about settler responsibility, constructed as follows: Battell Lowman and Barker (2015) suggest that,

> *For centuries, Indigenous people have had to learn to understand how Settler people think…[and] in order to find new ways of living together respectfully…Settler people need to take up the responsibility of learning about Indigenous ontologies…[including] the understandings and worldviews of the specific peoples on whose lands Settlers live.* (p. 20)

What do you think about the suggestion that Settlers need to take up the responsibility of learning Indigenous ontologies?

While the lengthier excerpts offered in my thesis are more interesting than the summary that follows, I will start by saying that although I anticipated diverse responses to the question of whether Settlers need to take up the responsibility of learning Indigenous ontologies, I was surprised that the 10 research participants could generate such an extensive range of ideas. Chatelet used the metaphor of "incommensurable paradigms" to express that it is "not possible," as well as seemingly "arrogant and privileged, to think that [settlers] would be able to…understand some other culture's ontologies and ways of being." Similarly, Browne suggested that settlers "cannot get" Indigenous knowledges. Instead, she proposed they could at least "be open to…accepting different ways of knowing as valuable." Ramanujan was positioned differently; he did not "necessarily agree…Settler people need to take up…learning about Indigenous ontologies." He referred to the possibility of achieving a third way for settlers and Indigenous peoples to live together with neither disrespect nor harm through notions such as diplomacy, protocol and respectfulness.

Noether and Agnesi seemed somewhat aligned in that each was quick to consider it "a no brainer" that Settlers should take up the responsibility of learning Indigenous onto-epistemologies, even though each also acknowledged issues of strain. Noether recalled a specific language learning experience where she witnessed an example of the anguish that cross-cultural learning can bring to Indigenous peoples. Similarly, Agnesi spoke of the population imbalance within academies that often results in overwhelming Indigenous peoples who work within such spaces, stretching them beyond their capacities. Hypatia's response equally reflected concern about the expectation of cross-cultural learning by taking into account that, presently, many Indigenous peoples are in the midst of healing themselves through their own learning of traditional knowledges. She noted that non-Indigenous "people…would have to humble themselves in order to…be accepting of another way of thinking." Likewise Euler highlighted that "it will take a certain person…[who] will pay the respect and everything that is needed" and that the mobility of people within the modern world perhaps erodes opportunities for cross-cultural learning.

Near the beginning of my doctoral studies, when I first showed interest in exploring Indigenous onto-epistemologies in relation to mathematics education, I was discouraged by more than one professor from heading along this research path. I never fully understood why. I assumed there was concern that it may not be possible for a Settler Canadian such as myself to explore Indigenous knowledges, or at least not in a respectful manner. After all, there is much history about "story-takers and story-makers…misrepresent[ing], misappropriate[ing], and misus[ing]" Indigenous stories and knowledges (Archibald et al., 2019, p. 5). Therefore, I think these discouragements rightly provided my first lessons in proceeding cautiously, however, I could not bring myself to shift to an alternative research route when my explorations were providing opportunities for my growth, beyond scholarship, by disrupting and challenging my worldview and Euro-Western onto-epistemologies. Time and again my experience in *trying* to understand led me to thinking differently, which in turn led me to knowing and being differently, especially in relation to mathematics education. Evidence of such change is provided throughout my thesis by offering various stories of my experiences. Consequently, like the research participant Gauss, I remain interested in working with the assumption that it is beneficial to at least "try to understand" multiple knowledges and onto-epistemologies because I enjoy growing through challenging status quos, be they my own normalizations or those imposed on me by others.

The idea of imposed normalizations related to the second guiding question that I selected for analysis after (re)reading and dwelling in the transcripts. This second question, though more leading than guiding, focused on (re)imaginings for mathematics education. I felt it necessary to phrase the question as I did in hopes that participants might then be able to challenge my interpretations and thinking. The second question read, as follows: It has been my perception that mathematics educators are applying or superimposing a mathematical lens from the dominant culture to Indigenous processes and artefacts to bring them into the mathematics classroom. What do you think about the ways Indigenous processes and artefacts are presently being viewed or used within mathematics classrooms? I keep hoping there is something else we can do instead of applying or superimposing a mathematical lens but what do you think?

One idea clearly reflected within the research participant responses was that reality is distinct from any mathematical representation of it. While this may not be a new idea, from my perspective it does seem to be a main issue when attempting to interface Indigenous and settler NUC-mathematical onto-epistemologies because settler Near-Universal, Conventional mathematical ways of knowing and being do not appear to value the diversity of reality. For instance, Ramanujan discussed the irony of actual variation in King and Coho salmon sizes verses the exact/static 1:2 relation that is portrayed within mathematics curricula. He explained that mathematical concepts were idealizations, often appearing as nonsense from Indigenous perspectives, perhaps because mathematics is purported as an "edifice of true knowledge [built] out of a foundation which isn't." Hypatia similarly mused about the emotion, spirit, creativity, personality and individuality that went into designing a ribbon skirt, as opposed to there being a perfectly mathematical way of making one. She described Indigenous knowledge as “more forgiving” than mathematics and brought to light many issues that could arise when trying to take Indigenous teachings into the classroom. Thus, when comparing Indigenous and settler NUC-mathematical perspectives, maybe it is not surprising that Gauss perceived Indigenous and mathematics lesson concepts were “two quite separate things, …difficult to bring together.”

RESEARCH SIGNIFICANCE

My research afforded me many understandings, but it also brought me to a crossroads between 1) approaches that keep Indigenous onto-epistemologies and mathematics education in a parallel relation so that Indigenous onto-epistemologies are not at risk of being recast through a superimposed mathematical lens; or 2) continued explorations of interfacings so that the body of knowledge presently referred to as mathematics can be challenged and eventually (re)framed by way of Indigenous lenses. Regardless, I gained valuable perspectives about mathematics education that continue to influence my work as an educator. As a result, whenever I am granted sessional instructor opportunities, I work at challenging and changing the content that is counted within my classrooms, along with my instructional strategies and evaluations of students’ understandings.

My doctoral experiences also afforded me practice in taking the personal and making it political. In using my personal desires and expectations to challenge and change myself, I pushed to make changes within broader educational contexts. For example, I requested to enroll in an undergraduate course on Cree culture and history during my doctoral studies. I also sought permission to use a final project instead of an exam when teaching Introductory Finite Mathematics. While such system challenges / changes may not appear striking, my work and commitment did not go unnoticed. I recently accepted a postdoc position with the Faculty of Science, and I am thrilled to be tasked, over the next two years, with rewriting the Introductory Finite Mathematics curricula. This is another chance for me to explore interfacing mathematics with Indigenous knowledges and to potentially influence others.

Prior to my thesis defence, at a Graduate Student Research Symposium, I was challenged as to why I did not pursue a line of research that involved direct collaboration / networking with local Indigenous communities from the onset of my data collection process. My response to this challenge felt weak and unprepared at the time, related to the idea that I did not know how to approach such a community at the onset of my data collection process. While that response may have been partially true, it was not reflective of a larger picture. During my formal period of data collection, I did not want to impose my research agenda on an Indigenous community, especially when I had not been “invited-in” (Jorgensen & Wagner, 2013, p. 2). Instead, I understood my doctoral research journey to be mainly reflective of my own “shifts towards becoming more culturally sensitive” (Smith, 2012, p. 179) and responsible for making changes within my own learning / teaching contexts / practices.

My postdoc can provide those collaborative / networking opportunities with Indigenous peoples and communities that my doctoral journey was missing, pushing my learning further now that I am more knowledgeable. I knew when I finished my doctoral research journey that learning / research can be a life-long process. I understood that I can choose to maintain a state of becoming, changing, evolving through engaging / participating. I claimed that though my thesis was complete, I was by no means finished with learning / researching and (re)imagining possibilities for mathematics education. I am so glad I was right. For other Settler Canadians who engage with my work, it may be of interest that when revisiting the question of whether it is even appropriate for me to continue trying to engage with Indigenous knowledges, I continue to feel encouraged that it is—so long as I simultaneously work towards partnering / networking with Indigenous peoples, while ensuring that I actively listen and not over-step the ways in which I have been invited to participate.

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DEEPENING PROSPECTIVE TEACHERS' UNDERSTANDING OF ELEMENTARY MATHEMATICS: THE CASE OF DEB

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INTRODUCTION

Classroom teachers are the key to implementing advancements in mathematics education. They are responsible for enacting reform-based learning and teaching recommendations that have emerged over the past two decades (National Council of Teachers of Mathematics [NCTM], 2000). Enacting reform-based practices for learning mathematics includes teachers anticipating multiple solution strategies, eliciting students' ideas, and orchestrating discussions to make connections between what students already know and new material (Smith & Stein, 2018). This is challenging for teachers, especially for those who have not had opportunities to connect their own understandings. And, if a teacher education program did not provide new mathematics experiences to offset over-reliance on standard algorithms (Kaasila et al., 2010), prospective teachers are likely to fall back on what they know and teach the next generation of students the way they were taught (McNeal & Simon, 2000). This study addresses the issue by illustrating how prospective elementary teachers can connect their mathematical ideas based on recommended strategies for teaching students.

RELEVANT LITERATURE

Previous research has found that when school students work with prior mathematics knowledge considered problematic—work impeding new learning—it actually promotes connections, and thus deeper understandings (McGowan & Tall, 2010; Martin & Towers, 2016). Like their future students, prospective mathematics teachers need to work with what they already know, identifying and analyzing prior knowledge that is problematic (McGowan, 2017). While school students typically work on progressively more advanced mathematics topics, prospective teachers need to work backwards to unpack their existing understanding of procedures in order to connect them with related concepts (Ball & Bass, 2000; Ma 1999).

This study builds on recommendations for prospective teachers to learn first from practical experiences (Gainsburg, 2012) in settings that are less complex than school classrooms (Grossman et al., 2009) that support practices that allow prospective teachers to rehearse eliciting and responding to others' ideas (Kazemi et al., 2016). At the same time, this study addressed specific concerns about elementary prospective teachers' fragmented mathematics understandings (e.g., Zazkis, 2011; Holm & Kajander, 2012) by explicitly focusing on unpacking existing mathematics understanding (e.g., rules and formulas) and connecting them to concepts (e.g., pictorial images). Previous studies that aim to deepen prospective teachers' understanding of elementary mathematics have investigated connecting activities like sorting multiple solutions (e.g., Eli et al., 2013), and implementing unit plans that show the relationship between rules and concepts (e.g., Whitehead & Walkowiak, 2017). However, what is often less clear is how the unpacking of prospective teachers' understanding affords opportunities to make connections that are necessary and important to each of them. In this study, prospective teachers experience a learning and teaching environment similar to what they can create for their future students, while unpacking and connecting their own mathematics understanding.

THEORETICAL FRAMEWORK

The Pirie-Kieren (P-K) theory for the Dynamical Growth of Mathematical Understanding (Pirie & Kieren, 1994) characterizes learning as constantly changing and moving forward and back in a non-linear fashion through eight *embedded* layers of understanding, from the learner's earliest informal representations, or *images*, to outer layers of formal understanding. This study used the first five layers of P-K's nested model—Primitive Knowing, Image Making, Image Having, Property Noticing, and Formalising—to describe how individual prospective teachers grew their understanding of elementary mathematics concepts. The first layer Primitive Knowing is the learner's starting point, everything that the learner knows and can do. Image Having and Image Making, the next two layers include multiple modes of expression: ideas, mental imagery, or pictorial representations that learners have or make about the topic under investigation. In Property Noticing, learners examine and reflect upon images, see connections between them, and notice their mathematical characteristics. Formalising embeds the previous four layers and is marked by an awareness that a method works for all relevant examples. At this point, the learner can explain and justify the method. Prospective teachers may start at any layer and move to other layers when working on mathematics tasks. Moving or *folding back* to other layers allows learners to re-experience the same material in a different light. This re-construction process describes how prospective teachers can work with formal and informal images to grow their existing understanding of elementary mathematics. The embedding of the P-K layers means that growing understanding in an outer layer is based on the learner understanding the related earlier layers. Recognizing that sometimes embedding does not occur, Pirie-Kieren's theory (1994) also includes the notion of *disjointed* understandings. The learner's algorithms, for example, might be disjointed or *disconnected* from their less formal understandings.

METHODS

A case study methodology was used in this study because it aligns well with an instrumental approach to investigate a contemporary phenomenon within its natural setting (Stake, 1995). The setting was a ten-day professional development course in elementary mathematics for 15 new graduates of a teacher education program at a large Canadian university. The prospective teachers who elected to take this course had previously completed two 36-hour methods courses in mathematics education. The course offered prospective teachers with opportunities to think and learn about making mathematics connections in multiple roles: as students working on elementary-level tasks, as co-teachers supporting the learning process, and as new teachers reading related research and reflecting on their experiences. Sharing roles and responsibilities with prospective teachers allowed me to do what Lampert (2000) described as researching from the inside. Using a case study aligns with her argument in favour of teachers being active participants in understanding the problems of practice.

As the course instructor, I focused on planning course activities, observing and listening to participants' mathematics ideas, and asking questions to elicit further work using tools. I shared these instructional responsibilities with the three participants who were designated co-teachers for that day. When planning, the co-teachers developed their own solutions for the next day's task and anticipated the possible solutions that their peers might develop. Later when participants worked on the day's task, the co-teachers and I asked questions to uncover their thinking, encourage them to communicate their work on the blackboard, and prompt them to construct or review different images so they could connect their ideas.

Each day prospective teachers explored a mathematics topic by working on a task. The co-teachers randomly assigned one or two working partners and a blackboard space to encourage interaction among peers. While working on the day's task, the groups generated multiple solution strategies and worked on comparing and connecting them and understanding the associated properties that explained how the concepts worked. They also recorded their work in personal notebooks. Course afternoons were devoted to participants looking back at their day's documentation, discussing solutions, and video-recording the details of connections they made. In addition, they worked on assigned readings and sometimes participated in co-planning for the next day. This paper will share Deb's (pseudonym) work on the subtraction question *how many ways can you figure out 91–79* on the fifth day of the course.

