

CANADIAN MATHEMATICS EDUCATION
STUDY GROUP

GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE
DES MATHÉMATIQUES

PROCEEDINGS / ACTES
2023 ANNUAL MEETING /
RENCONTRE ANNUELLE 2023



University of Regina
Regina, Saskatchewan
June 7 – 11, 2023

EDITED BY:
Jennifer Holm, *Wilfrid Laurier University*
Charlotte Megroureche, *Manchester Metropolitan University*

*Proceedings of the 2023 Annual Meeting of the
Canadian Mathematics Education Study Group /
Groupe Canadien d'Étude en Didactique des Mathématiques*
are published by CMESG/GCEDM.
Published in August 2025.

ISBN: 978-1-7771235-4-3

**PROCEEDINGS OF THE 2023 ANNUAL MEETING OF THE CANADIAN
MATHEMATICS EDUCATION STUDY GROUP / ACTES DE LA RENCONTRE
ANNUELLE 2023 DU GROUPE CANADIEN D'ÉTUDE EN DIDACTIQUE DES
MATHÉMATIQUES**

46th Annual Meeting
University of Regina
Regina, Saskatchewan
June 7 – 11, 2023

CONTENTS / TABLES DES MATIÈRES

LISA LUNNEY BORDEN	vii	<i>Introduction</i>
	ix	<i>Schedule / Horaire</i>

**PLENARY LECTURES / CONFÉRENCES
PLÉNIÈRES**

JENNIFER SUH	5	<i>Cultivating joy, wonder and power through community-based math modeling</i>
MARIEME NGOM	17	<i>Building the foundations of mathematical modeling and machine learning</i>

WORKING GROUPS / GROUPES DE TRAVAIL

ED DOOLITTLE, FLORENCE GLANFIELD, ELDER DR. ELMER GHOSTKEEPER, ELDER BETTY MCKENNA, CYNTHIA NICOL, & JENNIFER S. THOM	25	A • <i>Machi kis kiyih tam uske/nantaw ota: Iteyhtam mas kooch etikwe apekka tam eyiniw misiwe uske kawapahtik ake ihike win ewa kiskino huma kewina (in Bushland Cree) / Learning from/on/with land/place: Imagining possibilities for braiding Indigenous worldviews, mathematics, and teaching / Apprendre avec la terre/le lieu: imaginer des possibilités pour tresser les visions du monde autochtones, les mathématiques et l'enseignement</i>
MANON LEBLANC & ROBYN RUTTENBERG-ROZEN	31	C • <i>Sawubona. I see you. Je te vois.</i>
VINCENT BOUCHARD & ASIA MATTHEWS	39	D • <i>An undergraduate curriculum based on mathematical reasoning skills (wouldn't it be awesome?)</i>
SABRINA HÉROUX & JANELLE MCFEETORS	51	E • <i>Games for mathematical learning / Jeux pour l'apprentissage des mathématiques</i>

LYNN MCGARVEY & EVAN THROOP-ROBINSON	71	F • <i>Where's the math? Inquiring into early years mathematics curriculum / Où sont les mathématiques? Enquêter sur le programme de mathématiques dans les premières années</i>
---	----	--

TOPIC SESSIONS / SÉANCES THÉMATIQUES

NAT BANTING	91	<i>Hot dogs and mathematics education: A career sandwiched between research and practice</i>
ELENA POLOTSKAIA	95	<i>Représenter ou ne pas représenter – est-ce la question? Développement de la pensée mathématique à l'école primaire / Represent or not represent – Is this the question? Mathematical thinking development in elementary school</i>
BERNARDO GALVÃO-SOUSA	101	<i>What is an applied problem? ...a journey through time...</i>
ANNIE SAVARD	111	<i>Teaching financial numeracy in schools: An overview of mathematical practices at play</i>
RINA ZAZKIS	117	<i>Pedagogical encounters and mathematical detours</i>
EDWARD DOOLITTLE	125	<i>Better living through combinatorics</i>

NEW PHD REPORTS / PRÉSENTATIONS DE THÈSES DE DOCTORAT

GENEVIÈVE BARABÉ	129	<i>Étude de l'évolution de tâches mathématiques routinières à travers leur exploitation collective: le cas de la tâche des notes de musique en classe de 6^e année / Study of the evolution of routine mathematical tasks through their collective investigation : the case of the music notes task in a 6th grade classroom</i>
CANAN GÜNEŞ	137	<i>A tri-ple analysis of thinking multiplicatively around/with TouchTimes</i>
SABRINA HÉROUX	145	<i>Exploratory study of mathematical activity during game sessions in the elementary classroom / Étude exploratoire de l'activité mathématique lors de séances de jeux en classe du primaire</i>
JOSH MARKLE	151	<i>Students' embodied experiences of spatial capability</i>
SHEREE RODNEY	159	<i>Embodied curiosity in the mathematics classroom through the affordance of The Geometer's Sketchpad</i>
ZACK WOLSKE	167	<i>Reflective practices in a TA observation program</i>

AD HOC SESSIONS / SÉANCES AD HOC

EGAN J. CHERNOFF	175	<i>The slow death of math teacher journals in Canada: RIP, Journal of the SMTS</i>
TINA RAPKE, MARC HUSBAND, & CRISTINA DE SIMONE	177	<i>Para-imaging: Re-presenting students' mental images</i>
ZACK WOLSKE	179	<i>COD AND BAO; NBC CAT; NOT TBD</i>

MATHEMATICS GALLERY / GALERIE MATHÉMATIQUE

ELLEN CARTER	183	<i>Connecting math to our lives and communities: Measuring our impact</i>
--------------	-----	---

APPENDICES / ANNEXES

187	A • <i>Working Groups at each Annual Meeting</i>
195	B • <i>Plenary Lectures at each Annual Meeting</i>
199	C • <i>Proceedings of Annual Meetings</i>

INTRODUCTION

Dr. Lisa Lunney Borden – President, CMESG/GCEDM
St. Francis Xavier University

Thank you, Regina! We returned to an in-person annual meeting in 2023 following three interrupted years. It was so wonderful to be together again and to catch up with old friends and make new friends. Both our scientific and social programs left our hearts and minds filled. Our plenary speakers both discussed ideas relating to mathematical modelling, Dr. Jennifer Suh addressing this idea from her research with young children, and Dr. Marieme Ngom exploring this idea from her work as a research mathematician. Both talks had me wondering about what we teach in K–12 and post-secondary mathematics and whether we are still teaching the kinds of mathematics children need to be learning. When plenaries challenge the very nature of mathematics education, I believe that makes for a great conference. Our Working Groups also dug deeply into their respective topics and the working group summaries, as always, made me wish I could have been in all of these sessions. We were also inspired by some amazing topic session speakers and five new PhDs. We are grateful to have their ideas and words documented in these proceedings.

I want to thank the executive who have worked so hard over the past year to make this meeting happen. Securing speakers and working group leaders is a complex and often challenging task. Our executive spends many hours throughout the year ensuring everything is in place for this event. We were pleased to see the re-elections of Limin Jao as member-at-large and Alayne Armstrong as treasurer. They work hard for our team. Both Edward Doolittle, member-at-large, and I were continuing in our roles this past year. Additionally, coopted members Jhonel Morvan and Jeanne Koudogbo, rounded out our team. Together this team managed the scientific program while our social program was organized by the amazing local organizing committee of Kathy Nolan, Gale Russell, Edward Doolittle, Alayne Armstrong, and their team of helpers for such a well-organized conference. We particularly note Alayne and Edward doing double duty on both the executive and the Local Organizing Committee.

Following our meeting this year both Alexandre Cavalcante and Sabrina Héroux offered to take over the roles as newsletter editors, replacing the outgoing Sarah DuFour and Robyn Ruttenberg-Rozen who did wonderful work over the past four years. This is such an important role that allows us all to stay connected and well-informed in between meetings.

Thank you to the University of Regina, the Faculty of Education (U of R), First Nations University, and PIMS for their generous support of this meeting.

Finally, I want to thank all members of our wonderful CMESG community who came and gathered again and shared ideas, laughter, joy and even a few tears. Our community is strong because we truly come together to learn and laugh together.

Merci Regina ! Après trois ans d'interruption, nous sommes revenus à une réunion annuelle en personne en 2023. C'était merveilleux de se retrouver, de revoir de vieux amis et de s'en faire de nouveaux. Nos programmes scientifiques et sociaux ont rempli nos cœurs et nos esprits. Nos conférenciers en séance plénière ont tous deux abordé des idées liées à la modélisation mathématique. La Dre Jennifer Suh abordant cette idée à partir de ses recherches avec les jeunes enfants et la Dre Marieme Ngom explorant cette idée à partir de son travail en tant que mathématicienne chercheuse. Ces deux exposés m'ont amené à m'interroger sur ce que nous enseignons en mathématiques de la petite enfance jusqu'au postsecondaire et à me demander si nous enseignons toujours le type de mathématiques que les enfants devraient apprendre. Lorsque les plénières remettent en question la nature même de l'enseignement des mathématiques, je pense que ceci est signe d'une excellente conférence. Nos groupes de travail ont également

approfondi leurs sujets respectifs et les résumés des groupes de travail, comme toujours, m'ont fait regretter de ne pas avoir pu participer à toutes ces sessions. Nous avons été inspirés par des intervenants extraordinaires lors des sessions thématiques et par cinq nouveaux docteurs. Nous sommes reconnaissants pour leurs idées et leurs paroles documentées dans ces actes.

Je tiens à remercier les membres de l'exécutif qui ont travaillé très fort au cours de l'année écoulée pour que cette réunion puisse avoir lieu. La recherche de conférenciers et d'animateurs de groupes de travail est une tâche complexe et souvent difficile. Notre comité exécutif consacre de nombreuses heures tout au long de l'année pour s'assurer que tout soit en place pour cet événement. Nous avons été heureux de voir les réélections de Limin Jao en tant que membre actif et d'Alayne Armstrong en tant que trésorière. Elles travaillent fort pour notre équipe. Edward Doolittle, membre général, et moi-même avons continué à assumer nos fonctions l'année dernière. De plus, les membres cooptés Jhonel Morvan et Jeanne Koudogbo ont complété notre équipe. Ensemble, cette équipe a pris en charge le programme scientifique. Le programme social a quant à lui été organisé par le formidable comité d'organisation local composé de Kathy Nolan, Gale Russell, Edward Doolittle, Alayne Armstrong, et par leur équipe d'assistants qui ont permis une conférence si bien organisée. Nous soulignons tout particulièrement la double participation d'Alayne et d'Edward au comité exécutif et au comité d'organisation local.

Suite à notre réunion de cette année, Alexandre Cavalcante et Sabrina Héroux ont proposé de prendre en charge la rédaction du bulletin d'information, en remplacement de Sarah Dufour et Robyn Ruttenberg-Rozen, qui ont accompli un travail remarquable au cours des quatre dernières années. Il s'agit d'un rôle très important qui nous permet à tous de rester en contact et bien informés entre les réunions.