DATA SOURCES AND ANALYSIS

Video recordings and participants' journals provided both real-time and daily-summary data for the study. The videos recorded the live interactions at the blackboard as participants recalled, used, and discussed mathematics related to the task. This allowed me, as the instructor, to interact with learners while they were working and then revisit and analyze the video-recorded scenes later to support my research. By the end of the course, there were 75 hours of recorded live interactions among participants, plus 135 look-back videos that participants had recorded as they reflected on their work at the end of each day. I used this detailed and overlapping data to identify and analyze prospective teachers existing images, disjointed understandings, and how they noticed properties and made connections in terms of Pirie-Kieren theory (1994). The incidents that included Deb were watched several times and transcribed. The analysis of the video data was adapted from Powell et al.'s (2003) analytical model that includes the following seven phases: viewing the video data attentively, describing the video data, identifying critical events, transcribing, coding, constructing a storyline and composing a narrative. A researcher may go back and forth through these phases.

RESULTS

DEB'S GROWTH OF UNDERSTANDING FOR SUBTRACTION.

Deb worked with two of her colleagues, and a peer co-teacher who returned several times to listen and ask questions. Deb quickly recorded her initial image on the blackboard as follows: 91–79 = 90–80. Deb's idea was to use a friendly number strategy—change the numbers to make the question easier—that was part of her existing understanding. She knew this rule worked in addition, as she demonstrated later in her journal entry: 19 plus 5 can be changed to 20 plus 4 by adding one to 19 and taking away 1 from 5: (19+1)+(5–1) = 20+4. Deb's group worked further on this idea, first using a hundreds chart on the chalkboard and then physical blocks. Initially, Deb had expected that friendly number version 90–80 would give the same answer as the original task 91–79, but through counting—making a second image—she and her group quickly realized that the answer was 12, not 10. Deb was puzzled by the two different answers. In her journal entry in figure 1, she circled the two +1's, drew an arrow to the other side of the equal sign, and circled another +2. Later in her look-back video, she said, "I needed to add plus two, but I couldn't figure out why because to get from 91 to 90, I subtracted one." Deb's friendly number strategy elicited from her Formalising layer of understanding of addition was used to make her initial image for subtracting. She had expected adding 1 and subtracting 1 would preserve the difference, like it preserves the sum in addition.

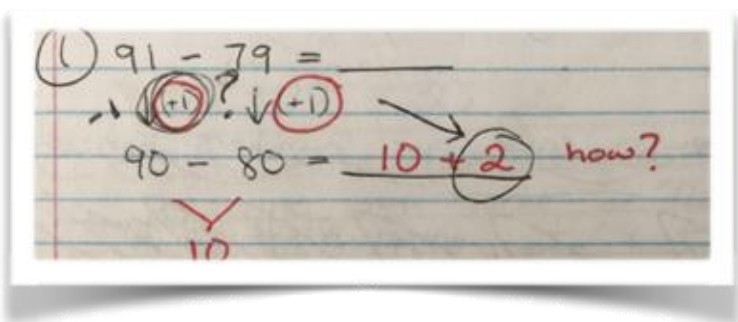

Figure 1. Deb's Image Making: Where is the plus two?

Then Deb began explaining the missing two. As she was talking to her peers at the blackboard, she recalled a rule for solving equations and said, "so whatever we do to this side, we have to do to this side," as she pointed to each side of the equal sign. In P-K terms (Pirie & Kieran, 1994), Deb's rule about the equal sign is also part of her Formalising layer of understanding. At one point in the discussion Deb said "I want to make this a 90 and I want to make this an 80 (*on the chalkboard, Deb records 90 above 91 and the 80 above 79*). So I'm taking one away from the big number and adding one to the small number, and whatever the answer is, I have to add 2 because I made the taking away number bigger by one and I made the initial number smaller by one…so that's two numbers (*circles the -1 and +1 on the board*). In her look-back video, Deb explained that a prompt: "Is there a tool you could use to check?" provoked her team to construct a number line on the blackboard to show her friendly number strategy for 91–79. Image Making (see Figure 2) enabled Deb to show her thinking on a number line: the difference of 10 between 80 and 90, the +1 on each end, and the total difference of 12. She could see the numerical relationships and why her initial friendly number strategy was not giving the expected answer.

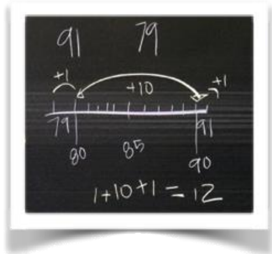

Figure 2. Deb's Image of 91–79 on a number line.

In her journal entry shown in Figure 3, Deb drew another number line using different numbers, 21–14 = 20–13 to show how the difference remained the same when it was shifted by one to the left. Deb noticed how shifting maintained "the difference between the two numbers," whereas her original image "had changed the difference." Later, in her look-back video, Deb explained this again by demonstrating how difference works physically. She held her hands 7 units apart, clasped a paper number line between two fingers at 21 cm and two fingers at 14 cm, and called on a peer to pull the number line all the way to 0 cm and 7 cm. She referred to this experience as an "aha moment" that "explained the idea of constant difference." Deb now recognized why her original image resulted in 10 rather than 12, and said, "I had changed the number in both directions and, therefore, had changed the difference." Deb continued to document her understanding at the Property Noticing layer in her journal. She wrote 91–79 = 92–80 and explained the balance between the two sides of the equal sign as a mental shift "in her head" as well as on the physical paper number line and said that she was "not counting anymore." By translating her use of the number line into a new equation, seeing the numerical relationships and showing this physically, Deb's Formalising layer of understanding for subtraction now embeds an image of constant difference.

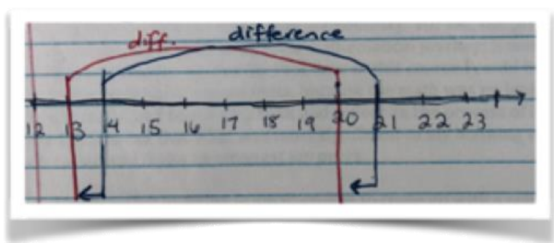

Figure 3. Deb shows shifting the difference by subtracting one.

DISCUSSION

This study illustrates how one prospective teacher recalled and worked with two rules: a number strategy she uses to make computing in addition friendlier, and what is done to one side of the equal sign needs to be done to the other side. Before unpacking and working with her rules, Deb's formal understanding was disjointed from an image of difference for subtraction. Deb's work suggests a process for how prospective teachers can make connections within their existing understanding of elementary mathematics. First, create two or more images by working on a math task, then compare these images and identify the presence or absence of connections among images or formula results, and ultimately resolve these disconnects to make new connections within their mathematics understanding.

When prospective teachers were presented with an elementary mathematics task, they naturally called upon their existing knowledge and experience. They were free to start with any solution. Some started by automatically recalling and using an algorithm, while others began with an informal approach, like Deb's choice to make the numbers friendlier. They were helpful simply because they reflected the participants' current understandings—what was being investigated and potentially connected.

The course was designed to ensure that participants worked with multiple images. It elicited multiple solution strategies by using instructions that asked for more than one approach, prompts by the instructor and co-teachers to try another method, and discussions among participants as they compared solutions. The key was that they unpacked their existing knowledge to specific tasks and put their ideas into action to see how well they worked. This created task-related images for them to work with while investigating connections.

Once prospective teachers had developed multiple images, they compared their solutions, looking for relationships, identifying the presence or absence of connections among images or formula results, and resolving disconnects. The connection-making process was dynamic. When the participants could not find the connections among images, they created yet another image by reviewing a peer's work, using another method to cross-check their solution, or working through a prompt from a colleague or the instructor. Despite different starting points and pathways in investigating the subtraction task, Deb ultimately grew her understanding of subtraction with the aid of physical materials, suggesting that adult learners, like children, may benefit from using concrete representations.

Working with existing mathematics knowledge to identify and resolve disconnects was hard, but gratifying work for the participants and their instructor. Though the participants often found these disconnects puzzling, and sometimes frustrating, they were motivated to keep working and find the answer. For Deb, resolving a disconnect was a 'now I see it' moment when she noticed properties related to the rules she recalled and previously relied on. Although Deb's understanding of her addition strategy could be viewed as problematic—it impeded her understanding of subtraction—the disjoint provided an opportunity. Working on it was essential for learning and deepening her understanding of subtraction.

CONCLUSION

In this study, prospective teachers' disconnects were viewed, discussed, and used as opportunities, not problems. Working on them was essential for learning and connecting their understanding of elementary mathematics. This renewal of existing knowledge is consistent with the Pirie-Kieren theory's (Pirie & Kieren, 1994) process of folding back to deepen earlier understandings. Deb's work illustrates how prospective teachers can connect their fragmented mathematics knowledge working on mathematics tasks that enable them to uncover and resolve the gaps in their understanding.

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THE USE OF STORY-BASED TASKS IN POST-SECONDARY STUDENTS' LEARNING OF STATISTICS

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INTRODUCTION

More than fifteen years ago, the first GAISE (Guidelines for Assessment and Instruction in Statistics Education) report was published to provide guidelines on best practices in statistics education (Garfield et al., 2005). They suggest instructors focus their courses on statistical thinking, broad statistical concepts and principles, foster active learning, promote the effective use of technology, and use real data that is relevant to students. Though there have been changes in how statistics has been taught since then, there continues to be difficulties in successfully converting these guidelines to practice (Tishkovskaya & Lancaster, 2012) and many innovations in statistics education are based on intuition rather than on research (Ramirez et al., 2012). Thus, there is still a need to research how to effectively support students' understanding of statistical concepts using suggestions of best practices in statistics education. This paper presents a portion of the findings of a study (Lemieux, 2020) that investigated an intervention using story-based tasks to explore statistics that was designed using the best practices of statistics education. This paper focuses on the portion of the study that investigated what features of the story-based tasks supported meaningful learning for the participants.

THEORETICAL PERSEPCTIVES

This study investigated the types of *understanding* students demonstrated of statistics concepts. The framework of understanding used was Skemp's (1976) perspective. Skemp's (1976) defined instrumental understanding as applying "rules without reasons" (p. 20) and relational understanding as "knowing both what to do and why" (p. 20).

This paper focuses on the aspects of the *stories* that supported meaningful learning. A story is a specific type of narrative that has a clear beginning and end and tells a sequence of events with a character that is driving the events towards a solution to a problem or conflict (Egan, 1986). Stories have characters, plots, context, conflict, imagery, emotions, and humor (Carter, 1993; Roberts & Stylianides, 2013; Zazkis & Liljedahl, 2009). Further, stories happen over a period—"it is not one single moment or a snapshot in time", have a plot, which is understood by making connections between the events and characters, and requires interpretation by the reader (Rossiter, 2008, p. 418). Defining these tasks as stories resulted in detailed description of the context and problem of the story; development of characters with distinct personalities; addition of dialogue between the characters; use of informal language that would fit with the personalities of the characters; physical descriptions of settings and characters; and sequential progression of events over time.

LITERATURE REVIEW

This review focuses on studies that investigated the use of stories to support students' learning of statistics. Only three such studies were found in the literature.

In Blackburn's (2016) study, he created a story about a fictionalized fish farm to contextualize concepts like the sampling distribution in an undergraduate business statistics course. The stories had characters such as "Freaky Fish" who explained statistical concepts (p. 3). To investigate the impact of the intervention, Blackburn used pre- and post-intervention tests, and open-ended questions about students' experiences with the story. Blackburn found that students reported that they were engaged in the learning through the story and believed that they would retain the material longer than if traditionally taught. But based on the quantitative analysis of the tests, Blackburn did not find a significant change in the scores post-intervention.

Sherwood (2018) examined the use of storytelling by undergraduate students in an economics program. In Sherwood's study, students wrote two stories: one about the normal distribution and the other about sampling distributions. Both stories had to use the same student-created context, reflect the key features of the topic, and have characters. After completing the stories, students were interviewed about their experiences writing the stories. Sherwood found that students went through three stages in writing the stories, where at each stage their understanding of the statistical concept changed and improved.

Smith (2014) developed a storytelling-questioning approach to aid young children in making informal statistical inferences from stories. Smith found that students made informal inferences using the stories and the questioning technique. Smith found that a feature of the story-telling questions that led students to engage in informal inferential reasoning was when the prompts specifically asked students to do so.

METHODOLOGY

This study employs a single qualitative case study (Stake, 1995) to investigate what features of the story-based tasks supported students' meaningful learning of statistics concepts.

THE BUSINESS STATISTICS COURSE

The course in this study is a multi-section algebra-based business statistics course that is intended for first-year business students at a university in southern Alberta, Canada. The course ran four hours per week during a thirteen-week semester, with two classes per week. There were four units in the course: sampling techniques and descriptive statistics, probability, formal statistical inference, and simple linear regression. The story-based tasks were designed for the first three units. The instructor for the course was a tenured faculty member with over fifteen years experience with the course.

STORY-BASED TASKS

The four stories in the intervention used in this study were pre-dominantly written by me in conjunction with the instructor for the course. Using the best practices as a guide, we aimed to develop stories that promoted statistical thinking, focused on broad concepts applied in real-world contexts, actively engaged students in their learning, and effectively used technology.

The instructor and I wrote the context, problem, characters, and the plot of the story. The stories' plots focused on a real-world business problem that would be resolved through some form of statistical analysis. Each story was around 10–12 pages long and focused on one major statistical topic. The stories were fictional but set in realistic situations. Table 1 provides a summary of each story and its related statistical topic.

	Problem presented in the story	**Key statistical topic**
Bob's Bikes	Determining if an inventory system is undervaluing items by on average more than $12	Sampling techniques and descriptive statistics
Can Dolphins Communicate?	Whether the dolphin Aries can understand an oral communication from the dolphin Daphne	Informal inferential statistics (with a focus on informal hypothesis testing)
The Dragon Lady	Determining if electric scooters are meeting the contractual obligations	Sampling distributions of sample means including the central limit theorem and normal distribution
Can They DIG It?	Determining if company should go forward with business expansion plan	Hypothesis testing and confidence intervals for mean and proportion

Table 1. Summary of the four story-based tasks in the intervention.

The stories were left intentionally incomplete. Within the story, the students were prompted to write dialogue between the characters. There were between 12 and 14 prompts per story. The prompts had the students produce and interpret statistical measures using technology, draw conclusions from the statistical analysis, explain their reasoning for why they choose specific statistical measures, and explain aspects of the statistical concepts covered in the stories. Thus, the students did not simply passively read the stories but were invited to actively engage with the story. As such, the stories were written both as a teaching tool (teacher as storyteller) and as a cognitive tool (student as storyteller / narrator; Roberts & Stylianides, 2013). The portion of the stories where students wrote dialogue for the characters are the story-based tasks. Each story also had an expert and novice character. That is, there was at least one character who was well-versed in statistics and at least one character who was new to the statistical topic presented in the story. The complete third story is provided in Lemieux (2020).

METHODS

PARTICIPANTS

The participants were 20 of the students enrolled in the business statistics course in the Winter 2017 term. Most students were in the first year of their business programs. This is the only mathematics course required for the business degree and the only pre-requisite for it is grade 12 mathematics. Thus, students did not have prior experience with most of the content covered in the course.

DATA

The data for the study were class artefacts that consisted of participants' written dialogues as responses to the prompts in the story-based tasks. As some of the story-based tasks were done in groups, only submissions where all group members agreed to be part of the study were collected. Though some participants stayed in the same group throughout the term, other participants changed groups. The size of the groups ranged from individuals to groups of four.

DATA ANALYSIS

The initial part of the analysis focused on determining what kinds of understanding students demonstrated using Skemp's (1976) framework of understanding. Once that analysis was complete, the data was revisited to examine how the specific aspects of the nature of stories impacted the dialogue written by the students for the story-based tasks. To do this, I coded the participants' responses to the story-based tasks for ways that the characters, plot, problem, context, imagery, emotions, and humor (Carter, 1993; Roberts & Stylianides, 2013; Zazkis & Liljedahl, 2009) impacted their dialogue and their understanding of the statistical topics.

To illustrate, when examining the impact of the nature of the characters on the participants' written dialogue, a theme that emerged was viewing the concept from the perspective of an expert character talking to a novice. For example, for the third story, one participant had the expert character use a gambling example to explain why we look at "even better evidence against the null hypothesis" when calculating the *p*-value. Here, the participant considered the perspective of the novice character by relating the abstract concept of *p*-values to a more concrete example of gambling.

Once the themes were identified, I went back to the initial analysis regarding understanding to determine the impact of the theme on the participants' understanding of the topics. To do this, it was determined what type of understanding was demonstrated using the story element. For example, the gambling example used by participant 17 demonstrated relational understanding of the concept as they correctly explained the reasoning behind the concept.

FINDINGS AND DISCUSSION

To discuss the findings in terms of supporting meaningful learning, it is important to establish if the story-based tasks were successful. As is outlined in Lemieux and Chapman (2020), there is evidence to suggest that the intervention can support the development of both instrumental and relational understanding of the four statistical topics investigated.

NATURE OF THE PROMPTS

A feature that arose from the data was how the nature of the prompts impacted the type of understanding demonstrated by students. The analysis revealed that the third story on sampling distributions of sample means elicited a specific type of relational understanding (named *basis understanding*) that was rare in the other three stories (Lemieux & Chapman, 2020). When students demonstrated basis understanding, they explained the reasoning (or basis) behind the concepts. While the other three stories, which covered the topics of descriptive statistics, informal inferential statistics, and formal inferential statistics, tended to elicit another type of relational understanding, which involved interpreting statistical measures in a context (named *contextual understanding*). By focusing on differences in the story elements, the findings suggest that a possible reason for these differences was due to the nature of the prompts in the stories. To illustrate, consider the prompt from the first story-based task that explored descriptive statistics:

> *Have at least two of the characters explain what the histograms and box plot mean in the context of the story. One of the characters should be Bart who will either ask a question or make an incorrect suggestion. Then either Jolene or Franca will answer the question or correct his misunderstanding.*

The prompt does not ask the students what a histogram or box plot is or to explain the different information that the two different visual descriptive statistics would provide. Instead, the focus is only on synthesizing the statistical and the contextual, that is, on developing contextual understanding. Though there were prompts in the story that asked students to generate statistical measures and to choose an appropriate measure and justify their choice, there were no prompts that asked students to consider the reasoning behind concepts for this topic.

As another example, consider the prompt from the second story-based task that explored informal inferential statistics:

> *Have both characters discuss what you think this probability means in the context of the initial research question. Again, because Sam is new to this material (as you are), his primary role is to ask questions or to state his own misconceptions. Be as thorough and concrete as possible.*

Again, the prompt does not ask about the reasoning behind the concept of the *p*-value, such as why it is the appropriate probability to examine for evaluating evidence. Instead, it, again, focuses on interpretation. Though there were prompts in this story that asked participants to explain aspects of informal inferential statistics, most focused on interpretation of the results within the context.

In contrast, the prompts for the third story-based task that explored sampling distributions of sample means asked students to explain the reasoning behind a concept, which resulted in a different type of relational understanding (i.e., basis understanding). For example, one prompt in the story-based task is

> *Have Jed and Reema discuss what a sampling distribution is. In particular, comment on how the data in the sample and the data on the sampling distribution are different and how the process of collecting a sample is different from the process of creating a sampling distribution.*

For this prompt, the focus is not on interpreting a sampling distribution but is instead on explaining the differences between them. Thus, the students were encouraged to go beyond interpretation and, instead, to focus on the theoretical differences between the two types of distributions. Unlike the other three story-based tasks, there were multiple prompts in the third one that asked students to delve into the reasoning behind the concepts. Thus, this was only the story-based task that consistently and repeatedly asked students to consider the reasoning behind the concepts, and it was the only story-based task where most participants demonstrated basis understanding. Further, all story-based tasks asked students to synthesize the statistical and contextual, and contextual understanding was the most common type of relational understanding that emerged from participants' work. Thus, there is evidence to suggest that the nature of the prompts is an important feature in the story-based tasks in eliciting specific and different types of understanding.

The role of the nature of the prompts in supporting meaningful learning is similar to the results found in other studies. For stories in statistics education, Smith (2014) suggested that a feature of prompts that led students to engaging in informal inferential reasoning and statistical thinking were that the prompts specifically asked students to do so. While Elia et al.'s (2010) study focused on stories in elementary mathematics education and found that for stories to generate learning appropriate prompting is required. Kaplan and colleagues (2018) did not explore prompts in stories but instead investigated whether changing the wording of the prompt impacted students' answers in statistics. They focused on how the prompts could "cue" the participants to provide a more complete description of a distribution (p. 97), while this finding focuses on how the prompts could "cue" the participants to focus on types of relational understanding of a concept. Thus, this study provides further evidence of the importance of prompts in statistics and mathematics education.

THE CONTEXT OF THE STORIES

A second feature of the story-based tasks that supported meaningful learning for the participants was the context of the tasks. Each story had a unique context that was relevant to the students' business program. The findings suggest a relationship between the nature of the contexts and students' development of relational understanding of the statistical topics. To illustrate, in their explanations of the reasoning behind the concepts, participants melded together the story-context and the abstract concept and, by doing this, demonstrated stronger explanations than participants who did not use the story-context. For example, one group inter-weaved their explanation of how to construct a sampling distribution with the story-context of quality control of solar powered scooters:

> Jed: *So a sampling distribution is used when the entire population is unknown, so we will take our sample of 320 and randomly select a sample of 30 from this larger sample and measure the mean. We will put these 30 back into the large sample and we will then randomly take another sample of 30 from the 320, measure the mean, and continue on until we have enough means to create a sufficient enough sampling distribution from all of our sample means. You said this process is called bootstrapping.*
> Reema: *Right on, it is important to note that there is a known difference between the data in the sample and the data on the sampling distribution. The sample of the 320 scooters collected were randomly sampled every 15 minutes, and then tested for peak speed which was then recorded and plotted on the curve. This is the actual sample containing raw data. The data collected from the process of bootstrapping, is the same data, except when we look at the sampling distribution, this is strictly made up from the sample means derived from bootstrapping. So it is the means of the 30 empirically sampled scooters.*

On the other hand, another group did not use the context at all:

> *Jed: A sampling distribution takes a large number of samples from the sample data we have collected, so essentially it's as though the sample itself is the population. But we use the mean from those samples, whereas for the original sample we only used the measured values.*

In doing so, as can be seen from the above examples, participants who used the story-context provided explanations that were more detailed and clearer compared to those who did not use the context in the explanation. In general, when participants balanced the abstract concepts with the concrete real-world context, the reasoning they presented behind the abstract concept was stronger and more complex than if they only focused solely on the abstract.

The importance of context-rich problems is well established in the field of statistics education. The benefits of exploring concepts through context-rich problems has been supported by multiple research studies (e.g., Nowacki, 2011). This study further supports this previous work and indicates that statistics embedded within a story is an additional method of providing context-rich problems to students.

THE EXPERT AND THE NOVICE CHARACTERS IN THE STORIES

A third feature of the story-based tasks that supported meaningful learning for the participants was the characters of the stories. Through the different types of characters (i.e., expert and novice), the story offered students the opportunity to view the concepts from different perspectives that likely played a role in the development of understanding of the selected statistical topics. All stories required the students to take on the characters of various expert and novice business professionals, which invited them to think about the concept from two different perspectives. Through these characters, the participants wrote dialogue between the characters that explained various statistical concepts from the perspective of an expert explaining the concepts to a novice and from the perspective of a novice new to the concept. The findings indicate that from the perspective of the expert talking to the novice, in general, the participants tended to use informal language, provide examples, and inter-weave the story-context into their explanations of the statistical concepts. From the perspective of the novice, in general, the participants asked questions about the concept and stated their understanding of the concept (which included both demonstrating understanding and purposefully stating misconceptions). By adopting the perspectives of these two types of characters throughout the four story-based tasks, the participants personalized their knowledge and demonstrated relational understanding of the statistical concepts. This suggests that incorporating such characters in the story-based tasks allowed the participants to explain their understanding of the concepts in a way that makes sense to someone else, which results in them considering the concepts relationally.

This consideration of engaging students in multiple perspectives of viewing the statistical concept forms a central part of the nature of the story-based tasks that is unique to this study. Though there are other studies that use stories in statistics education (e.g., Blackburn, 2016; Sherwood, 2018), within the literature review, there were none that required the students to consider the concepts from multiple perspectives. Thus, this study adds to the literature by demonstrating how the use of different perspectives in stories can support students' understanding of statistics.

LIMITATIONS

Though this study suggests that certain features of story-based tasks can support meaningful learning of statistical concepts, there are limitations. The data for this study was collected only for one class, in one term. This makes it difficult to generalize the results beyond the participants in the study. However, since case study methodology calls for naturalistic generalizations (Stake, 1995), it is possible for readers to learn and gain insight from the case.

FUTURE STUDIES

As the study was only conducted in one class, there are opportunities to study the story-based tasks more deeply. For example, the context in all stories was business related. Therefore, investigating stories using different contexts is worthwhile. As another example, the nature of the prompts was important to support learning and future studies could examine how best to write prompts to elicit certain types of understanding.

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EMBODYING UBUNTU, INVOKING SANKOFA, AND DISRUPTING WITH FELA: A CO-EXPLORATION OF SOCIAL ISSUES AND CRITICAL MATHEMATICS EDUCATION WITH AFRICAN YOUTH

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ABSTRACT

The purpose of this qualitative research was to study if and how Sub-Saharan African youth use mathematics in understanding, challenging, and disrupting social issues related to the African continent. I draw on decolonial theory from an African perspective and African [decolonial] frameworks (Ubuntu, Sankofa, Fela Anikulapo-Kuti Music [FAM]) to center the perspectives of the colonized Other and decenter power within the research space. The findings of this study highlight Sub-Saharan African youth's investment in re-reading and re-writing their African world with and without mathematics. Specifically, they reveal that youth were invested in re-writing false narratives about the African continent by calling forth African Indigenous ways of knowing. This re-writing led to epistemic freedom and cognitive justice—an essential component of social justice—that redress the loss of Indigenous knowledges.

INTRODUCING THE BAOBAB TREE

The Baobab tree is commonly referred to as the tree of life given the range of resources it provides to communities and due to its longevity. For that reason, when Soro, one of the African youths I collaborated with on this project, brought up the Baobab Tree during one of our early sessions, the group lit up. Everyone had a comment to make around the importance of the Baobab tree on the African continent. I honor this majestic tree by sectioning this paper using various aspects of the Baobab tree.

THE ROOT OF THE BAOBAB TREE

In this paper, I make visible the contribution of Sub-Saharan African (SSA) youth in critical mathematics education (CME) through theoretical and methodological lenses that have largely been invisible in CME. The purpose of this research was to understand if and how SSA youth used mathematics in understanding, challenging, and disrupting social issues related to the African context (Osibodu, 2020). This research is rooted and guided by African frameworks (Ubuntu, Sankofa, and Fela-Anikulapo Kuti Music [FAM]). These epistemological and methodological framings ground this research in a decolonizing paradigm and reflect my continual quest to elevate African Indigenous ways of knowing (Chilisa, 2011; Dei, 1994, 2012; Ndlovu-Gatsheni, 2018; Ngugi wa Thiong'o, 1986) instead of centering western and colonial worldviews. My decentering of western ways of knowing does not mean that I do not value western contributions or that I am choosing to essentialize African ways of knowing. Rather, I take up Bullock's (2018) statement that CME "has a commitment to…make visible what has been obscure and bring to the center what has been marginalized" (p. 123). I endeavoured to invoke the pre-colonial African education that infused "notions of culture, centring learners' histories, identities and experiences, and focusing on the learner's agency to bring about change in their personal and community lives" (Dei, 2012, p. 114).