Merci à l'Université de Regina, à la Faculté d'éducation (U of R), à l'Université des Premières Nations et PIMS pour leur soutien généreux pour cette réunion.

Enfin, je tiens à remercier tous les membres de notre merveilleuse communauté GCEDM qui sont venus et qui se sont réunis à nouveau pour partager des idées, des rires, des joies et même quelques larmes. Notre communauté est forte parce que nous nous réunissons pour apprendre et rire ensemble.

Thank you / Merci,

Lisa Lunney Borden
President CMESG / Présidente du GCEDM



Horaire

Mercredi 7 juin	Jeudi 8 juin	Vendredi 9 juin	Samedi 10 juin	Dimanche 11 juin
	8h45-10h15 Groupe de travail	8h45-10h15 Groupe de travail	8h45-10h15 Groupe de travail	8h45-9h30 Temps pour préparer la présentation de « clôture »
	10h15-10h45 Pause	10h15-10h45 Pause	10h15-10h45 Pause	9h45-10h30 Ad hoc
	10h45-12h15 Groupe de travail	10h45-12h15 Groupe de travail	10h45-12h15 Groupe de travail	10h30-11h00 Pause
				11h15-12h30 Clôture
12h00-15h30 FLM réunion des directeurs	12h30-13h45 Dîner	12h30-13h30 Dîner	12h30-13h45 Dîner	
	13h45-14h15 Petits groupes	13h00-14h00 Dessert, café et galerie mathématique	13h45-14h15 Petits groupes	
	14h25-15h25 Discussion de la plénière I	14h00-15h00 Plénière II	14h25-15h25 Discussion de la plénière II	
		15h00-19h00 Excursions		
14h30-18h45 Inscription	15h30-16h15 Séance thématique		15h30-16h00 Pause	
	16h15-16h45 Pause		16h00-16h45 Séance thématique	
	16h45-17h25 Thèses		16h45-17h15 Ad hoc / FLM Q&A	
17h00-18h30 Souper	17h30-18h45 Assemblée générale annuelle			
18h30-19h30 Ouverture GCEDM	18h45- Souper libre	19h00- Souper	18h00-22h00 Souper Social / Danse	
19h30-20h30 Plénière I				
20h30-22h00 Réception				

Schedule

Wednesday June 7	Thursday June 8	Friday June 9	Saturday June 10	Sunday June 11
	8:45-10:15 Working Groups	8:45-10:15 Working Groups	8:45-10:15 Working Groups	8:45-9:30 Time to Organize 'Closing'
	10:15-10:45 Break	10:15-10:45 Break	10:15-10:45 Break	9:45-10:30 Ad hoc
	10:45-12:15 Working Groups	10:45-12:15 Working Groups	10:45-12:15 Working Groups	10:30-11:00 Break
				11:15-12:30 Closing
12:00-15:30 FLM Meeting of the Directors	12:30-13:45 Lunch	12:30-13:30 Lunch	12:30-13:45 Lunch	
	13:45-14:15 Small Groups	13h00-14h00 Dessert, coffee & math gallery	13:45-14:15 Small Group	
	14:25-15:25 Plenary I Discussion	14:00-15:00 Plenary II	14:25-15:25 Plenary II Discussion	
14:30-18:45 Registration	15:30-16:15 Topic Session	15:00-19:00 Excursions	15:30-16:00 Break	
	16:15-16:45 Break		16:00-16:45 Topic Session	
	16:45-17:25 PhD		16:45-17:15 Ad hoc / FLM Q&A	
17:00-18:30 Dinner	17:30-18:45 Annual General Meeting			
18:30-19:30 CMESG Opening	18:45- Dinner on your own		19:00- Dinner	
19:30-20:30 Plenary I				
20:30-22:00 Reception				

Plenary Lectures

Conférences plénières

CULTIVATING JOY, WONDER AND POWER THROUGH COMMUNITY-BASED MATH MODELING

Jennifer M. Suh, PhD
George Mason University

ABSTRACT

This plenary talk detailed a process for designing community based mathematical modeling tasks where mathematics educators and teachers collaborated on co-creating tasks. Community-based Mathematical Modeling has the power to catalyze change by focusing on building awareness of social issues and using mathematics to take action. Educating the hearts and minds of our youth is at the core of our work as mathematics educators. A critical skill in mathematical modeling involves posing mathematical problems in authentic real-life contexts that are relatable to students' interests, knowledge, and skills that enable students to use mathematics to help make decisions, predict, and determine meaningful solutions to the problem. In this plenary talk, I share about a collaborative project where we focus on advancing equity and strengthening teaching of mathematical modeling using culturally responsive mathematical practices. By noticing students' mathematical strengths and assigning competence, teachers positioned students as mathematicians. This plenary talk invites the Canadian Mathematics Education Study Group to engage in dialogue with one another in cultivating joy, wonder and power through Community-based Mathematical Modeling.

RELATED RESEARCH

WHAT IS COMMUNITY-BASED MATH MODELING?

Mathematical Modeling (MM) involves posing mathematical problems in authentic real-life contexts that are relatable to students' interests, knowledge, and skills that enable students to use mathematics to help make decisions, predict, and determine meaningful solutions to the problem (Garfunkel & Montgomery, 2019). Mathematical Modeling is part of the school mathematics curriculum in a growing number of countries. In the United States, making mathematical models is a Standard for Mathematical Practice and specific modeling standards appear throughout high school as part of the Common Core State Standards (Common Core State Standards Initiative [CCSSI], 2010). The Guideline for Assessment and Instruction in Mathematical Modeling Education (GAIMME) reports (Consortium of Mathematics and Its Applications, Bedford, MA & Society for Industrial and Applied Mathematics [COMAP & SIAM], 2016) that mathematical modeling should be taught at every stage of a student's mathematical education and can be used to motivate curricular requirements and to highlight the importance and relevance of mathematics in answering important questions in the real world.

International communities of mathematics educators have been keenly interested in the many perspectives of mathematics modeling. For example, in the recent International Psychology of Math Education Plenary, scholars addressed three strands of research in math education: a) pedagogically oriented research perspective, where Blum et al. (2022) explored teaching approaches that promote math modeling competencies; b) socioculturally-oriented research perspective, where Yang et al. (2022) reported on ways to include ethnomathematics in modeling; and c) psychological-oriented research, where Schukajlow et al. (2022) examined the role of affective aspects and Lu et al. (2022) explored the influence of creativity on math modeling.

For this plenary talk, I highlight a collaborative project called EQSTEMM: Advancing Equity and Strengthening Teaching through Mathematical Modeling, where my team used a pedagogic approach called Community based Mathematical Modeling (CBMM). In this paper, I will use “we” to reflect the important collaboration among the other principal investigators on this project as well as the many teachers and coaches that collaborated on task design. This CBMM approach builds on the work of Anhalt et al. (2018), Aguirre et al. (2019) and Cirillo et al. (2016) introducing mathematical modeling as a vehicle to engage students in rigorous mathematics while bringing awareness of social issues in their community. CBMM is situated at the intersection between Teaching Math for Social Justice (TMfSJ) and Math Modeling (Cirillo et al., 2016) and has the potential to build critical consciousness in both teachers and students (Anhalt et al, 2018). In introducing CBMM to elementary teacher designers, we also focus keenly on a socio-critical perspective (Barbosa, 2006) by building awareness of social issues in students’ local communities by identifying a situation in the real world that students can connect with and that can be solved with mathematics. For CBMM, there is a critical component where the mathematical modeling and analysis leads students to using mathematics to make decisions and take action for change (Suh, Tate, et al., 2023).

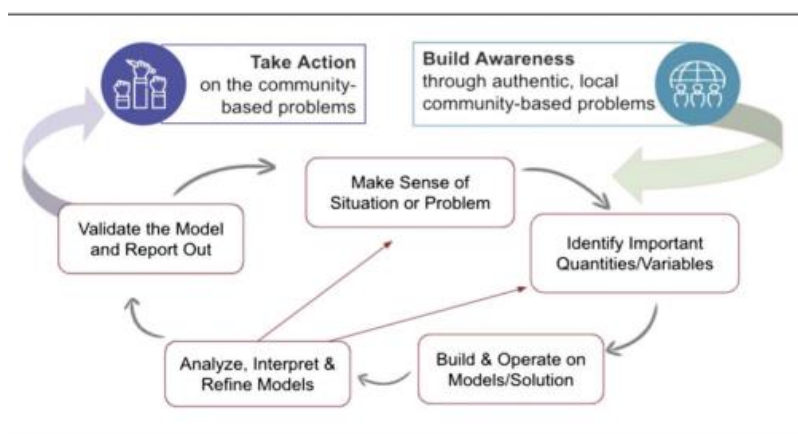


Figure 1. CBMM cycle.

Mathematics educators have emphasized the importance of TMfSJ as an approach to teaching mathematics that supports students in analyzing power dynamics and challenging injustices as they learn to read and rewrite their world with mathematics (Aguirre et al., 2019; Berry et al., 2020; Gutstein, 2006). TMfSJ is an ambitious practice which places importance on a specialized knowledge required for mathematizing socio-political issues. More specifically, as Felton et al. (2017) noted, “Designing and implementing sociopolitical mathematics lessons requires an understanding of both the sociopolitical and mathematical knowledge that a task is likely to evoke” (p. 452).

In the United States, the Standards for Preparing Mathematics Teachers developed by the Association of Mathematics Teacher Educators (AMTE, 2017, Standard C.2.1) places emphasis on preparing teachers of mathematics with equity-oriented practices where teachers “use mathematics to address problems and issues in their homes and communities” (p. 13), and yet not many practicing teachers have engaged in learning mathematics through this approach. Teachers need to experience learning mathematics through TMfSJ lessons (Gutstein, 2006) and to see mathematics as a human endeavor and develop critical consciousness (Anhalt et al., 2018). Other mathematics professional organizations in the United States including TODOS and NCSM (2016) call for a dual commitment to mathematics and social justice by “elevating the professional learning of mathematics teachers and leaders” (p.10). Community based mathematical modeling exemplifies a teaching approach that provides students and teachers a powerful way mathematics can be an analytical tool to understand complex situations and take action (Gustein, 2006). CBMM is important and authentic, yet challenging to design and ambitious to enact, and we need professional development support for teachers to integrate CBMM into their practice.

One of the critical steps to co-designing Community-based Math Modeling Tasks focused on access, fairness, representation, and justice is working with teachers to consider local community problems that are relevant to the students. This requires knowing deeply about the local contexts and issues to build students’ awareness and consciousness about contextual factors and that have *transformative potential* (Jemal, 2017) for students to take action on inequitable conditions to initiate positive change. An important caveat is that teachers do not simply take an activity

out of a book and implement it in their classrooms without first doing the self-reflective work. This kind of self-reflective work takes time for teachers and is more effectively done in a collective professional learning community and alongside standards that guide the mathematics and social justice connections (Learning for Justice, 2022).

DESIGN OF THE STUDY

CENTERING CULTURALLY RESPONSIVE MATH TEACHING IN CBMM

Teaching mathematics attending to both social justice and math modeling requires mathematics teacher educators to support teachers in this ambitious teaching. Our project called EQSTEMM has used the Culturally Responsive Mathematics Teaching framework (Zavala & Aguirre, 2023, Figure 2) involving a set of specific pedagogical knowledge, dispositions, and practices that privilege mathematics, mathematical thinking, cultural and linguistic funds of knowledge, and issues of power and social justice in mathematics education.

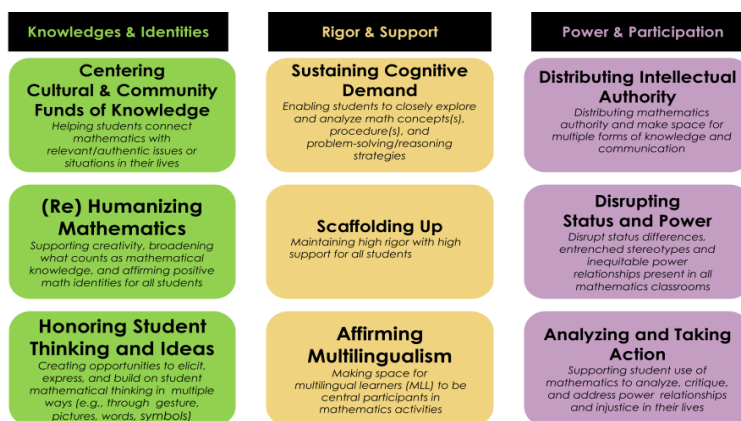


Figure 2. Culturally responsive math teaching framework (Zavala & Aguirre, 2023).

Zavala and Aguirre's (2023) comprehensive framework, Culturally Responsive Mathematics Teaching (CRMT), is theoretically grounded in the intersection of culturally responsive pedagogy (Ladson-Billings, 1995), pedagogical content knowledge, and humanizing mathematics. Table 1 shows the three key areas of CRMT: knowledge and identities; rigor and support; and power and participation, along with essential questions for teachers to integrate into their instruction. The first area centers children and families with attention to cultural/community funds of knowledge, rehumanizes mathematics with attention to curiosity, and explicitly focuses on children's mathematical thinking. The second area attends to cognitive demand, ways to scaffold for students to maximize access and sustained engagement, and affirms multilingualism by highlighting ways to uplift children learning mathematics in more than one language. The third area illustrates how intellectual authority and participation can be broadened, works to dismantle destructive stereotypes that pervades all mathematics classrooms, and provides opportunities to take action to make positive change in their communities.

GUIDING OUR WORK WITH DESIGN PRINCIPLES FOR CBMM

The design-based principles for CBMM, generated through an equity-focused professional learning project, illustrate how mathematics must go beyond traditional mathematics tasks often void of humanizing practices to mathematical tasks that elicit cultural and community knowledge and experiences, address power dynamics, and broaden student participation and what counts as mathematics (Civil, 2007; Turner et al., 2021). My current project, EQSTEMM: Advancing Equity by Strengthening Teaching through Math Modeling, with colleagues across three other regions of the United States, has demonstrated that CBMM tasks provide transformative humanizing math learning spaces that emphasize community care, equitable participation, and deep mathematical understanding (Suh et al., 2018; Turner et al., 2021; Turner et al., 2023).

To attend to the dual focus on mathematics and social justice (Gutiérrez, 2002, 2013; Kokka, 2022;) and support teachers in designing authentic tasks with math teacher educators (MTE), my colleagues and I shared five design

principles for CBMM tasks (Suh, Tate, et al., 2023). These design principles offered our Lesson Study teams a structure to support the co-construction design process focused on social justice and are included in Table 1.

Table 1. Design Principles for CBMM (Suh, Tate, et al., 2023).

Design Principles for CBMM	Prompts for Designers
<u>Design Principle #1:</u> <i>CBMM tasks are situated in authentic local community issues.</i> Together the teacher and students pose a mathematical modeling question that impacts the local community	How do you build meaningful connections and context focused on social justice issues?
<u>Design Principle #2:</u> <i>CBMM tasks explore data to identify and understand social issues.</i> Data related to the situation help students use mathematics to better understand the phenomenon or social issues.	How can mathematics be used to unearth factors related to social justice issues?
<u>Design Principles #3:</u> <i>CBMM tasks use rigorous mathematics to describe, predict, optimize, and make decisions about a situation centering issues of social justice.</i>	How do you sustain the rigor of the mathematics as you explore issues of social justice?
<u>Design Principles #4:</u> <i>CBMM tasks encourage the co-construction of ideas through collaboration, negotiation, and justification to build collective knowledge through community.</i>	How do you attend to important mathematics processes and practices as students engage and make meaning of the issue?
<u>Design Principles #5:</u> <i>CBMM tasks yield useful and solution-oriented action for community stakeholders.</i> Students feel empowered by bringing awareness, taking actions, and attaching the learning experience with civic empathy.	How do you move beyond awareness to taking action on social justice issues?

Research Questions

Through these co-designed lessons, my colleagues and I explored the research questions:

1. How do the three areas of CRMT (knowledge and identity, rigor and support, and power and participation) reflect in the enactment of a CBMM?
2. How did the design principles support teachers in co-designing justice-oriented CBMM tasks?

Setting and Participants

This study is part of a broader research and professional development program focused on culturally responsive mathematical modeling (MM) in the elementary grades. Teachers participated in a year-long, hybrid professional development program that included monthly in-person sessions and asynchronous activities to deepen learning. In-person sessions, facilitated by the authors, introduced frameworks for CRMT and included time to explore modeling tasks and routines, collaboratively plan activities, and reflect on classroom enactments. Asynchronous materials included readings, videos of modeling lessons, and reflection prompts.

This study focused on 18 elementary school teachers with students from kindergarten through 5th grade (ages 5 through 11). The teachers participated in our professional development program at one of four research sites in different regions of the United States throughout the Mid-Atlantic. The teachers were from two school sites from this Mid-Atlantic region that served racially and linguistically diverse students from underserved communities.

Description of the Case Studies

This plenary talk focused on three case studies of CBMM tasks. One of the tasks was called Diversifying the Library, which was implemented by a first-grade teacher who co-designed a CBMM task that investigated cultural and racial diversity of books in the class library and whether the library collection was fair. Another task, the inclusive

playground, was generated by a team of teachers whose school was going through renovation. The task focused on understanding what inclusive meant and how they could use a budget to plan for an inclusive playground in their existing school yard. The third task was called The Sports Clinic (Suh, Maxwell, et al., 2023) and was inspired by a teacher who lived in the community and experienced a lack of access to sports for her daughter and realized that the students in the community did not have access to fields and sports. Table 2 includes the tasks with the focus of the mathematical modeling; justice in terms of access, representation, diversity; and the team of teachers engaged in the Lesson Study.

Table 2. Description of CBMM Task.

CBMM Task	Diversifying Library	Inclusive Playground	Sports Clinic
Math Modeling	Descriptive modeling from class taking inventory of diversity representation in books to make decisions on book purchases to make the library “more fair”	Surveying students on accessibility of equipment. Budgeting and designing the optimal play area for an inclusive playground	Statistical investigation to understand the problem situation through survey and make decisions on which sports to offer at the sports clinic. Planning logistics of running a sports clinic after school.
Justice-focused	Representation and diversity of characters, authors, and illustrators by people of color	Access and Universal Design for handicapped and physically challenged students to enjoy playground	Access to sports clinics for student in elementary grades who may not have access to learning sports offered in high school
LS Teams with facilitators	Grade 1 teacher and grade 6 teacher from School 1 and three kindergarten teachers in School 2	Five Grade 3 teachers from School 1 and two Grade 3 and two Grade 4 teachers from School 2	Grade 3, 4, 5, 6 teachers with school-based coach at School 1

Data sources

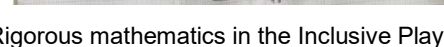
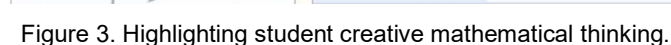
Our data source included transcripts from planning sessions and end-of-year interviews with teacher participants. Each teacher participated in an hour-long individual interview with project researchers after completing professional development activities. Interview topics included teachers’ perceptions of MM and connections to the CRMT framework; experiences implementing modeling lessons, student learning, and supports; and challenges for teaching modeling. Secondary data sources included reflective memos after each session and teacher journal prompts about/for how MM activities enacted in their classrooms supported components of CRMT.

RESULTS

HOW THE THREE AREAS OF CRMT WERE REFLECTED IN THE ENACTMENT OF THE TASKS

Through our study, we found community-based math modeling empowered the teacher to leverage equity, empathy and mathematical rigor in the early grades. The case study demonstrated how teachers can design lessons that help children make sense of complex social topics and advance their mathematical reasoning strategies. Children connected to an authentic situation that impacted their own community, they investigated possible inequities, and mathematized fairness which deepened their critical mathematical agency (Suh, Maxwell, et al., 2023) and civic empathy toward others now and in the future (Lee et al., 2021)

To engage in justice-oriented task design, we first invited our teachers in the self-reflective work of asking how they experienced or witnessed injustices and if they had discuss topics related to Access, Fairness, Representation and Justice. We engaged in a discussion around how mathematics could be included justice-oriented mathematics tasks. Most of our collection of community-inspired tasks that were codesigned with teachers generally fell into three broad categories including 1) school-based issues and events; 2) environment justice issues; and 3) historical and current social justice issues in their community.

1. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ 2. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ 3. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

10

As a mathematical modeling task, students posed the problem: what sports should we offer in our sports clinic and how can we run this successfully? This led to many mathematical modeling pathways including polling students in their community on their favorite sports and the sport they want to play in high school. Using the decision matrix, students created criteria to make a decision. The class considered different plans and logistics and ways they could take action and change the structural barriers that caused this inequity. They came up with an idea of offering an afterschool sports clinic where parent coaches or college graduates would teach basic skills that are needed for high school sports like lacrosse, baseball, and volleyball and wrote a letter to the school board shown in Figure 5.

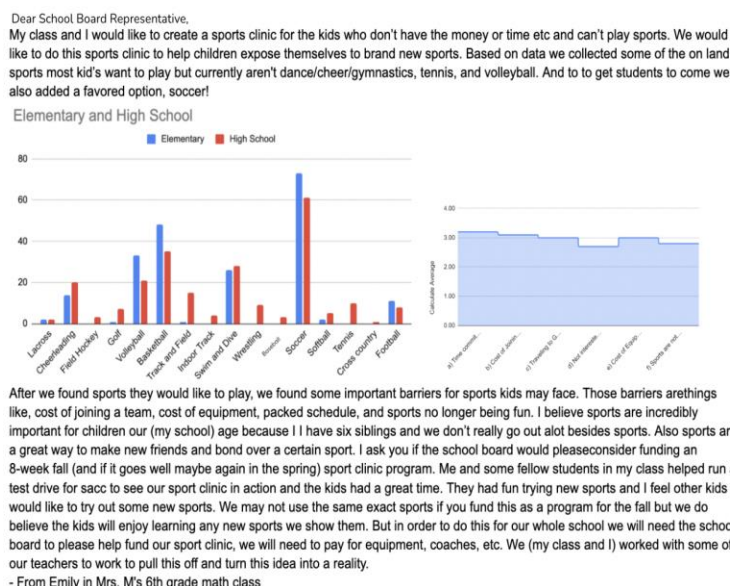


Figure 5. Taking action by writing a letter to the School Board.