GUIDING FRAMEWORKS

I use three African frameworks that speak to ways of being, knowing, and doing. The first, Ubuntu is a humanist perspective rooted in African thought. Archbishop Tutu (1999) explains the notion that "my humanity is caught up, is inextricably bound up, in yours" (p. 31). Exploring CME with Ubuntu in this study allowed us to focus on cultivating a space where we reflected on our shared African identity as we engaged in understanding these social issues. Ubuntu allowed us to recognize that our growth and knowledge was also dependent on the support we received from each other. Sankofa (Dei, 2012) comes from the Twi people of Ghana and roughly means *going back to retrieve what was lost* which pushes us to constantly reflect on the past before moving forward. Invoking Sankofa in this study allowed us to consider our past, present, and future as we sought to disrupt social issues. In addition, it allowed us to examine what was lost in our prior mathematics learning as we considered studying the social issues in this study. Concurrently, Sankofa allowed us to *go back* to our various African contexts in considering, questioning, critiquing, and seeking truth that we had not had the space to consider previously. Lastly, I developed the Fela Anikulapo-Kuti music (FAM) methodology based on his Anikulapo-Kuti (1986) song, *Teacher don't teach me nonsense*. FAM centers co-learning, disruption, and joy (Osibodu, forthcoming). The research questions that guided this study are

1. What knowledges do Sub-Saharan African youth draw upon in their investigation of social issues? How might these knowledges advance our understanding of CME?
2. When and how do Sub-Saharan African youth draw on mathematics, broadly, to understand and to disrupt social issues? How do they view the role of mathematics in disrupting these social issues?
3. What affordances does an African epistemological grounding have as youth and I engage in the research process?

THE SEEDS OF THE BAOBAB TREE

CRITICAL MATHEMATICS EDUCATION (CME)

Gutstein (2016), notes that "although multiple interpretations exist for these terms, U.S. educators mainly refer to 'mathematics for social justice,' and those in other countries refer to 'critical mathematics,' often with roughly the same meaning" (p. 455). Common across these contexts is the purposeful role of CME in research, teaching, and teacher education in proffering possibilities of mathematics as a tool toward social change (Osibodu, 2021). In this research, I use CME to mean scholarship that engages people (particularly students) in considering ways to use mathematics to understand and address social (in)justices often beyond the classroom.

DECOLONIAL THEORY

I also draw on decolonial theory to theorize CME through the lens of coloniality of power, knowledge, and being (Maldonado-Torres, 2016; Ndlovu-Gatsheni, 2015). Ndlovu-Gatsheni (2018) asserts that decolonial theory widens our view on social justice to include cognitive justice to shift towards epistemic freedom as "colonial education has never been inclusive" (Wane, 2009, p. 161). Cognitive justice acknowledges the epistemicide that was wrought upon the colonized Other by the intentional attempt of killing and disvaluing Indigenous knowledges. Thus, epistemic freedom "is fundamentally about the right to think, theorise, interpret the world, develop own methodologies and write from where one is located and unencumbered by Eurocentrism" (Ndlovu-Gatsheni, 2018, p. 17). Engaging in this process of epistemic freedom engenders what Ndlovu-Gatsheni termed *critical decolonial consciousness*.

PLANTING THE BAOBAB TREE

Decolonizing methodologies build from decolonial theories as an approach to engage in research that centers the worldviews of the colonized Other and that is relational (Chilisa, 2011; Osibodu, forthcoming; Patel, 2015; Smith, 1999; Wilson, 2008). In other words, decolonizing methodologies shift from a western way of conducting research by foregrounding and elevating multiple knowledges. A decolonizing approach to research also allows research participants with healing, transformation, and self-determination (Smith, 1999). In this research, I ensured that I offered space for myself and the youth co-researchers to share from our own experiences and ways of knowing embedded in our African communities.

THE STORIES WE TELL: DATA GENERATION

I am intentional in calling this section 'data generation' as opposed to 'data collection' because the youth / co-researchers and I collectively generated the data that made this study possible. The data consists primarily of the stories and memos I wrote throughout semester-long exploration. The five co-researchers I partnered with during the Spring 2019 semester are Soro from Cote d'Ivoire, Njo from The Gambia, Sanyu from Uganda, Manyoni from Zimbabwe, and Mendrika from Madagascar. They were all in their second year of university at the time of data generation. The SSA youth and I met for 12 weeks for approximately two hours each session during the spring 2019 semester. All sessions were video, and audio recorded.

MAKING SENSE OF OUR STORIES: DATA ANALYSIS

I transcribed all the data generated that noted the tone of voices and facial expressions. I wanted to humanize the transcription process as much as possible. As I transcribed, I made comments that connected to other sessions or ideas we generated. Using Wilson's (2008) notion of intuitive logic, I returned to the memos I wrote after each session to determine themes, starting from my memos starting with the narratives around what Africa meant to us that we shared at the first full session. Though it would have been ideal not only to analyze the data but also to write these stories with co-researchers, they were, understandably, unable to commit that much time. I did, however, meet with them to review the entire findings chapter together to ensure that I accurately reflected our collective research.

THE FRUITS OF THE BAOBAB TREE

CHALLENGING AFRICA'S SINGLE STORY

In our first full FAM session, I asked each of us to select a picture of their choice on the internet that represents "What Africa Means to Us." After we shared our picture, we each took turns sharing why we picked the picture and we chimed in on each other's' stories. Sanyu's picture represented colonization where she described Africa as gold leaves that are constantly plucked by the west. Manyoni's picture showed three older African women and spoke about the strength of African elders particularly of Black African women. Mendrika's picture was a rainbow. She asserted that despite the wars and challenges on the African continent, the rainbow, which appears after a storm, represents pan-Africanism. I used the street food in Nigeria called Suya to describe African creativity. Soro's picture was a Baobab tree which he described as a signifier of community. Lastly Njo's picture was an African woman with a baby wrapped on her back with the west African Ankara fabric. She described Africa as our motherland and shared that just like a woman's vulnerability to violence, she is not defined by it. I share this because these pictures really set the stage, unintentionally I might add, for the types of conversations that arose in our FAM sessions.

"I NEVER LIKED MATHS IN SCHOOL BUT I NEVER KNEW I ALREADY KNEW MATHS!": INVESTIGATING AFRICAN INDIGENOUS KNOWLEDGES IN MATHEMATICS

A couple of FAM sessions later, as we were summarizing a session that Njo had missed, Sanyu made the following comment: "wherever you see a right-angle, it means a white man has been there", led us to examining artefacts in our communities. This comment led me to ask the group to bring in artefacts from home to our next session. Similar to the previous prompt, I could not predict what might come out of our conversation, but I was committed to centering our voices. Examining artefacts allowed us to discuss practices and ways of knowing that are liberatory for us because according to Dei (2012), "Indigenous cultural knowledge is about searching for wholeness and completeness" (p. 112). We discussed the symmetrical pattern on a bowl Sanyu brough from Uganda and the shapes and possible formation of the Pascal triangle on the Zulu necklace.

Njo was unable to bring any physical artefacts, but she wanted to discuss the traditional home found in most sub-Saharan countries, as well as the traditional Gambian boat. In discussing the home, Njo remarked that,

> *When it comes to the men, with the construction, the thatched [roof] of the house…they have to know the circumference on top and now they go find the grasses. The grasses have to be shaped in a in a way that it will cover the circumference and totally like it will not sink in. So there's I don't know, like, I'm just realizing that I never really really thought…Because they already knew what the round thing is. We just [call it], its circumference!"*

Mendrika wondered if this knowledge could not be considered common sense-knowledge, but Njo quickly countered, explaining that she felt that this common-sense knowledge is "ingrained in us." To Njo, common sense felt lesser than formal (e.g. school mathematics) knowledge; a binary she opposed. She rejected the notion of illiteracy because as she argued that it supported the idea that Africans were expected to "consume" colonial education. Njo rejected deficit notions of African Indigenous ways of knowing particularly as she connected this to the knowledge of the environment; another aspect of African Indigenous knowledges (Dei, 2012). She referenced that they learned from their "own hands" and "own ways" that existed before colonization. As we finished this exploration, Njo voiced that "I never liked maths in school, but I never knew that I already knew maths in my own way!" (Osibodu, forthcoming, p. 12).

While Sanyu's comment on the right-angle might have seemed unimportant in that moment, it led to many moments of joy as we recognized the knowledges in seemingly simplistic artefacts we had not previously realized. My goal in this dissertation had no initial direct connection to ethnomathematics—the study of mathematics and culture (D'Ambrosio, 1985)—but centering the voices of my co-researchers led the deep engagement that arose. Engaging in this activity illuminated African Indigenous ways of knowing and thinking that, to my co-researchers and I, had only been apparent in school mathematics.

"AS AFRICANS, WE HAVE OUR OWN WAYS THAT OUR ANCESTORS HEALED PEOPLE": OUR PARTICIPATORY RESEARCH PROJECT

Though we discussed issues such as the impact of colonization on African education and the gendered spaces of STEM classrooms, we decided to hone in on one social issue that we could systematically analyze. It dawned on me that I had not seen issues like these mentioned in prior research concerning mathematics and social justice. Issues discussed in this literature tend to be those related to gross injustices faced by minoritized communities, such as racial discrimination in housing, driving while black / brown, or the prison industrial complex. In this research, the issues discussed included addressing brain drain of youth in Africa, lack of African history in the school curriculum, and the challenges of inter-Africa travel.

Ultimately, we voted on one issue and decided to pursue Effective Public Health Strategies. The youth decided that they wanted to disrupt the notion that manufactured medicine was more valuable than traditional medicine. Manyoni noted that "as Africans, we have our own ways that our ancestors healed people." She also expressed her concern towards manufactured medicine because as she said, "the western side will continue to capitalize on people's health and what better place than Africa."

To compare manufactured and traditional medicine, we designed a survey to collect information from youth in their respective social circles. Questions on the survey included asking how African youth define traditional and manufactured medicine, their experiences with each, and the type of medicine youth were likely to consider passing down to their offspring. At the end of our exploration, the group recognized that the salient reason the 17 youth who responded to the survey might have shifted from a reliance on traditional medicines to manufactured medicines was because of the precision of dosages in manufactured medicine. This was similar to what we found when we spoke to elders in our lives. Dosages are a numerical measurement, and it appeared there was an inherent trust in precise measurement over the feeling, intuition, or the 'just knowing' that is a part of African Indigenous ways of knowing.

Similar to the exploration around the mathematics embedded in our communities, Njo brought back the notion of different types of mathematics involved in manufactured medicine and traditional medicine. For Njo, elders who use traditional medicine informally tell their customers the appropriate amount to take as they have also learned what 'dosage' to recommend. This measurement, however, is not the 'calibrated mathematics' used in manufactured medicine that many youths are accustomed to. Njo posited the notion of "their math" and "our math." They defined "their math" as African Indigenous ways of knowing while "our math" is the dominant mathematics learned in formal school spaces. Njo is drawing on African Indigenous knowledge practices that do not rely solely on naming and numbering. She later shared that the heavy reliance of naming and numbering is a western concept. More generally, this separation of formal (*our* math) and informal knowledge (*their* math) is an important dichotomy to attend to in decolonizing work.

Though the youth found ways mathematics could be harnessed as a tool for disruption, there was concern that mathematics has its limitations because there are African Indigenous practices that cannot be quantified. For example, Mendrika talked about a practice in the farming community in which she was raised. After sowing beans, the farmers take a few seconds to sit on the land before leaving. She shared that the farmers believed that if they did not sit on the land, their harvest would not be plentiful. Our conversation veered into other traditional practices like Mendrika's example, that elders 'just know' will produce the intended result, even when no can fully explain why. This type of knowledge, or way of knowing, usually comes from traditions often passed down orally and, though it might not have scientific value, is still valid in African Indigenous knowledge systems and practices.

LASTING LEGACY OF THE BAOBAB TREE

Gutstein's (2007) framework connects the critical, community, and classical knowledges, towards a *goal* of critical consciousness and to racial justice within a U.S. context. My findings suggest that a *goal* of CME in a sub-Saharan African context is to *render visible that which has been made invisible*. In other words, youth are not simply *reading* and *writing* an already existing world but are invested in *re-reading* and *re-writing* their African world by *shifting the center*. Dei's (1994) early work helps me contextualize this focus as he stated that centering African Indigenous knowledges is "a weapon of liberation" (p. 17) to counter western ideologies in schooling. Furthermore, he asserted that it is not enough for an African-centered education to be emancipatory or liberatory as is the case with critical pedagogy from a Freirean perspective. Instead, African Indigenous knowledges offer "a language of possibility through which to deconstruct and reclaim, not only new forms of knowledge", as well as to reconstruct, challenge, and contest identities, histories, voices, and place (Dei, 1994, p. 19).

Moreover, the findings point to the necessity of expanding our understanding of social justice as not only considering issues of justice at present but also cognitive justice that calls to the past. Cognitive justice works at redressing epistemicides (loss of Indigenous knowledges) caused by colonization and push us towards epistemic freedom (Ndlovu-Gatsheni, 2015, 2018). Epistemic freedom expands knowledges to include those forms of knowledges that are not restricted to philosophical and scientific forms. These forms of knowing also include Indigenous ways of knowing that emphasize "relational knowing, intuition-reasoning and empathy" (Greene, 2019, p. 95) which have been labelled "myths, superstitions, and non-science" (Boutte et al., 2019, p. 15). When Sanyu stated that "whenever you see a right-angle, it means a white man has been there," it changed the course of our research space. Prior, I had not considered bridging CME, ethnomathematics, and Indigenous ways of knowing in mathematics in this work, as there was no precedent in the literature for doing so.