HOW THE DESIGN PRINCIPLES SUPPORTED TEACHERS IN CODESIGNING JUSTICE ORIENTED TASKS

For Research question 2, we examined how the five design principles supported teachers in their design and enactment of the CBMM.

In terms of Design Principle #1, *CBMM tasks are situated in authentic local community issues*. Together the teacher and students pose a mathematical modeling question that impacts the local community. All three tasks were motivating for the students because the problem was situated in their community. One of the teachers recounted how codesigning the Sports Clinic task was important to her because she lives in the school community and her children attended elementary, middle, and high school in the neighborhood. She witnessed this inequity firsthand as a parent and felt that students could feel empowered by this issue.

In terms of Design Principle #2, *CBMM tasks explore data to identify and understand social issues*. Data related to the situation help students use mathematics to better understand the phenomenon or social issues. As a facilitator, it was important for the professional development facilitator to support teachers in finding data that allowed teachers to dig in deeper with the social issues. Tasks that are social justice oriented unveil factors from multiple perspectives. For example, using a [report on access to youth sports](#) with disparities across household income and race/ethnicity, then understanding the many facets of barriers to access was important for teachers to understand. With this knowledge, teachers felt more equipped to have critical conversations about issues that were systemic and ways to take actions to address them.

In terms of Design Principles #3, *CBMM tasks use rigorous mathematics to describe, predict, optimize, and make decisions about a situation centering issues of social justice*. The rigorous mathematics that played a role in each of the CBMM tasks differed and were appropriate to the grade levels in which the task was implemented. For example, for the Diversifying the Library, the first graders worked on optimizing the purchase of \$20 to buy books that would

make their library ‘more fair.’ In the Inclusive Playground, students in 3rd and 4th grade worked on budget as well as mapping out the equipment within a given area. Finally in the Sports Clinic, students used a decision matrix to weigh in on important criteria for determining which sports to offer. As part of the consideration, they needed to make sense of and analyze the survey data they collected from their peers in Grades 3–6. Each of these tasks presented messiness in the real-world data, but because the task was authentic and relevant to students, they persevered and used all the mathematics skills they possessed.

In terms of Design Principles #4, *CBMM tasks encourage the co-construction of ideas through collaboration, negotiation, and justification to build collective knowledge through community.* An important phase of the modeling cycle after building the model is analyzing and interpreting and refining their model. In each of the CBMM tasks, teachers allowed small groups to present their model and receive feedback from peers. In the case of the Inclusive Playground, students had to describe how their playground was truly ‘inclusive.’ Some of the justification and design thinking demonstrated a level of empathy that was impressive. Students justified why they placed certain inclusive equipment and gave a rationale that they did not want to just purchase equipment to add to their existing playground—not just integrated—but to make it really inclusive so that children can play ‘side by side.’



Figure 6. Justification for the Inclusive Playground.

In terms of Design Principles #5, *CBMM tasks yield useful and solution-oriented action for community stakeholders.* Students feel empowered by bringing awareness and taking actions and attaching the learning experience with civic empathy. This design principle took the PD facilitators and teachers a significant amount of time to consider. Teachers engaged in deep conversation to list the ways we can impact change that goes beyond charity to bringing some structural changes. For the Diversifying Library task, this discussion led us to thinking about how getting book donations of books with diverse characters would improve the situations but would be more impactful if we worked with the school librarian to take stock of current books and examine book purchasing practices to diversity books. Additionally, we could start a campaign to write to book publishers to take inventory of their book collection and encourage authors and illustrators of color to write more books to diversify literature.

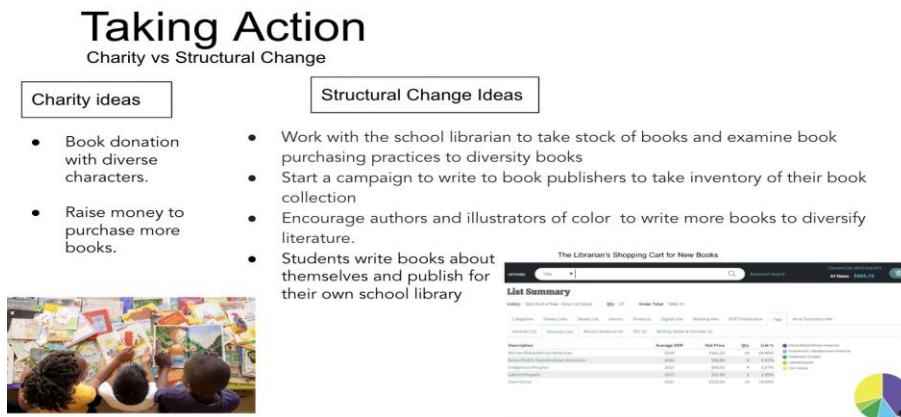


Figure 7. Taking action to go beyond charity to promoting structural change.

CONCLUSIONS AND IMPLICATIONS FOR MATH TEACHER EDUCATORS AND RESEARCHERS

In summary, the three cases of Community-based Math Modeling tasks cultivated joy, wonder, and power in teachers and students. One teacher shared, “*Students will be able to show what is important to them, as well as see that they can make a difference in their communities and in our world. They CAN make a Difference!*” Another shared, “*I would like to think that this could be the start of a visibility campaign and bring about real change in all schools, but disappointment is a part of life too. I will demonstrate resilience by encouraging students to not give up!*” In this way, we heard our teachers’ radical hope, a commitment to the belief that circumstances can eventually improve even in the midst of oppression (French et. al., 2020).

Some important implication for mathematics teacher educators is coming to this work by engaging in community building and trust with teacher co-designers. Teachers come with great assets as being community informants who not only know more about the families, community and students but also come with lived experiences that play in important role in planning and teaching a justice-oriented task. Working with young children around topics of social justice also brought a level of sensitivity and care. The team of teachers working on the Diversifying the Library had deep conversations with one another about talking about race and culture. As a team we talked about the notion of multidimensional identities:

During this community task it was challenging to decide how to approach the conversation of identity and fairness. My team wondered what was most appropriate for our grade level. Ultimately, we decided to broaden the idea of identity beyond race and ethnicity and that allowed all students to access the task from many perspectives.

CBMM mathematical modeling has potential to exemplify a practice that aligns with the four recommendations from National Council of Teachers of Mathematics (NCTM, 2020) and act as a catalyst for change that mathematics educators should consider in early elementary mathematics education.

In reference to *broadening the purposes of learning mathematics* (NCTM, 2020), students who engaged in mathematical modeling experienced the wonder and joy of solving a problem to improve their environment, which broadens the purpose of learning mathematics. They felt a sense of agency as they worked on improving their school community problem. Mathematical modeling aims to provide students with community-based tasks that they can connect to personally and make meaning of the mathematics within the world they are living in.

In reference to *creating equitable structures in mathematics* (NCTM, 2020), CBMM focuses on the tenets of Culturally Responsive Math Teaching Framework (CRMT, Zavala & Aguirre, 2023), immersed in the task with peers who complemented one another with the multiple knowledge bases that each student brought. The task piqued students’ interest because it was set in their community and generated collaboration and meaningful discourse, while students designed their solution. The classroom communities built through mathematical modeling embodied high expectations, and empowered students as the holders of mathematical knowledge.

In reference to *implementing equitable mathematics instruction* (NCTM, 2020), teachers created participation structures with partner talk, carousel walks, “give one, get one,” and sentence stems, to nurture students’ positive mathematical identities and a strong sense of agency. By having thought partners with their peers, students had a mathematical community behind them, as they dared to take risks, challenge each other’s ideas, and persevere in finding a solution. The mathematical modeling structure poses the teacher with the crucial role of facilitating the learning that is owned by the students. As teachers work in the background to further push the groups of students in their thinking, every student’s mathematical ideas take the forefront of the instruction, as they help their peers tackle a meaningful challenge.

Finally, in reference to *developing deep mathematical understanding* (NCTM, 2020), the elementary students used the mathematics that they learned to describe the plan, take action and tackle complex issues in their community and rigorous mathematics. As showcased in this case study, the realistic and messy nature of the modeling task set in the real world allows students to encounter mathematical ideas many times beyond the grade level objective. Ultimately, students are enticed to engage in more rigorous mathematics. In our project, we heard students say, “Teach us the math so we can solve this problem!”, the empowered disposition that we want to cultivate in every student.

In closing, I would like to share a quote from Gholdy Muhammad (2023):

Criticality asks students to evaluate and dismantle systems of oppression while building sociopolitical consciousness. It allows them to think outside of themselves and consider how to love others in humanity. (p. 11)

Community based math modeling allowed teachers to cultivate mathematical doers and thinkers who could apply their learning and problem solving to question and examine multiple perspectives on issues in their local community to develop their voice and become young change agents.

ACKNOWLEDGEMENTS

The work reported here is supported by a grant from the National Science Foundation (EQSTEMM; Advancing Equity and Strengthening Teaching through Mathematical Modeling www.eqstemm.org Grants, 2010269, 2008997, 2010202, 2010178). We would like to thank all our courageous educators, children and families who continue to impress upon us the importance of making our world a more just and humanizing place.