In this research, I demonstrate the value in bridging these ideas: My research revealed that doing so engendered what Ndlovu-Gatsheni (2018) called *critical decolonial consciousness*. Drawing on decolonial theory and frameworks, I argue that critical consciousness raising requires embedding decolonial notions of questioning the past, present, and future repercussions of colonization and settler colonialism (Tuck & Yang, 2014). Thus, critical decolonial consciousness in the context of CME involves a call to dismantle settler colonialism and continued effects of colonization. This is because in the sub-Saharan African context, one cannot underestimate the damage and continued vestiges brought on by colonization.

Lastly, the influence of African epistemological grounding was invaluable. Engaging with Sankofa, Ubuntu, and FAM within the framing of decolonial theory and methodologies opened the space for co-learning, reflecting on previous experiences before moving forward, and finding the joy within this work. The sharing of African narratives at the start was a defining moment, as it was in doing so that we began to recognize our overlapping commitments to the African continent. Instead of starting from problems and issues that exist in our different African countries, we all embodied Ubuntu by recognizing the strengths in our shared humanity. FAM also pushed us to disrupt the belief that Indigenous African ways of knowing, in particular within the context of mathematics, is illegitimate. This research shows that, when SSA youth see the embedded Indigenous [mathematical] knowledges within their cultural practices and / or artefacts, they begin to feel ownership of their school mathematics.

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VOICES FROM THE FIELD: EXPLORING THE SECONDARY-TERTIARY TRANSITION INTO CALCULUS

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ABSTRACT

This study investigated factors affecting the secondary-tertiary transition into calculus for students in STEM disciplines. Drawing from a larger study, this paper presents qualitative data obtained through semi-structured interviews with professors, instructors, and administrators associated with first-year calculus at two large Ontario-based universities (n_1=12; n_2=10). Findings highlight critical ways participants have observed students facing challenges as they transition into first-year calculus. Participants feel that students enter tertiary mathematics with a fundamentally different idea of what calculus and mathematics are, compared to tertiary educators, resulting in major disconnects in learning. Further, participants believe that students experience dissonance in feelings of belonging within mathematical spaces. Students struggle with mathematics, often for the first time, in first-year calculus, and this causes major challenges in their conceptions of who they are. This study highlights the need to consider academic and personal challenges specific to mathematics learners as students navigate the secondary-tertiary transition.

INTRODUCTION

The national conversation around encouraging students to pursue careers in science, technology, engineering, and / or mathematics (STEM) fields is not new and has, in fact, been commonplace for at least 10 years (e.g., Munroe & Stuckey, 2013; Science, Technology and Innovation Council, 2014). Yet, tertiary graduation rates in STEM fields in Canada are not substantially rising (Orpwood et al., 2012), pushing stakeholders to consider reasons why. Among the many possible factors impacting this trajectory, some scholars have suggested that mathematics, and more specifically, calculus, acts as a gatekeeper within STEM fields (Gainen, 1995), in part because first-year calculus is a requisite (or 'service') course in these degree programs. This is of particular concern because, while the transition from secondary to tertiary education is generally challenging, there is significant research that suggests that the transition into mathematics occupies a unique position of difficulty (e.g., Clark & Lovric, 2009; Pyzdrowski et al., 2013). Given this, this paper seeks to broadly consider the question, "what are the factors that impact student success in first year calculus?"

Though this question has been addressed quantitatively, particularly in the United States context (e.g., Burton, 1989; Sadler & Sonnert, 2018), perspectives of those who interact with students at this transition point are woefully limited in the literature (Wade et al., 2016), especially in the Canadian context. Hence, in this paper, I seek to broadly illuminate some of the major themes that emerged from a large-scale, multi-site study that focused on the transition from secondary to tertiary calculus for students in STEM disciplines in Ontario. Though the overarching study used a mixed methodology, I will focus on a subset of the qualitative results from this work, namely through sharing the perspectives of instructors, professors, and administrators (all of whom occupied the role of 'tertiary educators') on this challenging time for students. This paper's primary focus is on elucidating the participants' views on how students' prior experiences in, and conceptions of, mathematics impact their transitions from secondary to tertiary calculus.

LITERATURE REVIEW

THE SECONDARY-TERTIARY TRANSITION: A BRIEF OVERVIEW

This study is informed by Clark and Lovric's (2008) model theorizing the changes students experience as they transition from secondary to tertiary mathematics. They suggest that it is helpful to view this transition as a 'rite of passage', where change is necessary for students to experience. This model (Clark & Lovric, 2009) proposes three stages of transition: separation (from secondary school mathematics), liminal (engaging in new beginnings in tertiary mathematics), and incorporation (adjusting to tertiary mathematics). Crucially, the liminal phase is where conflict occurs for students and is thus the focus of many studies that examine this secondary-tertiary transition (e.g., Cullinane, 2011; Hong et al., 2009; Zietara, 2016).

While studies across the world have examined the impact of prior mathematics experience on first-year success (e.g., Adkins & Noyes, 2018; Burton, 1989; Sadler & Sonnert, 2018), there are fewer that have investigated the perspectives of those impacted by this transition point in first-year calculus. Some have spoken to those involved in the endpoints of this transition (i.e., secondary mathematics teachers and tertiary mathematics professors) and found a marked disconnect between their expectations, particularly in terms of the presence of calculus in the secondary classroom (e.g., Hong et al., 2009; Wade et al., 2016). Other studies focusing on the student perspective have found that the pace of first-year calculus was challenging (Kouvela et al., 2015; Sierpinska et al., 2007) and that waning confidence in their mathematical abilities exacerbated their struggles in the course (Zietara, 2016). Interestingly, an emerging body of work has begun to investigate student perspectives on this transition through the lens of changing identities over time (Di Martino & Gregorio, 2019; Hernandez-Mendez et al., 2011). Research has found that some students interpret the challenges they face as an opportunity to forge a new identity (Hernandez-Mendez et al., 2011), while others have found that the new series of 'firsts' (i.e., first time not understanding a concept, first time failing, etc.) proves to be pivotally difficult for some students (Di Martino & Gregorio, 2019). This developing scholarship suggests additional factors for stakeholders to focus on in the examination of the secondary-tertiary transition in mathematics.

METHODS

This study draws on data from a broader research project that used mixed methods (Creswell & Plano Clark, 2007) to explore the secondary-tertiary mathematics transition. The proceeding sections describe the qualitative method used. Because much of the literature on the secondary-tertiary transition centres specifically on quantitative paradigms (e.g., Fayowski et al., 2009; Sonnert & Sadler, 2018), I believed it to be critical to investigate this issue through a qualitative lens as well. This was of particular importance because a qualitative view of this phenomenon will help illuminate deeper questions of 'why' and 'how' trends manifest (Creswell, 2007).

CONTEXT & RESEARCH DESIGN

A multi-case study (Stake, 1995; Yin, 2002) approach was used to collect data from two research sites (Institution #1 and Institution #2). Institution #1 and Institution #2 are both large, research-intensive universities in Ontario, Canada. Both institutions have among the highest admissions average requirements for each of their STEM degree programs, and first-year calculus courses are generally among the largest courses on each campus (lecture sections often exceed 100 students). Additionally, at both institutions, first-year calculus differs across programs, so specialized versions of calculus are offered to, for example, science students and engineering students.

DATA COLLECTION

The primary method of qualitative data collection at both institutions was semi-structured interviews (Teddlie & Yu, 2007). Semi-structured interviews were the chosen method of qualitative inquiry because they provided flexibility with interviews where questions offered a focused framework within which participants could consider their experiences but were also able to elaborate and delve deeper into other, sometimes unexpected or divergent lines of thought (Gubrium & Holstein, 2001). All interviews took place between fall 2018 and summer 2019. At each institution, the aim was to recruit participants who were either instructors, professors, and / or administrators who had experience with first-year undergraduate calculus. Thus, purposive sampling (Teddlie & Yu, 2007) was necessary to employ. In total, 12 participants were interviewed from Institution #1 and 10 participants were interviewed from Institution #2. The interview protocol was developed based on themes from a smaller pilot study (Sahmbi, 2014) and centred around participants' perspectives on student successes, challenges, and personal and institutional adaptations to this transition. Data was collected concurrently from both institutions. Pseudonyms were provided for all participants. To further protect the anonymity of participants, the pseudonyms selected do not necessarily reflect the demographic identity indicators of participants, and instead are the first names of various Bollywood stars. Importantly, as an outsider to the academic mathematics spaces at each institution, it was critical that trust was established with participants. Thus, in addition to conversations prior to data collection, participants were given the choice to member check transcripts prior to dissertation publication.

DATA ANALYSIS

Data were transcribed verbatim and analyzed iteratively through multiple cycles using the general inductive approach (Thomas, 2006). Qualitative data analysis software (*NVivo 12*) aided in the organization and subsequent coding of data. Coding of the data occurred through the constant comparison method (Miles et al., 2013) and iterative cycles of analysis where reflexive memos were created to aid in subsequent cycles of analysis (Falk & Blemenreich, 2004). This was particularly crucial to keep track and make sense of the volume of data across both institutions. Indeed, while *a priori* codes were used based on big ideas that guided the development of the interview protocol, series of analysis allowed for new and often-unexpected themes to emerge. In this way, data analysis was a living process that evolved over time to ensure that the data from interviews were rigorously and intensively analyzed.

FINDINGS

Though institutional differences existed, there was considerable overlap in themes that emerged in both case studies. Thus, findings shared in this paper will be presented as an interwoven story of perspectives from participants at both Institution #1 and Institution #2.

INCONGRUENT CONCEPTIONS OF CALCULUS

Participants at both institutions were eager to discuss the impact of prior experiences in mathematics on student success in tertiary calculus. Notably, several participants indicated that students in their calculus courses typically arrive with strong 'basic' computational skills and are particularly adept at rote learning. Farhan explained that students exhibit strong skills conducive to "successfully mastering algorithms, differentiation…all this stuff where there is a set procedure, it's a recipe, it's an algorithm." This was corroborated by participants at both institutions, where many were concerned that students appear to believe that the preceding skills are what makes one successful in mathematics. Sidharth expanded on this, indicating that for calculus in particular, he wants students to recognize that "[calculus is not] just a set of rules to memorize…[but] rather, a concept to understand", yet he has found that this runs contrary to students' understanding of mathematics as a discipline. In addition to specific topics in calculus (e.g., limits, infinity, etc.), many participants highlighted that problem solving, contextual learning, and application of understanding are crucial for success. Rani explained, "I want [students] to be able to solve complex and new problems that they have not seen before" and to apply concepts learned in class to novel situations. This emphasis on ways of thinking and doing mathematics was prominent in many of the discussions I had with participants, aptly summarized by Sidharth as participants "want[ing] to encourage students to think about what [a] problem means and how to understand how to use mathematical thinking" in these situations.

While this might be the aim of many instructors and professors, participants indicated that there is an incongruence across expectations between educators and students. Ranveer suggested that "most [students] come thinking that calculus is going to be about computations", rather than a subject-area focused on conceptual (as opposed to mechanical or algorithmic) learning. Contrary to societal frames of mathematics, "math is [also] words, not just formulas" (Anil), and this can be a challenging fact to accept for many students. The disconnect in views of mathematics—particularly for students who have been successful in mathematics while holding a differing view of the discipline—concerned participants because they feared that students would be lulled into a false sense of confidence about their understandings. Saif noted, "there is a danger when students think they know something and then they do not pay attention", or they do not know how to adjust their way of learning to meet different expectations. Thus, different conceptions of what mathematics and, more specifically, calculus are—and, by extension, what it means to understand them—impact the way students engage with course material. Further, they also impact students' expectations of how to succeed in mathematics at large.

RECONSIDERING ONE'S PLACE IN STEM, IN MATHEMATICS

As participants delved into the many factors they perceive to impact students' success across the secondary-tertiary transition into calculus, several identified a particular issue affecting high-achieving STEM students. Because the institutions and programs that each of the participants taught at require high admissions averages, students who enter first year necessarily have previously experienced significant success in mathematics. Indeed, for many, 'being good at school', especially mathematics, is a central part of their identity. While this can be positive, it also can result in significant dissonance when they experience challenges in mathematics for the first time.

Referring to students in first-year mathematics courses, Dilip explained,

> *They never had to ask for help [before now]…you have the fact that most of these kids were [at] the top of their school, mathematics wise…so, suddenly, they have got things they do not know how to do right away…they were always the best [at]…but now [they are] just one of the, one of the people study[ing] mathematics [here]. You may be the best, but chances are, you aren't, right?…[and] that is a big kick in the head for some people here.*

Experiencing these mathematical challenges for the first time, as well as readjusting their own identity in this space can sow doubts in students. Madhuri shared that many students began to question themselves and their place in their programs when they struggled, saying things like, "it shouldn't be this hard…maybe I'm not good at math after all." Shahrukh expanded on this, suggesting that "this is the first time [these students] have ever had to struggle with mathematics…[but] it's not like they've lost their ability to do math over the summer before they came here, but it is [that] they are now one of a lot of people like them." Shahrukh highlights two major ideas here: despite their newfound lack of mathematical self-confidence, students are still mathematically capable, but they now exist and learn amongst many students who are just as mathematically capable and might now be experiencing the same challenges in their learning. While one might consider this a comforting thought, many participants indicated that this is not necessarily encouraging for students to feel because so much of who they have been up to this point is tied with excelling in mathematics. Indeed, participants suggested that this leads students to question their sense of belonging in STEM disciplines, and in some cases, negatively affects their mental health. Consequently, this is a major obstacle for students to overcome in their first year.

DISCUSSION & CONCLUSION

This paper offers some insight into the perspectives of key frontline stakeholders—namely, professors, instructors, and administrators—on the transition from secondary mathematics into first-year calculus. Specifically, participants highlighted two major ideas: 1) there is incongruence between students' and tertiary educators' perspectives on what calculus and more broadly, mathematics, are as fields of study, as well as what it means to be successful in them; and 2) students who struggle across the transition from secondary to tertiary mathematics experience a specific instability in their sense of belonging in mathematics and STEM disciplines at the tertiary-level. Clark and Lovric's (2008; 2009) work offer a broad contextualization of these findings, particularly in terms of the liminal stage of the secondary-tertiary transition. It is evident that participants recognize that when they meet students in their first year, they are in this period of transition where they are experiencing significant changes and challenges on both a mathematical and a personal level, and that this influences how they navigate the transition.

Many participants stressed the disconnect they feel between their aims and understandings of calculus, and mathematics as a discipline, and how students conceptualize it, which suggests that there is a gap in understanding between K-12 and tertiary systems (Wade et al., 2016). Indeed, tertiary educators in this study suggest that, though computational skill is important, deeper, conceptual skills and creative ways of thinking are critical for success in mathematics. This comports with recent analyses from the Education Quality and Accountability Office (EQAO) of students' mathematical performance in Ontario, where they found that elementary students "have stronger knowledge and understanding of fundamental math skills than they have the ability to apply their skills and to think critically about them" (Education Quality and Accountability Office, 2019, p. 2). Recognizing these shared observations is crucial in advancing better ways of teaching and learning in mathematics, as there is a suggestion that, across the continuum of formal mathematics education (i.e., K-16+), students must be better supported in developing both application and critical thinking skills in mathematics.

Importantly, participants in this study also offered observations that align with what some scholars have identified as a struggle with an identity shift for students in mathematics (Di Martino & Gregorio, 2019; Hernandez-Mendez et al., 2011). These previous studies have focused on the student voice, and while it is certainly critical to elevate student perspectives on their own experience, this study demonstrates an awareness from professors, instructors, and administrators of the importance of this identity shift and its presence in their courses. To date, this perspective is relatively underexplored, and there is limited scholarship that suggests that frontline stakeholders recognize this particular challenge that students face. I believe that the acknowledgement of this facet of the secondary-tertiary transition from the tertiary-level is an important part of the large project of supporting students on their journey towards success. With consideration of the importance of belonging in higher education (e.g., Allen & Bowles, 2012), it is crucial that educators recognize how and why students in their classrooms may feel that they do not belong, and how we can all better support their efforts to persist and find themselves in mathematics.

The secondary-tertiary transition into mathematics is complex and dynamic, and as these findings suggest, academic and personal factors specific to being a mathematics learner must be considered if we are to better support students across this transition. I encourage institutions, particularly mathematics departments and faculties, to capitalize on findings that suggest that there are overlapping concerns across stakeholders, including students, on some of the factors that impact the secondary-transition, and find ways to collaborate to develop solutions. There is much that folks at each stage of the transition can learn from each other, including teachers, professors, and students, all in the service of achieving the same goal: improving access, engagement, and success in mathematics.

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UNDERSTANDING MATH TEACHERS' PARTICIPATION IN A PROFESSIONAL LEARNING NETWORK

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ABSTRACT

This study investigates the interactive conversations in a Professional Learning Network (PLN) to understand its affordances. It used interpretive inquiry as the methodology and complexity thinking as the theoretical framework. One PLN was targeted to collect the archived data from which four blog posts and their comments were selected as illustrative examples. The results presented the diverse conversation structures through conversation weaving and conversation expanding as well as the multiple types of knowing emergent from the conversations including mathematics-for-teaching, beliefs about teaching, social relationships, blog resources, *and* recounting experiences. *The knowing of* mathematics-for-teaching *was enacted in the moments of mathematics teachers' participation in the conversations. The other four types of knowing were implicated with the emergence of* mathematics-for-teaching*, the teachers' participation in the PLN, and the evolvement of the PLN itself. However, they have not yet been explored in the predominant research on teachers' disciplinary knowledge of mathematics.*

BACKGROUND

Online professional learning communities have become prominent in teachers' professional development in recent years (Beach & Willows, 2014; Borba & Llinares, 2012; Dash et al., 2012; Trust, 2012). As a new form of them (Trust, 2016), professional learning networks (PLNs) have the potential to make teachers' professional learning more "participatory, grassroots and supportive" (Carpenter & Krutka, 2015, p. 708) and make it possible for teachers to access important resources that they could not afford or even access in the local communities (Dede et al., 2005). It is not surprising, then, that a growing number of mathematics teachers have participated in PLNs to extend their professional learning. Yet, what their conversation structures look like in PLNs and what could emerge from their conversations in relation to mathematics-for-teaching remains unknown. This study addresses this gap by investigating the collective conversations in a PLN to understand its affordances.

METHODOLOGY

This research used interpretive inquiry as the methodology and complexity thinking as the theoretical framework. One PLN was targeted to collect the archived data—blog posts and comments—from which four blog posts and their comments were selected as illustrative examples.

I commenced the data analysis as soon as I collected the first post because collecting guided me to make further inquiry possible, as Patterson and Williams (2002) suggest. To put it differently, I engaged simultaneously in the processes of data selection, collection, and analysis.

Specifically, data analysis and collection were interwoven in an unfolded spiral and the hermeneutic circle (Figure 1). The unfolded spiral in Figure 1 shows the interplay between the data collection and the data analysis. Each of the four selected examples went through several loops, and all the sets of example collection and analysis were eventually taken as the forward arc (the left side of the diagram circle) of the whole hermeneutic circle (the diagram circle) and examined through the backward arc (the right side of the diagram circle) as a whole to see the relationship between the whole and the part of data collection and analysis.

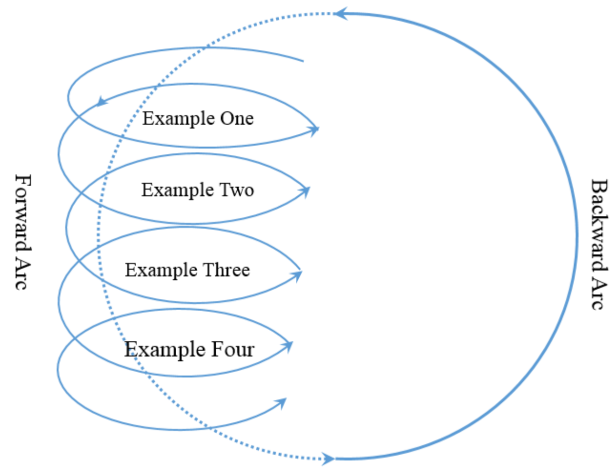

Figure 1. A diagram showing the process of data collection and analysis.

In addition, writing was used as a way of engaging my thinking in terms of its significant role in interpretive inquiry. I wrote out whatever emerged from the data collection and analysis and rewrote it over and over again as a part of my reflection to uncover what "insights and connections emerged from the very process of the writing itself" (Ellis, 1998, p. 6).

The four illustrative examples are concerned with 1) teaching improvement, 2) textbook presentations, 3) introducing rational functions, and 4) solving problems about chord lengths. Each of them was named after its content. For instance, the example dealing with the application of writing improvement to teaching improvement was titled *Teaching Improvement*; the example with the presentations of the Handshake Problem from two textbooks was titled *Textbook Presentations of the Handshake Problem*; the example with introducing the concepts of rational functions in a graphical way to facilitate students' conceptual understanding was named *Introduction of Rational Functions*; and the one about solving the problem(s) of the chord lengths was called *Solving Problems about Chord Lengths*.

Several data analysis techniques and conceptual frameworks (see Table 1) including recursive dynamics, the features of fractal images, thematic analysis, mathematics-for-teaching, and necessary conditions for complex systems were adopted in this study.

Data	**Purpose**		**Analysis Techniques and Conceptual Frameworks**
Posts and Comments	The Emergence of Knowing	The emergent topics	Thematic analysis
		The collective knowing	The model of mathematics-for-teaching
	The Structure of the Conversations	Weaving	Recursive dynamics
		Extending	Fractal images
	The Environment of the PLN		The necessary conditions for complex systems

Table 1. An overview of techniques and conceptual frameworks.

RESULTS

The above data analysis of the selected four illustrative examples resulted in the identification of structures of conversations among the participants and the emergence of knowing from the conversations in the PLN. To be more specific, the analysis on the recursions and the conversation extensions revealed the diversity of conversation structures, while the analysis on the emergent topics and the collective knowing reveals the emergence of the multiple types of knowing from the conversations: *mathematics-for-teaching*, *beliefs about teaching*, *blog resources*, *recounting experiences*, and *social relationships* (see Figure 2).

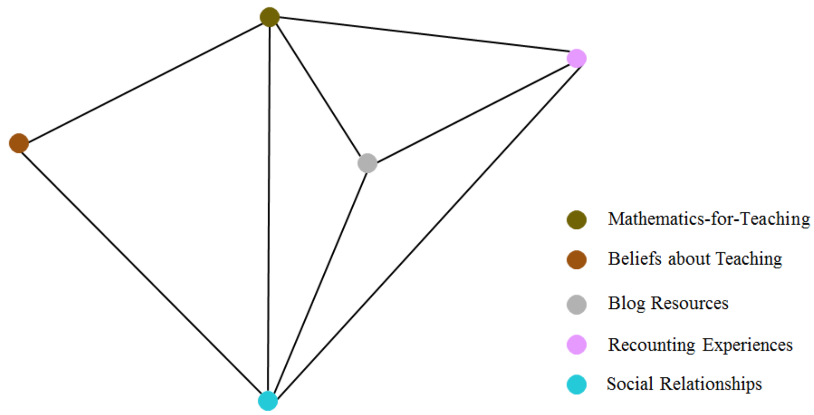

Figure 2. The emergent knowing.

The emerged knowledge is calling me to re-examine the theories about teacher professional knowledge. My re-examination makes clear that the knowledge of mathematics-for-teaching is available and could be enacted in the moments of mathematics teachers participating in the PLN. That means participating in PLN is a means of enacting mathematics-for-teaching. My re-examination also reveals that such types of knowledge as beliefs about teaching, blog resources, recounting experiences and social relationships have not yet been involved in the predominant research of mathematics teacher's professional knowledge such as Ball and colleagues' (Ball et al., 2008) knowledge of mathematics-for-teaching, Ma's (1999) profound understanding of fundamental mathematics, and Davis and Renert's (2014) profound understanding of emergent mathematics. That means those types of knowledge are not yet considered elements of conventional teacher's professional knowledge, but in this PLN, they are indispensable for the emergent knowledge, teacher's participation, and the evolvement of PLN itself.

The PLN provided an open space for participants to explore in their own ways. They had space to satisfy individual needs and interests, such as reflecting on their individual learning and / or teaching experiences, expressing their feelings, and sharing or searching for resources. Certainly, the openness and sociality of the PLN meant individual doings with regards to reflections, expressions, or resources available to others. This initiated collective learning from which different kinds of knowing emerged. For instance, through the collective learning in the PLN, participants were able to know mathematics-for-teaching, argue about teachers' beliefs about teaching, build up social relationships, and share blogs and experiences as resources.

In addition, the individual doings and the collective learning interacted with each other in the open space. The collective learning brought more conversations on the topics of individual doings. This encouraged the individuals to do more and share more. The more doings or sharing done by individuals, the more collective learning occurred. There are reasons to believe that such interplay is conducive to the development of the PLN and, in turn, the well-being of the PLN will nourish the individual doings and the collective learning.

CONCLUSIONS

This study contributed to the research areas relevant to teacher professional learning through PLNs and teachers' disciplinary knowledge of mathematics. First, this study contributed to the rapidly growing literature on teacher professional learning through PLNs. It could help people better understand the structures of conversations among participants and the emergence of knowing from conversations within a PLN. Second, the study contributed to the theorization on teachers' disciplinary knowledge of mathematics. Its results uncovered the five types of knowing emergent from the conversations in the PLN. As an open learning site, a PLN makes it possible to observe how learning occurs (Bates et al., 2018) and allows different types of knowing to present explicitly. Except for the knowing of *mathematics-for-teaching*, the other four types of knowing inclusive of *beliefs about teaching*, *blog resources*, *recounting experiences*, and *social relationships* have not yet been addressed in the dominant research on teachers' disciplinary knowledge of mathematics. However, they were implicated with the emergence of *mathematics-for-teaching*, the teacher's engagement in the PLN, and even the sustainability and development of the PLN itself. Therefore, the other four types of knowing are proposed to be elements of the teachers' disciplinary knowledge of mathematics from the systematic and dynamic perspective.

The study helps me to better understand mathematics teachers' professional learning through their participation in the professional learning networks and online learning communities (Dash et al., 2012). Additionally, it offers a valuable reference for reviewing online and even conventional teacher professional development. Looking forward, the study will inform further exploration of the nature of the relatively new form of teacher professional learning when we come to realize the affordances of our digitally connected world and the intricacies of teachers' professional growth as indicated by Brooks and Gibson (2012).

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Ad Hoc Sessions

Ad hoc sessions

LAUGHABLE OR LAUDABLE: MATHEMATICS TEACHER EDUCATORS' AIMS AND POTENTIAL NEEDS FOR SUPPORT

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The Canadian Journal of Science, Mathematics and Technology Education (2019) published a special issue that focused on many of the issues facing mathematics teacher educators. The issue was filled with articles discussing the Math Wars in Canada (Chernoff, 2019), and the challenges mathematics educators face when explaining these issues to the media (Abtahi & Barwell, 2019; McFeetors & McGarvey, 2019). Boaler (2012) has made public the personal attacks that she has faced from two particular mathematicians in the field about her research into mathematics teaching and learning. Even though the Milgrim and Bishop paper that was used to 'disprove' Boaler's work has since been discredited, it is still being used today by some Canadian mathematicians to negate the work of Canadian mathematics teacher educators. Similarly, others have become emboldened to attack publications that are targeted at changing the public narrative. As an example, to follow are two of the many inflammatory comments made in the ensuing online discussion in response to Holm & Kajander (2019), which was an attempt to explain the goals of modern mathematics education research to a more generalist audience, as was advocated by the CMESG working group of Savard and Simmt (2017).