REFERENCES

- Aguirre, J. M., Anhalt, C., Cortez, R., Turner, E. E., & Simi-Muller, K., (2019). Engaging teachers in the powerful combination of mathematical modeling and social justice. *Mathematics Teacher Educator*, 7(2), 7–26.
- Anhalt, C., Staats, S., Cortez, R. & Civil, M., (2018). Mathematical modeling and culturally relevant pedagogy. In Y. J. Dori, Z. Mevarech, & D. Baker (Eds.), *Cognition, Metacognition, and Culture in STEM Education* (pp. 307–330). Springer: the language of science, Publishing Editor Education. http://dx.doi.org/10.1007/978-3-319-66659-4_14
- Association of Mathematics Teacher Educators. (2017). *Standards for preparing teachers of mathematics*. <http://www.amte.net/standards>
- Barbosa, J. C. (2006) Mathematical modelling in classroom: a socio-critical and discursive perspective. *Zentralblatt für Didaktik der Mathematik*, 38, 293–301. <https://doi.org/10.1007/BF0265281>
- Berry, R. Q., III, Conway, B. M., IV, Lawler, B. R., & Staley, J. W. (2020). *High school mathematics lessons to explore, understand, and respond to social injustice*. Corwin.
- Blum, W., Berta Barquero, B., & Durandt, R. (2022). Innovative research approaches for exploring teaching environments designed to promote mathematical modeling competency In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 149–152). PME.
- Cirillo, M., Bartell, T. G., & Wager, A. (2016). Teaching mathematics for social justice through mathematical modeling. In C. Hirsch & A. Roth McDuffie (Eds.), *Annual perspectives in mathematics education: Mathematical modeling and modeling with mathematics* (pp. 87–96). National Council of Teachers of Mathematics.
- Civil, M. (2007). Building on community knowledge: An avenue to equity in mathematics education. In N. Nasir & P. Cobb (Eds.), *Improving access to mathematics: Diversity and equity in the classroom* (pp. 105–117). Teachers College Press.
- Common Core State Standards Initiative (CCSSI). (2010). *Common Core State Standards for Mathematics*. National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf
- Consortium of Mathematics and Its Applications (COMAP) & Society for Industrial and Applied Mathematics (SIAM). (2016). *Guidelines for Assessment and Instruction in Mathematical Modeling Education (GAIMME, 1st ed.)*. COMAP, Inc. & SIAM. <http://www.siam.org/reports/gaimme.php>
- Felton-Koestler, M. D., Simic-Muller, K., & Menéndez, J. M. (2017). *Reflecting the world: A guide to incorporating equity in mathematics teacher education*. Information Age Publishing

- French, B. H., Lewis, J. A., Mosley, D. V., Adames, H. Y., Chavez-Dueñas, N. Y., Chen, G. A., & Neville, H. A. (2020). Toward a psychological framework of radical healing in communities of color. *The Counseling Psychologist*, 48(1), 14–46. <https://doi.org/10.1177/0011000019843506>
- Garfunkel, S., & Montgomery, M. (2019). *Guidelines for Assessment and Instruction in Mathematical Modeling Education (GAIMME)*. Consortium of Mathematics and Its Applications (COMAP) and Society for Industrial and Applied Mathematics (SIAM). <http://www.siam.org/reports/gaimme.php>
- Gutiérrez, R. (2002). Enabling the practice of mathematics teachers in context: Towards a new equity research agenda. *Mathematical Thinking and Learning*, 4(2&3), 145–187.
DOI: https://doi.org/10.1207/S15327833MTL04023_4
- Gutiérrez, R. (2013). The sociopolitical turn in mathematics education. *Journal for Research in Mathematics Education*, 44(1), 37–68. <https://doi.org/10.5951/jresmetheduc.44.1.0037>
- Gutstein, E. (2006). *Reading and writing the world with mathematics: Toward a pedagogy for social justice*. Routledge. <https://doi.org/10.4324/9780203112946>
- Jemal, A. (2017). Critical consciousness: A critique and critical analysis of the literature. *The Urban Review*, 49(4), 602–626.
- Kokka, K. (2022). Toward a theory of affective pedagogical goals for social justice mathematics. *Journal for Research in Mathematics Education*, 53(2), 133–153. <https://doi.org/10.5951/jresmetheduc-2020-0270>
- Lanson Billing, G. (1995). Toward a theory of culturally relevant pedagogy. *American Educational Research Journal*, 32(3), 465–491.
- Learning for Justice. (2022). *Social justice standards: The learning for justice anti-bias framework*. <https://www.learningforjustice.org/>
- Lee, C. D., White, G., & Dong, D. (Eds.). (2021). *Educating for civic reasoning and discourse*. Committee on Civic Reasoning and Discourse. National Academy of Education.
- Lu, X., Kaiser, G., & Leikin, R. (2022). The influence of creativity on mathematical modeling and its role within mathematical modeling activities. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 162–165). PME.
- Muhammad, G. (2023). On criticality. *Voices from the Middle*, 30(4), 12–14.
- National Council of Teachers of Mathematics (NCTM). (2020). *Catalyzing change in early childhood and elementary mathematics*. NCTM.
- Schukajlow, S., Krawitz, J., & Carreira, S. (2022). The influence and role of affective aspect in mathematical modeling. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 158–162). PME.
- Suh, J. M., Burke, L., Britton, K., Matson, K., Ferguson, L., Jamieson, S., & Seshaiyer, P. (2018). Every penny counts: Promoting community engagement to engage students in mathematical modeling. In R. Gutierrez & I. Goffney (Eds.), *Annual perspectives in mathematics education: Rehumanizing mathematics for students who are Black, Indigenous, and/or Latin@* (pp. 63–78). National Council of Teachers of Mathematics.
- Suh, J. M., Maxwell, G., Roscioli, K., Tate, H., Seshaiyer, P., & Martinen, R. (2023). Young mathematicians take action through sport clinics. *Mathematics Teacher: Learning and Teaching PK-12*, 116(11), 845–855.
- Suh, J. M., Tate, H., Rossbach, M., Green, S., Matson, K., Aguirre, J., Seshaiyer, P., & Steen, S. (2023). Dilemmas and design principles in planning for justice-oriented community-based mathematical modeling lessons. *Mathematics Teacher Educator*, 11(3), 210–230. <https://doi.org/10.5951/MTE.2022-0025>
- Turner, E., Roth McDuffie, A., Aguirre, J., Foote, M. Q., Chapelle, C., Bennett, A., Granillo, M., & Ponnuru, N. (2021). Upcycling plastic bags to make jump ropes: Elementary students leverage experiences and

- funds of knowledge as they engage in a relevant, community-oriented mathematical modeling task. In J. Suh, M. Wickstram, & L. English (Eds.), *Exploring mathematical modeling with young learners* (pp. 235–266). Springer. doi:10.1007/978-3-030-63900-6_11
- Turner, E., Suh, J. M., Tate, H., Soltelo Ocampo, D., Carlson, M., Aguirre, J., & Fulton, E. (2023). Cultivating equity and empathy in community focused elementary math modeling. In M. Strutchens, G. Krause, D. Y. White, & J. Bay-Williams (Eds.), *Antiracist mathematics education: Stories of acknowledgement, action, and accountability* (pp. 121–132). TODOS Mathematics for All.
- TODOS and NCSM. (2016). Mathematics education through the lens of social justice: Acknowledgment, actions, and accountability. *Social Justice Position Statement*. <https://www.todos-math.org/socialjustice>
- Yang, X., Schwarz, B., & Rosa, M. (2022). Cultural and socio cultural influences on the implementation of mathematical modeling education and consequences for mathematical modeling education. In C. Fernández, S. Llinares, A. Gutiérrez, & N. Planas (Eds.), *Proceedings of the 45th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 155–158). PME.
- Zavala, M. R., & Aguirre, J. M. (2023) *Cultivating mathematical hearts: Culturally responsive math teaching in elementary classrooms*. Corwin.

BUILDING THE FOUNDATIONS OF MATHEMATICAL MODELING AND MACHINE LEARNING

A PREPRINT

Marieme Ngom

Argonne Leadership Computing Facility Argonne National Laboratory

ABSTRACT

Mathematical sciences are widely used in a plethora of industry and business applications such as healthcare, finance, or meteorology. With the rise of machine learning, artificial intelligence, and big data analytics, novel and innovative use of mathematics are being established. Early exposure to the concepts of computational mathematics can help students develop the skills to solve real-world problems and better understand the mathematical principles that underlie our modern world. Moreover, learning to create and analyze mathematical models at a young age fosters critical thinking and analytical skills that can be applied across various disciplines beyond STEM. This talk will explore the importance of introducing mathematical modeling in the classroom at a young age. We will provide practical, easy to grasp examples of how educators can incorporate these concepts in their curricula.

INTRODUCTION

The aim of this report is to explore the role of mathematical modeling and machine learning in modern education from the point of view of an applied mathematician. We show how introducing fundamental concepts from these areas early can empower future generation in a data-driven world. We will first give a complete definition of mathematical modeling and machine learning, focusing in particular on the intrinsic relation between the two. Then, we will discuss how important and relevant it is to introduce fundamental concepts of mathematical modeling, programming and machine learning in the lower grades. Finally, we will provide hands-on, easy to follow exercises educators can implement or adapt in their classrooms.

MATHEMATICAL MODELING

DEFINITION OF MATHEMATICAL MODELING

The Consortium for Mathematics & its Applications (COMAP) and the Society for Industrial and Applied Mathematics (SIAM) defines mathematical modeling as “is a process that uses mathematics to represent, analyze, make predictions or otherwise provide insight into real-world phenomena” in the guidelines for assessment and instruction in mathematical modeling education (GAIMME; Consortium for Mathematics and Its Applications & Society for Industrial and Applied Mathematics, 2016, p. 8). This mathematical representation typically consists of equations and functions that capture the essential relationships between the real-world system’s component. The resulting models serve as powerful tools that are used in various disciplines from physics and engineering to economics and social sciences. For instance, mathematical modeling played a central role in the response against the recent Covid global pandemic. Mathematical models were not only used to predict the spread of the virus (Diop et al., 2020) within a population but also to measure the effectiveness of the vaccines and predict Covid variants (Zvyagin et al., 2022).

Another application where mathematical modeling is widely used is in Numerical Weather Prediction (NWP). In this field, the physical laws governing the atmospheric processes (pressure, temperature, etc.) are represented with mathematical models which are used to predict weather conditions. The different steps involved in setting and solving a mathematical model are illustrated in Figure 1.

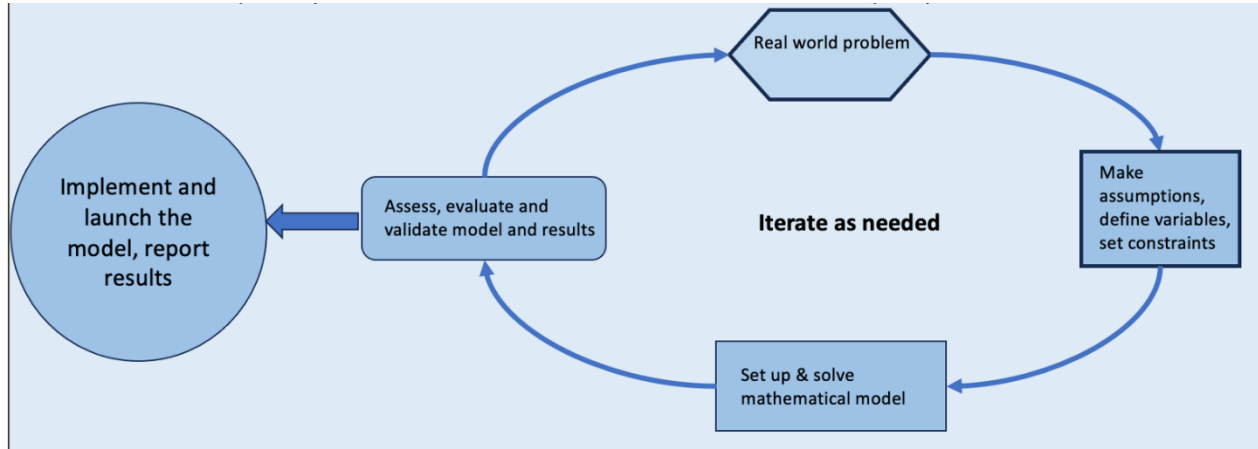


Figure 1. Chart describing the different steps of a mathematical model.