> *Are these 2 professors, "math professors", i.e. have a background/degree in mathematics, or "math education professors", which [sic] have a background in education? Clarifying this specific point is crucial to the legitimacy of the content of this article. Why? Because most of the damage inflicted on math curricula and resources over the past 2-3 decades hails from education departments, which have been in charge of training our teachers and advising agencies and consultancy firms which publish math books and foster the ed consulting industry.*
>
> *To illustrate why, Jo Boaler, famed guru of YouCubed, is constantly being mistaken for a "math professor", when what she really does, is teach math teachers. Her claims that mistakes make the brain grow, and memorization harms children, is why we now have millions of schoolchildren across North America receive sub par math education.*

and even

> *If you can't see that the whole thing is laughable on its face, and abusive when applied to children, God help you.*

Our *ad hoc* proposed a purposeful discussion of a possible new platform that would support changing the narrative via working together and explicitly supporting one another as mathematics educators. Attempts at creating a specifically Canadian group for mathematics teacher educators have not been successful in the past, partially due to lack of funding. With the advent of online forums and social media, such a group might now be initially feasible with very little funding, and this initial discussion sought to discuss the interest, needs and potential for the creation of such an organisation. We support a move forward by bringing together mathematics teacher educators and their allies in identifying where the problems are and making a plan for how to move forward to explicitly support each other. Our goal is to strengthen this Canadian Mathematics Teacher Educator community and work to start changing the narrative.

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TEACHER BELIEFS ABOUT TEACHING MENTAL MATH

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A review of the literature indicates a broad range of definitions for mental math and consensus about the possible affordance of teaching mental math. For example, researchers have found that when students are faced with solving computational problems orally, they engage in sense-making activities (Proulx, 2013). Furthermore, studies analyzing number talks (e.g., Pourdavood et al., 2020) have found enhancements to students' problem solving, reasoning, and communication skills. Olsen (2015) asserts that solving mental math questions has the potential to grow students' conceptual understandings of numbers and support algebraic reasoning. It seems that the benefits of mental math are far reaching and the teaching of mental math should be a focus.

With one of the obvious changes of Ontario's 2020 Curriculum (Ontario Ministry of Education, 2020) being the explicit mention of mental math, we feel it is timely and worthwhile for mathematics educators and researchers to collaboratively consider how practicing teachers conceptualize mental math in their classrooms. More specifically, to direct effort towards the beliefs school teachers hold about teaching mental math and how these beliefs may support and hinder the teaching of mental math that would align with current mathematics education research recommendations. The *ad hoc*, reported here, is hopefully the beginning of such a discussion and endeavour.

Considering the range of held beliefs and what the teaching of mental math might look like in school classrooms, it is not unreasonable to consider how elementary teachers might have traditionally used mental math activities, like mad-minutes, to develop students' computational skills such as speed and accuracy. However, with the inclusion of *flexibility* and *strategy choice* as necessary components for developing number fluency and the rise of teaching practices involving oral talks about computations in classrooms (e.g., number talks), a multitude of perceptions about mental math and how it is taught are important to recognize.

During this *ad hoc*, the authors offered three teacher beliefs they could imagine teachers holding on the teaching of mental math based on their work in school classrooms and collaborations with teachers. *Ad hoc* attendees considered these beliefs, gave feedback, and imagined more based on familiar research and their direct experience with teachers and pre-service teachers. Attendees were presented with example videos of what the authors deemed to be research-inspired/infused teaching of mental math. Small nuances of teacher moves in the videos were discussed and a broader discussion of how researchers could support teachers in the teaching of mental math ensued.

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Appendices

Annexes

Appendix A / Annexe A

WORKING GROUPS AT EACH ANNUAL MEETING / GROUPES DE TRAVAIL DES RENCONTRES ANNUELLES

1977 *Queen's University, Kingston, Ontario*

- Teacher education programmes
- Undergraduate mathematics programmes and prospective teachers
- Research and mathematics education
- Learning and teaching mathematics

1978 *Queen's University, Kingston, Ontario*

- Mathematics courses for prospective elementary teachers
- Mathematization
- Research in mathematics education

1979 *Queen's University, Kingston, Ontario*

- Ratio and proportion: a study of a mathematical concept
- Minicalculators in the mathematics classroom
- Is there a mathematical method?
- Topics suitable for mathematics courses for elementary teachers

1980 *Université Laval, Québec, Québec*

- The teaching of calculus and analysis
- Applications of mathematics for high school students
- Geometry in the elementary and junior high school curriculum
- The diagnosis and remediation of common mathematical errors

1981 *University of Alberta, Edmonton, Alberta*

- Research and the classroom
- Computer education for teachers
- Issues in the teaching of calculus
- Revitalising mathematics in teacher education courses

1982 *Queen's University, Kingston, Ontario*

- The influence of computer science on undergraduate mathematics education
- Applications of research in mathematics education to teacher training programmes
- Problem solving in the curriculum

1983 *University of British Columbia, Vancouver, British Columbia*

- Developing statistical thinking
- Training in diagnosis and remediation of teachers
- Mathematics and language
- The influence of computer science on the mathematics curriculum

1984 *University of Waterloo, Waterloo, Ontario*

- Logo and the mathematics curriculum
- The impact of research and technology on school algebra
- Epistemology and mathematics
- Visual thinking in mathematics

1985 *Université Laval, Québec, Québec*

- Lessons from research about students' errors
- Logo activities for the high school
- Impact of symbolic manipulation software on the teaching of calculus

1986 *Memorial University of Newfoundland, St. John's, Newfoundland*

- The role of feelings in mathematics
- The problem of rigour in mathematics teaching
- Microcomputers in teacher education
- The role of microcomputers in developing statistical thinking

1987 *Queen's University, Kingston, Ontario*

- Methods courses for secondary teacher education
- The problem of formal reasoning in undergraduate programmes
- Small group work in the mathematics classroom

1988 *University of Manitoba, Winnipeg, Manitoba*

- Teacher education: what could it be?
- Natural learning and mathematics
- Using software for geometrical investigations
- A study of the remedial teaching of mathematics

1989 *Brock University, St. Catharines, Ontario*

- Using computers to investigate work with teachers
- Computers in the undergraduate mathematics curriculum
- Natural language and mathematical language
- Research strategies for pupils' conceptions in mathematics

1990 *Simon Fraser University, Vancouver, British Columbia*

- Reading and writing in the mathematics classroom
- The NCTM "Standards" and Canadian reality
- Explanatory models of children's mathematics
- Chaos and fractal geometry for high school students

1991 *University of New Brunswick, Fredericton, New Brunswick*

- Fractal geometry in the curriculum
- Socio-cultural aspects of mathematics
- Technology and understanding mathematics
- Constructivism: implications for teacher education in mathematics

1992 *ICME–7, Université Laval, Québec, Québec*

1993 *York University, Toronto, Ontario*

- Research in undergraduate teaching and learning of mathematics
- New ideas in assessment
- Computers in the classroom: mathematical and social implications
- Gender and mathematics
- Training pre-service teachers for creating mathematical communities in the classroom

1994 *University of Regina, Regina, Saskatchewan*

- Theories of mathematics education
- Pre-service mathematics teachers as purposeful learners: issues of enculturation
- Popularizing mathematics

1995 *University of Western Ontario, London, Ontario*

- Autonomy and authority in the design and conduct of learning activity
- Expanding the conversation: trying to talk about what our theories don't talk about
- Factors affecting the transition from high school to university mathematics
- Geometric proofs and knowledge without axioms

1996 *Mount Saint Vincent University, Halifax, Nova Scotia*

- Teacher education: challenges, opportunities and innovations
- Formation à l'enseignement des mathématiques au secondaire: nouvelles perspectives et défis
- What is dynamic algebra?
- The role of proof in post-secondary education

1997 *Lakehead University, Thunder Bay, Ontario*

- Awareness and expression of generality in teaching mathematics
- Communicating mathematics
- The crisis in school mathematics content

1998 *University of British Columbia, Vancouver, British Columbia*

- Assessing mathematical thinking
- From theory to observational data (and back again)
- Bringing Ethnomathematics into the classroom in a meaningful way
- Mathematical software for the undergraduate curriculum

1999 *Brock University, St. Catharines, Ontario*

- Information technology and mathematics education: What's out there and how can we use it?
- Applied mathematics in the secondary school curriculum
- Elementary mathematics
- Teaching practices and teacher education

2000 *Université du Québec à Montréal, Montréal, Québec*

- Des cours de mathématiques pour les futurs enseignants et enseignantes du primaire/Mathematics courses for prospective elementary teachers
- Crafting an algebraic mind: Intersections from history and the contemporary mathematics classroom
- Mathematics education et didactique des mathématiques : y a-t-il une raison pour vivre des vies séparées?/Mathematics education et didactique des mathématiques: Is there a reason for living separate lives?
- Teachers, technologies, and productive pedagogy

2001 *University of Alberta, Edmonton, Alberta*

- Considering how linear algebra is taught and learned
- Children's proving
- Inservice mathematics teacher education
- Where is the mathematics?

2002 *Queen's University, Kingston, Ontario*

- Mathematics and the arts
- Philosophy for children on mathematics
- The arithmetic/algebra interface: Implications for primary and secondary mathematics / Articulation arithmétique/algèbre: Implications pour l'enseignement des mathématiques au primaire et au secondaire
- Mathematics, the written and the drawn
- Des cours de mathématiques pour les futurs (et actuels) maîtres au secondaire / Types and characteristics desired of courses in mathematics programs for future (and in-service) teachers

2003 *Acadia University, Wolfville, Nova Scotia*

- L'histoire des mathématiques en tant que levier pédagogique au primaire et au secondaire / The history of mathematics as a pedagogic tool in Grades K–12
- Teacher research: An empowering practice?
- Images of undergraduate mathematics
- A mathematics curriculum manifesto

2004 *Université Laval, Québec, Québec*

- Learner generated examples as space for mathematical learning
- Transition to university mathematics
- Integrating applications and modeling in secondary and post secondary mathematics
- Elementary teacher education – Defining the crucial experiences
- A critical look at the language and practice of mathematics education technology

2005 *University of Ottawa, Ottawa, Ontario*

- Mathematics, education, society, and peace
- Learning mathematics in the early years (pre-K – 3)
- Discrete mathematics in secondary school curriculum
- Socio-cultural dimensions of mathematics learning

2006 *University of Calgary, Calgary, Alberta*

- Secondary mathematics teacher development
- Developing links between statistical and probabilistic thinking in school mathematics education
- Developing trust and respect when working with teachers of mathematics
- The body, the sense, and mathematics learning

2007 *University of New Brunswick, New Brunswick*

- Outreach in mathematics – Activities, engagement, & reflection
- Geometry, space, and technology: challenges for teachers and students
- The design and implementation of learning situations
- The multifaceted role of feedback in the teaching and learning of mathematics

2008 *Université de Sherbrooke, Sherbrooke, Québec*

- Mathematical reasoning of young children
- Mathematics-in-and-for-teaching (MifT): the case of algebra
- Mathematics and human alienation
- Communication and mathematical technology use throughout the post-secondary curriculum / Utilisation de technologies dans l'enseignement mathématique postsecondaire
- Cultures of generality and their associated pedagogies

2009 *York University, Toronto, Ontario*

- Mathematically gifted students / Les élèves doués et talentueux en mathématiques
- Mathematics and the life sciences
- Les méthodologies de recherches actuelles et émergentes en didactique des mathématiques / Contemporary and emergent research methodologies in mathematics education
- Reframing learning (mathematics) as collective action
- Étude des pratiques d'enseignement
- Mathematics as social (in)justice / Mathématiques citoyennes face à l'(in)justice sociale

2010 *Simon Fraser University, Burnaby, British Columbia*

- Teaching mathematics to special needs students: Who is at-risk?
- Attending to data analysis and visualizing data
- Recruitment, attrition, and retention in post-secondary mathematics
  Can we be thankful for mathematics? Mathematical thinking and aboriginal peoples
- Beauty in applied mathematics
- Noticing and engaging the mathematicians in our classrooms

2011 *Memorial University of Newfoundland, St. John's, Newfoundland*

- Mathematics teaching and climate change
- Meaningful procedural knowledge in mathematics learning
- Emergent methods for mathematics education research: Using data to develop theory / Méthodes émergentes pour les recherches en didactique des mathématiques: partir des données pour développer des théories
- Using simulation to develop students' mathematical competencies – Post secondary and teacher education
- Making art, doing mathematics / Créer de l'art; faire des maths
- Selecting tasks for future teachers in mathematics education

2012 *Université Laval, Québec City, Québec*

- Numeracy: Goals, affordances, and challenges
- Diversities in mathematics and their relation to equity
- Technology and mathematics teachers (K-16) / La technologie et l'enseignant mathématique (K-16)
- La preuve en mathématiques et en classe / Proof in mathematics and in schools
- The role of text/books in the mathematics classroom / Le rôle des manuels scolaires dans la classe de mathématiques
- Preparing teachers for the development of algebraic thinking at elementary and secondary levels / Préparer les enseignants au développement de la pensée algébrique au primaire et au secondaire

2013 *Brock University, St. Catharines, Ontario*

- MOOCs and online mathematics teaching and learning
- Exploring creativity: From the mathematics classroom to the mathematicians' mind / Explorer la créativité : de la classe de mathématiques à l'esprit des mathématiciens
- Mathematics of Planet Earth 2013: Education and communication / Mathématiques de la planète Terre 2013 : formation et communication (K-16)
- What does it mean to understand multiplicative ideas and processes? Designing strategies for teaching and learning
- Mathematics curriculum re-conceptualisation

2014 *University of Alberta, Edmonton, Alberta*

- Mathematical habits of mind / Modes de pensée mathématiques
- Formative assessment in mathematics: Developing understandings, sharing practice, and confronting dilemmas
- Texter mathematique / Texting mathematics
- Complex dynamical systems
- Role-playing and script-writing in mathematics education practice and research

2015 *Université de Moncton, Moncton, New Brunswick*

- Task design and problem posing
- Indigenous ways of knowing in mathematics
- Theoretical frameworks in mathematics education research / Les cadres théoriques dans la recherche en didactictique des mathématiques
- Early years teaching, learning and research: Tensions in adult-child interactions around mathematics
- Innovations in tertiary mathematics teaching, learning and research / Innovations au post-secondaire pour l'enseignement, l'apprentissage et la recherche