CHALLENGES AND COMPUTATIONAL SOLUTIONS

In many real-world situations, mathematical models are too complex to be solved exactly. In NWP for example, the governing equations, the famous Navier-Stokes equations, cannot be solved analytically. Additionally, measuring all relevant quantities within a model's domain can be impractical. Consider the NWP example again, it would not be feasible to measure every atmospheric process at every point. To circumvent this, a technique called discretization is often used. We can represent the system (e.g., the atmosphere) as a grid, sampling a finite number N of points for numerical approximations and inference. These approximations, often achieved through numerical methods like finite differences (equation to follow) which is often used to approximate the derivative of a function f in a grid of points $x_i, i = 1 \dots N$

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h}$$

Where $h = x_{i+1} - x_i$ is the discretization step, allow us to translate the model into a form suitable for computer calculations.

Computers play a crucial role in this process. They automate calculations, provide speed and efficiency, and enable visualization of complex models. High-performance computing facilities like the Argonne Leadership Computing Facility host several supercomputers such as Aurora and Polaris that possess the processing power and memory capacity to handle massive datasets, facilitating large-scale numerical simulations.

In essence, mathematical modeling relies heavily on two components in practice: numerical approximations and computer-aided numerical simulations. When developing these computer programs, considerations like efficiency, scalability, accuracy, and reproducibility are paramount.

MACHINE LEARNING

Machine learning (ML) is a subfield of artificial intelligence (AI) that allows computers to learn from data without explicit programming. ML algorithms can identify patterns, make predictions, and improve their performance over time as they are exposed to more data. Figure 2 illustrates the different steps involved in a ML workflow. This makes them valuable tools for analyzing complex datasets and extracting meaningful insights. The applications of machine learning are vast and ubiquitous in our daily lives. Recommendation systems on e-commerce platforms and streaming services leverage machine learning to suggest products or content tailored to user preferences. Similarly, weather

forecasting models are increasingly incorporating machine learning techniques to improve prediction accuracy by learning from historical data patterns (Nguyen et al., 2023). ML-powered software programs are transforming the educational landscape. Software like Carnegie Learning's LiveLab (n.d.) leverage machine learning algorithms to tailor learning materials and difficulty levels based on each student's performance and progress. This personalized approach fosters inclusivity by catering to diverse learning styles and paces. Furthermore, machine learning can be used to identify students at risk of falling behind, allowing educators to provide timely interventions and support.

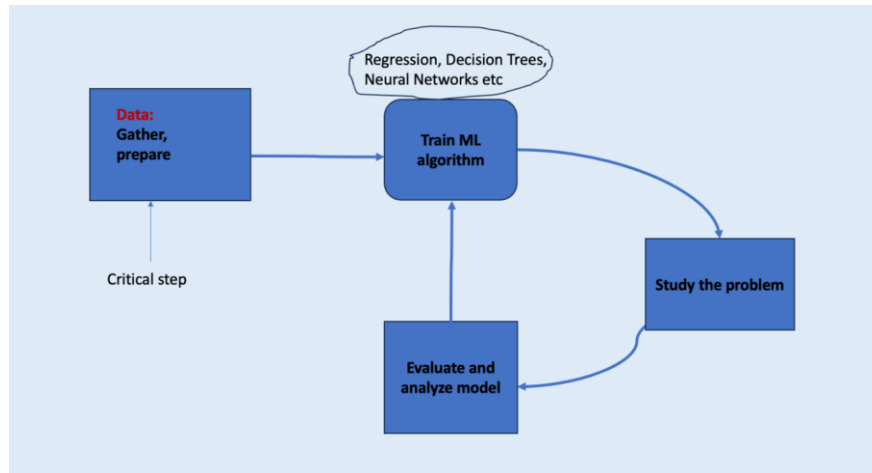


Figure 2. Machine learning procedure.

Beyond direct student interaction, machine learning offers valuable tools for educators themselves. Machine learning-aided research can assist in analyzing student data to identify potential biases or representation issues within educational materials, such as children's books. This data-driven approach allows educators to curate more inclusive learning environments that cater to the diverse needs of their students.

Two among the most common ML approaches are

- **Supervised learning:** In this approach, algorithms are trained on labeled datasets where each data point has a corresponding label or category. These labeled examples serve as a guide, allowing the algorithm to learn the underlying relationships between features (data points) and labels. Once trained, the algorithm can then classify new, unseen data points based on the learned patterns.
- **Unsupervised learning:** In contrast to supervised learning, unsupervised learning deals with unlabeled data. Here, the objective of the algorithm is to uncover inherent structures or patterns within the data itself. Clustering algorithms, a common type of unsupervised learning, group similar data points together without any predefined labels.

THE INTERSECTION OF MATHEMATICAL MODELING AND MACHINE LEARNING

As opposed to mathematical modeling, machine learning does not require a clear and explicit understanding of the real-world problem of interest. In fact, it only needs data to learn properties and make predictions. However, in practice, mathematical modeling and ML are intrinsically related.

For instance, in climate modeling, complex mathematical models represent the physical processes governing the Earth's climate system. Machine learning can then be used to analyze vast datasets of climate observations and identify subtle patterns or trends that may not be readily apparent in the traditional model. This combined approach can lead to more accurate climate predictions.

Additionally, linear algebra, probability and statistics methods, differential and integral calculus are fundamental in defining and solving a ML algorithm and in using it to make predictions.

By introducing both mathematical modeling and machine learning concepts early on, we teach students the necessary skills for understanding and interacting with the data-rich world around them. The next section of this report will explore practical, classroom-ready exercises that can introduce these concepts in an engaging manner.

HOW TO INTRODUCE MATHEMATICAL MODELING AND MACHINE LEARNING CONCEPTS EARLY?

To equip students with tools that are becoming increasingly crucial in every professional field, it is important to introduce many of the concepts discussed earlier at an early stage. This will build a strong mathematical foundation that will benefit them in higher grades and everyday life. Fundamental concepts such as derivatives and finite differences can be effortlessly incorporated into an elementary school curriculum. For example, students can record a quantity periodically and observe its change. Grids can be introduced using garden grids with different flowers/plants or floor tiles. Sampling methods can be easily demonstrated by placing candies of different colors in a jar. Students can then draw a sample of candies and categorize them by color.

Several resources can be used to introduce programming in the classroom, even for lower grades. This can be initially done without a screen by utilizing educational robots. Students could, for example, try to program a robot to move from point A to point B. This activity can be complexified by asking students to find the shortest path. Younger students (K–2) can be introduced to important computer science concepts like algorithmic thinking (e.g., by having them create a step-by-step process for everyday activities like baking) or hardware components by showing them a computer and having them build a model out of cardboard boxes.

Machine learning concepts can also be effortlessly taught early on. Students can learn to analyze data, identify patterns, and make predictions. For example, K-2 students could record the weather conditions (sunny, cloudy, rainy, or snowy) over a specific period, then count and label each occurrence. In grades 3-5, students could record daily temperatures, plot them on an xy-plane, and analyze the relationship between temperature fluctuations and rain or snow. They could then attempt to predict future weather events (e.g., the next rain) using the collected data (this will, of course, require a sufficient amount of data).

CONCLUSION

Early exposure to mathematics modeling fosters engagement with mathematics, critical thinking and problem-solving. As we progress further into the digital age, familiarity with AI/ML concepts from a young age will prepare students for a future where these skills could be crucial. The goal is not to push every child to become a future mathematician, computer scientist or ML specialist but to equip them with the skills and understanding to navigate a rapidly evolving data-driven world.

ACKNOWLEDGMENTS

This work was supported by the U.S. Department of Energy, Office of Science, Office of Advanced Scientific Computing Research, Scientific Discovery through Advanced Computing (SciDAC) program and the FASTMath Institute under Contract No. DE-AC02-06CH11357 at Argonne National Laboratory. Government License.

This research used resources of the Argonne Leadership Computing Facility, a U.S. Department of Energy (DOE) Office of Science user facility at Argonne National Laboratory and is based on research supported by the U.S. DOE Office of Science-Advanced Scientific Computing Research Program, under Contract No. DE-AC02-06CH11357.

Government License. The submitted manuscript has been created by UChicago Argonne, LLC, Operator of Argonne National Laboratory (“Argonne”). Argonne, a U.S. Department of Energy Office of Science laboratory, is operated under Contract No. DE-AC02-06CH11357. The U.S. Government retains for itself, and others acting on its behalf, a paid-up nonexclusive, irrevocable worldwide license in said article to reproduce, prepare derivative works, distribute copies to the public, and perform publicly and display publicly, by or on behalf of the Government. The Department of Energy will provide public access to these results of federally sponsored research in accordance with the DOE Public Access Plan. <http://energy.gov/downloads/doe-public-access-plan>.

REFERENCES

- Carnegie Learning. (n.d.). *LiveLab* [computer program]. Carnegie Learning, Inc.
- Consortium for Mathematics and Its Applications & Society for Industrial and Applied Mathematics (2016). *Guidelines for assessment & instruction in mathematical modeling education* (1st ed.). COMAP, Inc. & SIAM.
- Diop, B. Z., Ngom, M., Biyong, C. P., & Biyong, J. P. (2020). The relatively young and rural population may limit the spread and severity of covid-19 in africa: a modelling study. *BMJ Global Health*, 5(5). doi: 10.1136/bmjgh-2020-002699
- Nguyen, T., Brandstetter, J., Kapoor, A., Gupta, J. K., & Grover, A. (2023). ClimaX: A foundation model for weather and climate. *International Conference on Machine Learning*.
<https://doi.org/10.48550/arXiv.2301.10343>
- Zvyagin, M., Brace, A., Hippe, K., Deng, Y., Zhang, B., Bohorquez, C. O., Clyde, A., Kale, B., Perez-Rivera, D., Ma, H., Mann, C. M., Irvin, M., Pauloski, J. G., Ward, L., Hayot, V., Emani, M., Foreman, S., Xie, Z., Lin, D., ... Ramanathan, A. (2022). GenSLMS: Genome-scale language models reveal SARS-CoV-2 evolutionary dynamics. *bioRxiv*. doi: <https://doi.org/10.1101/2022.10.10.511571>

Working Groups



Groupes de travail

**MACHI KIS KIYIH TAM USKE/NANTAW OTA: ITEYIHTAM MAS KOOCH ETIKWE
APEHKA TAM EYINIW MISIWE USKE KAWAPAHTIK AKE IHIKE WIN EWA
KISKINO HUMA KEWINA (IN BUSHLAND CREE)**

**LEARNING FROM/ON/WITH LAND/PLACE: IMAGINING POSSIBILITIES FOR
BRAIDING INDIGENOUS WORLDVIEWS, MATHEMATICS, AND TEACHING**

**APPRENDRE AVEC LA TERRE/LE LIEU : IMAGINER DES POSSIBILITÉS POUR
TRESSER LES VISIONS DU MONDE AUTOCHTONES, LES MATHÉMATIQUES ET
L'ENSEIGNEMENT**

Ed Doolittle, *First Nations University of Canada*
Florence Glanfield, *University of Alberta*
Elder Dr. Elmer Ghostkeeper, *University of Alberta*
Elder Betty McKenna, *First Nations University of Canada*
Cynthia Nicol, *University of British Columbia*
Jennifer S. Thom, *University of Victoria*

COLLABORATORS

Lisa Lunney Borden
Layne Burns
Brody Causley
Natasha Davidson
Kirsten Dyck
Ed Doolittle
Lisa Eberharter
Florence Glanfield
Elmer Ghostkeeper

Shana Graham
Jennifer Holm
Marc Husband
Limin Jao
Eva Knoll
Stephanie La France
Betty McKenna
Richelle Marynowski
Joyce Mgombelo

Cynthia Nicol
Kelly Paton
Tina Rapke
Gale Russell
Garry Strick
Jennifer Suh
Jennifer Thom
Jo Towers

INTRODUCTION

In our Working Group description, we invited the CMESG/GCEDM community to explore relationships with place/land, community, and mathematics education. Keen to explore possibilities for weaving Indigenous perspectives, mathematics, and land, our goal was to create a generative space where ideas of relations; listening to the land; art, stories and storywork; being shaped by mathematics, Indigenous statistics, technology/ies, relationality; patterning; transformation; movement, spirituality; measurement; location; growth; and identity could be explored. The following questions provided some direction:

- How can we re-member or navigate relationships with mathematics, community, land and place?
- How can we unlearn sometimes problematic ways of relating to each other, our students, and mathematics from/on/with land and place?
- What might it mean for mathematics teaching and learning if the notion of ‘learning from/on/with place’ were central?
- Robin Wall Kimmerer (2013) describes land as library. What might this mean for mathematics teaching and learning?
- How can mathematics teaching and learning be informed by Indigenous principles? (e.g., Archibald’s (2008) Indigenous Storywork principles such as responsibility, reverence, respect, and reciprocity, inter-relatedness, synergy and holism; or gratitude, care, compassion; spiritual, physical, emotional, cognitive; 7 grandfather teachings; grandmother pedagogies and/or methodologies)
- How might we restructure our teaching practices around experiences?
- What are ways in which teachings from/on/with the land play a role?

Dans la description de notre groupe de travail, nous avons invité la communauté CMESG/GCEDM à explorer les relations avec le lieu/terre, la communauté et l’enseignement des mathématiques. Notre objectif était de créer un espace génératif où les idées de : relations ; écouter la terre; l’art, les histoires et le storywork; façonner et être façonné par les mathématiques, les statistiques autochtones, la ou les technologies, la relationnalité; modelage; transformation; mouvement, spiritualité; la mesure; emplacement; croissance; et l’identité pourraient être explorées à travers des questions telles que :

3. Comment pouvons-nous nous souvenir ou naviguer dans les relations avec les mathématiques, la communauté, la terre et le lieu ?
4. Comment pouvons-nous désapprendre des manières parfois problématiques de nous relier les uns aux autres, à nos élèves et aux mathématiques à partir/sur/avec la terre et le lieu ?
5. Qu’est-ce que cela signifierait pour l’enseignement et l’apprentissage des mathématiques si la notion « d’apprendre depuis/sur/avec le lieu » était centrale ?
6. Robin Wall Kimmerer (2013) décrit la terre comme une bibliothèque. Qu’est-ce que cela pourrait signifier pour l’enseignement et l’apprentissage des mathématiques ?
7. Comment l’enseignement et l’apprentissage des mathématiques peuvent-ils être éclairés par les principes autochtones ? (par exemple, les principes de narration d’Archibald (2008) tels que la responsabilité, la révérence, le respect et la réciprocité, l’interdépendance, la synergie et l’holisme; ou la gratitude, l’attention, la compassion; spirituel, physique, émotionnel, cognitif; 7 enseignements de grand-père; pédagogies et/ou méthodologies de grand-mère)
8. Comment restructurer nos pratiques pédagogiques autour des expériences?
9. De quelles façons les enseignements de/sur/avec la terre jouent-ils un rôle?

While all other Working Groups for CMESG/GCEDM 2023 met at the University of Regina, our Working Group came together at First Nations University of Canada, a 10–15 minute walk from the University of Regina buildings that hosted CMESG/GCEDM. With our focus on mathematics, land/place, and Indigenous perspectives, worldviews, and knowledge systems, the stunning and powerful architecture and lands of First Nations University of Canada became the perfect space for our Working Group discussions and activities. Each of our meeting days incorporated a slightly longer break so that participants could join CMESG/GCEDM members from other Working Groups during break time. Although the extended break shortened our time for the Working Group, we found that the walking time between institutions provided increased opportunities for members to continue discussing working group ideas as well as deepen relationships with each other.

DAY 1: RELATIONSHIP BUILDING & ELDER TEACHINGS

A key focus of the working group was relationships. To prepare us for (re)imagining our relationships with mathematics and place/land we, collectively, dedicated the first day to relationship building. Guided by Indigenous pedagogies we formed a circle (pushing tables to the classroom walls and bringing chairs to the centre) so that everyone could contribute to the discussion, share ideas, and respectfully listen to each other. Elder Betty began with a prayer and welcome inviting participants to the lands of Treaty 4 and Treaty 6 territories. Elder Betty asked participants to feel and acknowledge the land where their feet are planted and all that needed to happen in the human and more-than-human worlds for each of us to be here. Each member of the circle introduced themselves in relation

to an object, item, or story of significance to them. Elder Betty shared experiences of learning from land/place while Ed, Florence, Cynthia and Jennifer participated in the first podcast of the series [Indigenous Math](#) focused on research for conceptualizing mathematics education for STEM as place.

For Day 1 it was planned that participants take a mathematics walk outside around the lands of First Nations University of Canada and bring what they noticed to our next meeting. Elder Betty, instead, suggested this be an activity for Day 2, as Day 1 focused on building relationships through sharing stories and experiences with mathematics.

DAY 2: MATHEMATICS, MOTHER EARTH, AND LIVING WALK

Elder Dr. Elmer Ghostkeeper joined virtually on Day 2. Elder Elmer shared wisdom around the ways in which Indigenous worldviews invite humans to live with the land rather than living off the land. In the Indigenous languages that Elder Elmer speaks there is a deep relationship to land, and, in Elmer's teachings, languages come from the land.

While we had prepared a seven-page document inviting participants to participate in a 'STEM' walk with different 'lenses', Elder Betty invited us to be humans on the walk without a particular lens. Elder Betty 'knew' that as a collective we were not yet ready to engage in a 'STEM' walk with different 'lenses'; we needed to be quiet and come to be present with all that Mother Earth might teach. Elders 'see' what we need in the present.

Elder Betty asked us to walk with ourselves; to walk quietly so that we could fully listen to Mother Earth using all our senses. We spent an hour walking, collecting images and recordings that spoke to us as we strengthened our relationship with Mother Earth.

We took photos and recordings of our walk and shared these through a Padlet digital platform (Figure 1).

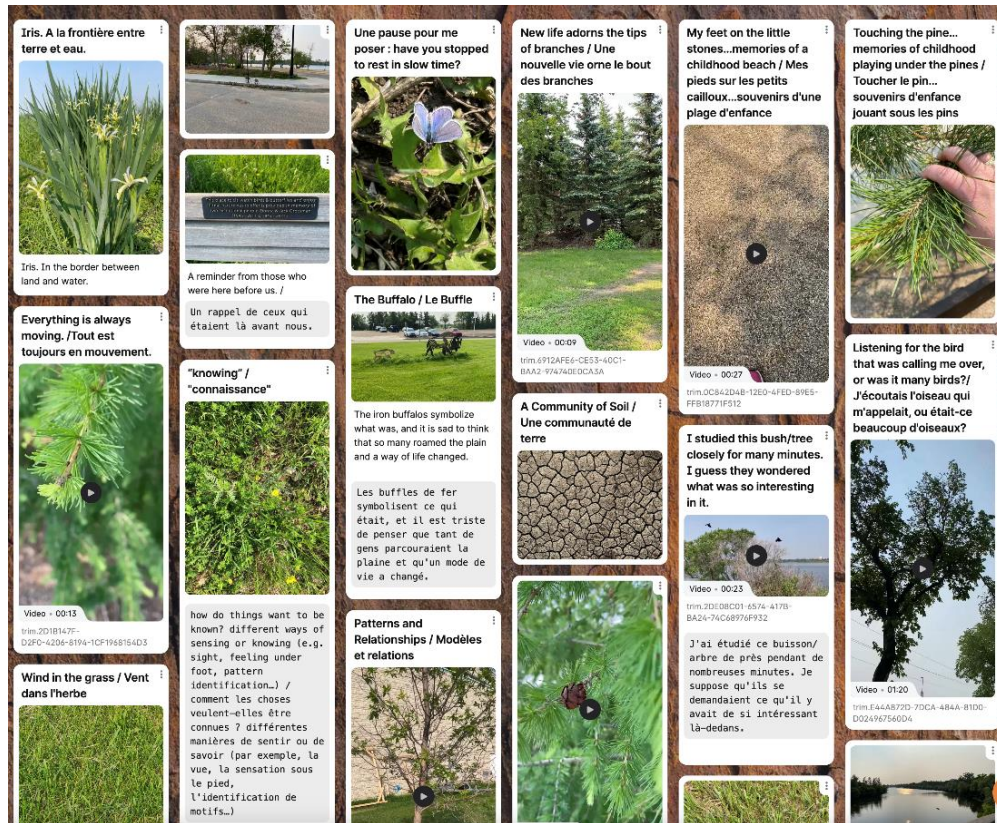


Figure 1. Images and recordings posted by collaborators of their living walk listening to Mother Earth ([Living Walk padlet](#)).

DAY 3: BEING MATHEMATICAL IN RELATION WITH MOTHER EARTH

Elder Betty joined in person and Elder Dr. Elmer joined virtually to listen to the ways in which we each ‘made sense’ of the Day 2’s walk in relation to being mathematical.

Day 3’s circle was shaped by the following questions and statements:

- In what way(s) are we preparing to bring our whole ‘selves’ into a mathematical relationship with Mother Earth and our human and more-than-human kin?
- (Re)membering and navigating relationships with mathematics, community, land and place.
- Un/learning ways of relating with others, including mathematics from/on/with land and place.
- (Re)cognizing land teachings for mathematical education.
- If we are kin with mathematics then what does that mean?

Emotional, physical, mental, and spiritual components of the medicine wheel were present in the three days. What we share with you here as the conclusion to the report is a found poem of the collective comments noted in Day 3 in the circle. No one voice is identified as these ideas emerged in the circle. In our understanding of the circle, ideas shared in the circle become a part of a collective voice.

No one is breathing ahead of us or behind us

Time via Blackfoot—past, future, and present altogether. They are not separate or in sequence but together.

Past isn’t behind us, and future ahead. We are always in the present, in the now.

Scale.

I plan with mathematics that is smaller than me. In the classroom we try to be at the same size. But we also need to be thinking of things that are bigger than us.

My students saved my life.

What is my relationality with my students and ethical obligation?

What does it mean to think of children as past, present, and future?

What does this mean for mathematics?