2016 Queen's University, Kingston, Ontario

- Computational thinking and mathematics curriculum
- Mathematics in teacher education: What, how… and why / Les mathématiques dans la formation des enseignants : quoi, comment… et pourquoi
- Problem solving: Definition, role, and pedagogy / Résolution de problèmes : définition, rôle, et pédagogie associée
- Mathematics education and social justice: Learning to meet the others in the classroom / Éducation mathématique et justice sociale : apprendre à rencontrer les autres dans las classe
- Role of spatial reasoning in mathematics
- The public discourse about mathematics and mathematics education / Le discours public sur les mathématiques et l'enseignement des mathématiques

2017 McGill University, Montréal, Québec

- Teaching first year mathematics courses in transition from secondary to tertiary
- L'anxiété mathématique chez les futurs enseignants du primaire : à la recherche de nouvelles réponses à des enjeux qui perdurent / Elementary preservice teachers and mathematics anxiety: Searching for new responses to enduring issues
- Social media and mathematics education
- Quantitative reasoning in the early years / Le raisonnement quantitatif dans les premières années du parcours scolaire
- Social, cultural, historical and philosophical perspectives on tools for mathematics
- Compréhension approfondie des mathématiques scolaires / Deep understanding of school mathematics

2018 Quest University, Squamish, British Columbia

- The 21st century secondary school mathematics classroom
- Confronting colonialism / Affronter le Colonialisme
- Playing with mathematics / Learning mathematics through play
- Robotics in mathematics education
- Relation, ritual and romance: Rethinking interest in mathematics learning

2019 St. Francis Xavier University, Antigonish, Nova Scotia

- Problem-based learning in postsecondary mathematics / L'apprentissage par problems en mathématiques au niveau postsecondaire
- Teaching primary school mathematics…what mathematics? What avenues for teacher training? / Enseigner les premiers concepts mathématiques à l'école primaire…quelles mathématiques? Quelles avenues pour la formation à l'enseighnement?
- Humanizing data / Humaniser les données
- Research and practice: Learning through collaboration / Recherche et pratique : apprendre en collaborant
- Interdisciplinarity with mathematics: Middle school and beyond
  Capturing chaos? Ways into the mathematics classroom / Capturer le chaos ? Entrées sur la classe de mathématiques

2021 Online (Virtual)

- Learning Theories / Théories (de l') apprenant
- Pour ou contre les tests : est-ce la bonne question ? / To test or not to test: Is this the question?
- The rewards and challenges of video in the field of mathematics education: Looking back in order to prepare for the future / Les apports et défis de la vidéo pour (la formation à) l'enseignement-apprentissage des mathématiques : regard du passé pour préparer le futur
- How can we be creative with large classes? / Comment composer avec les grands groupes ?
- Returning to our roots: Exploring collaborative possibilities for research and teaching in mathematics and mathematics education

Appendix B / Annexe B

PLENARY LECTURES AT EACH ANNUAL MEETING / CONFÉRENCES PLÉNIÈRES DES RENCONTRES ANNUELLES

1977	A.J. COLEMAN	The objectives of mathematics education
	C. GAULIN	Innovations in teacher education programmes
	T.E. KIEREN	The state of research in mathematics education
1978	G.R. RISING	The mathematician's contribution to curriculum development
	A.I. WEINZWEIG	The mathematician's contribution to pedagogy
1979	J. AGASSI	The Lakatosian revolution
	J.A. EASLEY	Formal and informal research methods and the cultural status of school mathematics
1980	C. GATTEGNO	Reflections on forty years of thinking about the teaching of mathematics
	D. HAWKINS	Understanding understanding mathematics
1981	K. IVERSON	Mathematics and computers
	J. KILPATRICK	The reasonable effectiveness of research in mathematics education
1982	P.J. DAVIS	Towards a philosophy of computation
	G. VERGNAUD	Cognitive and developmental psychology and research in mathematics education
1983	S.I. BROWN	The nature of problem generation and the mathematics curriculum
	P.J. HILTON	The nature of mathematics today and implications for mathematics teaching
1984	A.J. BISHOP	The social construction of meaning: A significant development for mathematics education?
	L. HENKIN	Linguistic aspects of mathematics and mathematics instruction
1985	H. BAUERSFELD	Contributions to a fundamental theory of mathematics learning and teaching
	H.O. POLLAK	On the relation between the applications of mathematics and the teaching of mathematics
1986	R. FINNEY	Professional applications of undergraduate mathematics
	A.H. SCHOENFELD	Confessions of an accidental theorist
1987	P. NESHER	Formulating instructional theory: the role of students' misconceptions
	H.S. WILF	The calculator with a college education

1988	C. KEITEL	Mathematics education and technology
	L.A. STEEN	All one system
1989	N. BALACHEFF	Teaching mathematical proof: The relevance and complexity of a social approach
	D. SCHATTSNEIDER	Geometry is alive and well
1990	U. D’AMBROSIO	Values in mathematics education
	A. SIERPINSKA	On understanding mathematics
1991	J .J. KAPUT	Mathematics and technology: Multiple visions of multiple futures
	C. LABORDE	Approches théoriques et méthodologiques des recherches françaises en didactique des mathématiques
1992	ICME-7	
1993	G.G. JOSEPH	What is a square root? A study of geometrical representation in different mathematical traditions
	J CONFREY	Forging a revised theory of intellectual development: Piaget, Vygotsky and beyond
1994	A. SFARD	Understanding = Doing + Seeing ?
	K. DEVLIN	Mathematics for the twenty-first century
1995	M. ARTIGUE	The role of epistemological analysis in a didactic approach to the phenomenon of mathematics learning and teaching
	K. MILLETT	Teaching and making certain it counts
1996	C. HOYLES	Beyond the classroom: The curriculum as a key factor in students’ approaches to proof
	D. HENDERSON	Alive mathematical reasoning
1997	R. BORASSI	What does it really mean to teach mathematics through inquiry?
	P. TAYLOR	The high school math curriculum
	T. KIEREN	Triple embodiment: Studies of mathematical understanding-in-interaction in my work and in the work of CMESG/GCEDM
1998	J. MASON	Structure of attention in teaching mathematics
	K. HEINRICH	Communicating mathematics or mathematics storytelling
1999	J. BORWEIN	The impact of technology on the doing of mathematics
	W. WHITELEY	The decline and rise of geometry in 20^{th} century North America
	W. LANGFORD	Industrial mathematics for the 21^{st} century
	J. ADLER	Learning to understand mathematics teacher development and change: Researching resource availability and use in the context of formalised INSET in South Africa
	B. BARTON	An archaeology of mathematical concepts: Sifting languages for mathematical meanings
2000	G. LABELLE	Manipulating combinatorial structures
	M. B. BUSSI	The theoretical dimension of mathematics: A challenge for didacticians

2001	O. SKOVSMOSE	Mathematics in action: A challenge for social theorising
	C. ROUSSEAU	Mathematics, a living discipline within science and technology
2002	D. BALL & H. BASS	Toward a practice-based theory of mathematical knowledge for teaching
	J. BORWEIN	The experimental mathematician: The pleasure of discovery and the role of proof
2003	T. ARCHIBALD	Using history of mathematics in the classroom: Prospects and problems
	A. SIERPINSKA	Research in mathematics education through a keyhole
2004	C. MARGOLINAS	La situation du professeur et les connaissances en jeu au cours de l'activité mathématique en classe
	N. BOULEAU	La personnalité d'Evariste Galois: le contexte psychologique d'un goût prononcé pour les mathématique abstraites
2005	S. LERMAN	Learning as developing identity in the mathematics classroom
	J. TAYLOR	Soap bubbles and crystals
2006	B. JAWORSKI	Developmental research in mathematics teaching and learning: Developing learning communities based on inquiry and design
	E. DOOLITTLE	Mathematics as medicine
2007	R. NÚÑEZ	Understanding abstraction in mathematics education: Meaning, language, gesture, and the human brain
	T. C. STEVENS	Mathematics departments, new faculty, and the future of collegiate mathematics
2008	A. DJEBBAR	Art, culture et mathématiques en pays d'Islam (IX^{e}-XV^{e} s.)
	A. WATSON	Adolescent learning and secondary mathematics
2009	M. BORBA	Humans-with-media and the production of mathematical knowledge in online environments
	G. de VRIES	Mathematical biology: A case study in interdisciplinarity
2010	W. BYERS	Ambiguity and mathematical thinking
	M. CIVIL	Learning from and with parents: Resources for equity in mathematics education
	B. HODGSON	Collaboration et échanges internationaux en éduction mathématique dans le cadre de la CIEM : regards selon une perspective canadienne / ICMI as a space for international collaboration and exchange in mathematics education: Some views from a Canadian perspective
	S. DAWSON	My journey across, through, over, and around academia: "...a path laid while walking..."
2011	C. K. PALMER	Pattern composition: Beyond the basics
	P. TSAMIR & D. TIROSH	The Pair-Dialogue approach in mathematics teacher education
2012	P. GERDES	Old and new mathematical ideas from Africa: Challenges for reflection
	M. WALSHAW	Towards an understanding of ethical practical action in mathematics education: Insights from contemporary inquiries
	W. HIGGINSON	Cooda, wooda, didda, shooda: Time series reflections on CMESG/GCEDM

2013	R. LEIKIN	On the relationships between mathematical creativity, excellence and giftedness
	B. RALPH	Are we teaching Roman numerals in a digital age?
	E. MULLER	Through a CMESG looking glass
2014	D. HEWITT	The economic use of time and effort in the teaching and learning of mathematics
	N. NIGAM	Mathematics in industry, mathematics in the classroom: Analogy and metaphor
2015	É. RODITI	Diversité, variabilité et convergence des pratiques enseignantes / Diversity, variability, and commonalities among teaching practices
	D. HUGHES HALLET	Connections: Mathematical, interdisciplinary, electronic, and personal
2016	B. R. HODGSON	Apport des mathématiciens à la formation des enseignants du primaire : regards sur le « modèle Laval »
	C. KIERAN	Task design in mathematics education: Frameworks and exemplars
	E. MULLER	A third pillar of scientific inquiry of complex systems—Some implications for mathematics education in Canada
	P. TAYLOR	Structure—An allegory
2017	Y. SAINT-AUBIN	The most unglamorous job of all: Writing exercises
	A. SELDEN	40+ years of teaching and thinking about university mathematics students, proofs, and proving: An abbreviated academic memoir
2018	D. VIOLETTE	Et si on enseignait la passion?
	M. GOOS	Making connections across disciplinary boundaries in preservice mathematics teacher education
2019	J-M. DE KONINCK	Découvrir les mathématiques ensemble avec les étudiants
	R. GUTIERREZ	Mathematics as dispossession: Reclaiming mental sovereignty by living mathematx
2021	S. MAYES-TANG	Teaching on empty: Trauma, achievement, and what's next in our math education community

Appendix C / Annexe C

PROCEEDINGS OF ANNUAL MEETINGS / ACTES DES RENCONTRES ANNUELLES

Past proceedings of CMESG/GCEDM annual meetings have been deposited in the ERIC documentation system with call numbers as follows:

Proceedings of the 1980 Annual Meeting . . . . ED 204120

Proceedings of the 1981 Annual Meeting . . . . ED 234988

Proceedings of the 1982 Annual Meeting . . . . ED 234989

Proceedings of the 1983 Annual Meeting . . . . ED 243653

Proceedings of the 1984 Annual Meeting . . . . ED 257640

Proceedings of the 1985 Annual Meeting . . . . ED 277573

Proceedings of the 1986 Annual Meeting . . . . ED 297966

Proceedings of the 1987 Annual Meeting . . . . ED 295842

Proceedings of the 1988 Annual Meeting . . . . ED 306259

Proceedings of the 1989 Annual Meeting . . . . ED 319606

Proceedings of the 1990 Annual Meeting . . . . ED 344746

Proceedings of the 1991 Annual Meeting . . . . ED 350161

Proceedings of the 1993 Annual Meeting . . . . ED 407243

Proceedings of the 1994 Annual Meeting . . . . ED 407242

Proceedings of the 1995 Annual Meeting . . . . ED 407241

Proceedings of the 1996 Annual Meeting . . . . ED 425054

Proceedings of the 1997 Annual Meeting . . . . ED 423116

Proceedings of the 1998 Annual Meeting . . . . ED 431624

Proceedings of the 1999 Annual Meeting . . . . ED 445894

Proceedings of the 2000 Annual Meeting . . . . ED 472094

Proceedings of the 2001 Annual Meeting . . . . ED 472091

Proceedings of the 2002 Annual Meeting . . . . ED 529557

Proceedings of the 2003 Annual Meeting . . . . ED 529558

Proceedings of the 2004 Annual Meeting . . . . ED 529563

Proceedings of the 2005 Annual Meeting . . . . ED 529560

Proceedings of the 2006 Annual Meeting . . . . ED 529562

Proceedings of the 2007 Annual Meeting . . . . ED 529556

Proceedings of the 2008 Annual Meeting . . . . ED 529561

Proceedings of the 2009 Annual Meeting . . . . ED 529559

Proceedings of the 2010 Annual Meeting . . . . ED 529564

Proceedings of the 2011 Annual Meeting . . . . ED 547245

Proceedings of the 2012 Annual Meeting . . . . ED 547246

Proceedings of the 2013 Annual Meeting . . . . ED 547247

Proceedings of the 2014 Annual Meeting . . . . ED 581042

Proceedings of the 2015 Annual Meeting . . . . ED 581044

Proceedings of the 2016 Annual Meeting . . . . ED 581045

Proceedings of the 2017 Annual Meeting . . . . ED 589990

Proceedings of the 2018 Annual Meeting . . . . ED 595075

Proceedings of the 2019 Annual Meeting . . . . ED 610111

NOTES

There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year. There was no Annual Meeting in 2020 due to COVID-19.