We aren’t inventing math we are learning to appreciate what Mother Nature is offering,

We’ve been living with math—it offers us tools or objects to think with.

I’m not doing it to colonize but as an offer to Mother Earth,

How are we a part of the world in being mathematical?

Challenge becomes to feel and think with your heart.

The longest journey of your life is from your heart to your head and back again.

It is time to re-think curriculum—it wasn’t meant for Indigenous children.

It was meant for the children of the colonizers.

Take back the classroom and reconnect with nature.

To have kinship with mathematics in kinship relationship.

Learn the sound of the environment.

To have kinship with mathematics in kinship relationship.

RESOURCES AND REFERENCES

We invite you to view/read these resources:

VIDEOS:

Janine Benyus—Biologist and Innovation Consultant. (2022). *On biomimicry* (3:14 min) [video]. YouTube.

https://www.youtube.com/watch?v=M_5Xxy2T3kA

- JC Cahill—University of Alberta Plant Biologist. (2012). *Smarty plants: Uncovering the secret world of plant behaviour* (44 min) [video]. Curio. <https://curio.ca/en/video/smarty-plants-uncovering-the-secret-world-of-plant-behaviour-1443/>
- Hermit Crabs. (2021). *LINE UP to swap shells!* (4:04 min) [video]. YouTube. <https://www.youtube.com/watch?v=zpjklLt1qWk>
- Robin Wall Kimmerer—Teaching Professor of Environmental and Forest Biology State University of New York. (2014). *Humans and nature talk. What does the Earth ask of us?* (16:56 min) [video]. YouTube. <https://www.youtube.com/watch?v=y4nUobJEEWQ>
- Suzanne Simard—UBC forest ecologist. (2016). *How trees talk to each other* (18 min) [video]. TED. https://www.ted.com/talks/suzanne_simard_how_trees_talk_to_each_other

LITERATURE:

- Archibald, J. (2008). *Indigenous storywork: Educating the heart, mind, body, and spirit*. UBC Press.
- Glanfield, F., Thom, J. S., & Ghostkeeper, E. (2020). Living landscapes, archi-text-ures, and land-guaging algo-rhythms. *Canadian Journal of Science, Mathematics, and Technology Education*, 20(2), 246–263. <https://doi.org/10.1007/s42330-020-00085-8>
- Ghostkeeper, E. (2007). *Spirit gifting: The concept of spiritual exchange* (2nd ed.). Writing on Stone Press.
- Kimmerer, R. W. (2013). *Braiding sweetgrass: Indigenous wisdom, scientific knowledge and the teachings of plants*. Milkweed Editions.
- Nicol, C., Thom, J. S., Doolittle, E., Glanfield, F., & Ghostkeeper, E. (2023). Mathematics education for STEM as place. *ZDM-Mathematics Education*, 55(7), 1231–1242. <https://doi.org/10.1007/s11858-023-01498-z>
- Styres, S. D. (2017). *Pathways for remembering and recognizing indigenous thought in education: Philosophies of iethi'nihténha ohwentsia'kékha (land)*. University of Toronto Press.
- Walter, M., & Andersen, C. (2016). *Indigenous statistics: A quantitative research methodology*. Routledge.

SAWUBONA. I SEE YOU. JE TE VOIS.

Manon LeBlanc, *Université de Moncton*
Robyn Ruttenberg-Rozen, *Ontario Tech University*

Sawubona is a Zulu term that is “as an invitation to a deep witnessing and presence. This greeting forms an agreement to affirm and investigate the mutual potential and obligation that is present in a given moment... this ‘seeing’ is essential to human freedom” (Global Oneness Project, 2023, para. 1)

As you might have noticed, there is no list of our community members at the top of the page. Instead, they are embedded in our invitation to you to join us in our study of belonging in a (mathematics) (education) community. We are also using the present tense in this article (some might argue that keeping this so open would unmake our piece of writing from being an article—but can we see it differently?) as an invitation to you, dear reader. Come join us on our journey. We see you. Bienvenue !

What does it mean to belong?¹

Que signifie l'appartenance ?

How can we push OUR thinking? OUR BOUNDARIES? What are boundaries? And if we (know) (understand) (define) those boundaries, can we push them outwards? Do we need boundaries? Is(not) there more belonging (to our mathematics community) if there are no boundaries? Wait...is defining the community a boundary? If the community is undefined is it so inclusionary that it becomes exclusionary? What are the risks to me? To you? To OUR community?

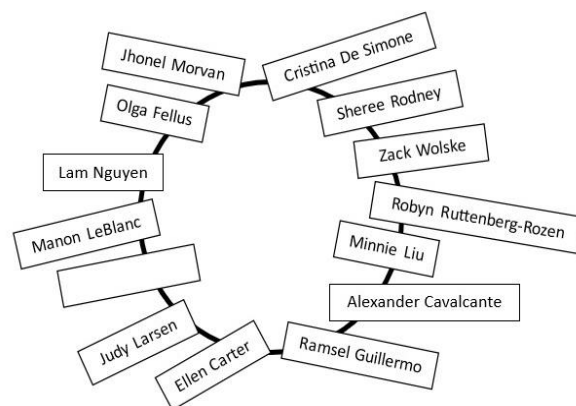


Figure 1. Participant list.

¹ We used blogger Julian Stodd's *14 questions about belonging* to guide the discussions of belonging in our working group. Julian's questions and contribution can be found [here](#). We add these questions throughout this article, and we finish the article with answers to Julian Stodd's questions from our working group community.

Invitation to you dear reader: We invite you to add your name to the circle of belonging.

WARNING!! WARNING!! WARNING!!

READ AT YOUR OWN RISK!!

Reading this article may (will? hopefully!) shake some of your foundations of understanding.

We invite you into our tensions and (of course) we do not share solutions.

Dear community member. Please stop reading for a moment. We want to draw your attention to the [circle of human concern](#).

If values construct the borders of the circle, what are the specific values that construct the mathematics community circle and create the parameters for belonging in the mathematics community? (e.g., belief in beauty of math? or math knowledge?)

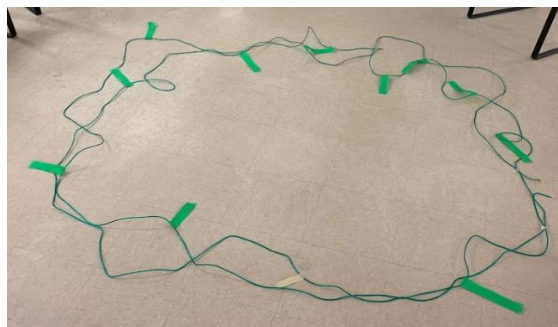


Figure 2.

So what do you think? Do you want a few minutes?

What values would you add on the border? Go ahead—write in the margins!!

We brainstormed some values in Figure 3....(Did the formatting for publication keep the pictures on the same page? Or do you need to look to the next page to find them?)

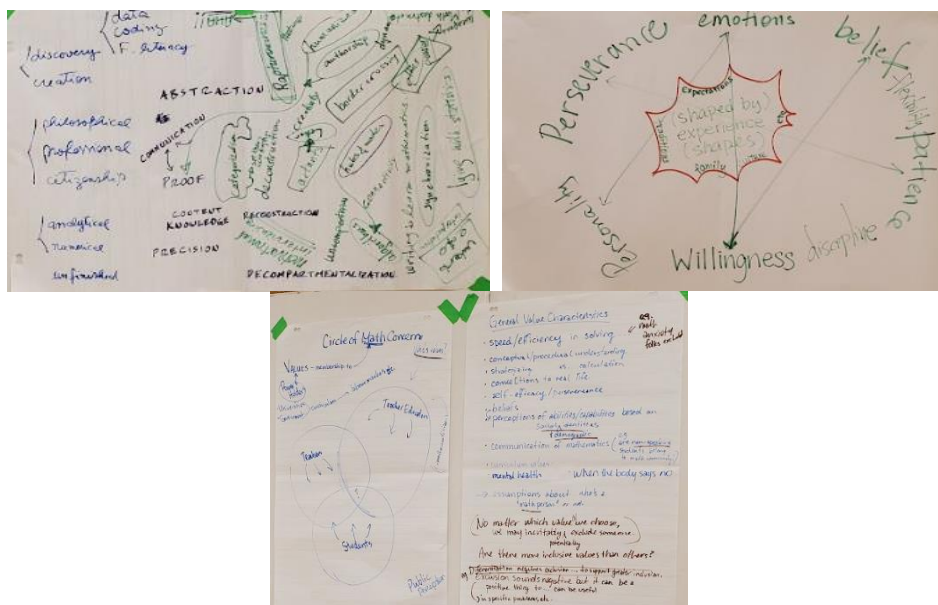


Figure 3.

The circle itself was a curiosity though:

"The circle felt like a bounded, connected space. I was curious about the intertwined nature of the ropes and if there was a meaning behind it. Why green? Was it twine or wire?"

Manon,

Oui,

do you think we should have two articles, one translated in English and one translated French, so our whole community feels a sense of belonging when reading this?

I am not sure... let's try it out...

"I appreciate the way the ideas of the community are built upon within the circle. Tensions of where a particular concept or idea "fits" are being negotiated, and it is clear that not everything is "taken as shared". Nonetheless, there is a genuine curiosity into the thinking of others that is fostering a true exploration of ideas. I think this is beautiful and I wonder if this is helping to set the tone of an inclusive space where we can grapple with the bounds, boundaries, and definitions of what it means to belong to a math community?"

« J'apprécie la manière dont les idées de la communauté sont construites au sein du cercle. Les tensions concernant la place d'un concept ou d'une idée particulière sont négociées, et il est clair que tout le monde n'a pas nécessairement la même compréhension d'une idée. Néanmoins, il existe une véritable curiosité à l'égard de la façon de penser des autres, ce qui favorise une véritable exploration des idées. Je trouve cela magnifique et je me demande si cela ne contribue pas à donner le ton d'un espace inclusif où nous pouvons débattre des limites, des frontières et des définitions relativement à ce que signifie appartenir à une communauté mathématique ? »

Manon, this is a quote from one of our community members. Should we attribute it to them?

Manon, il s'agit d'une citation d'une personne membre de notre communauté. Devrions-nous la lui attribuer ?

Good question, Robyn. But if we do, do we need to include a quote from everyone? Do we even have quotes from everyone?

Bonne question Robyn. Mais si nous le faisons, devons-nous inclure une citation de chaque membre ? Et avons-nous des citations de tout le monde ?

We do not have quotes from everyone. I wonder if we name a few people do we make invisible everyone else?

Nous n'avons pas de citations de tout le monde. Je me demande si, en citant quelques personnes, nous rendons invisibles toutes les autres ?

I think so. But I am not sure. Reader, what do you think? Should we cite the quotes, risking making other group members invisible? Or can we just attribute it to the community? This way, one voice is not prioritized over another voice.

Je pense que oui. Mais je n'en suis pas sûre. Vous qui lisez ce texte, qu'en pensez-vous ? Devrions-nous citer les sources, au risque de rendre les autres membres du groupe invisibles ? Ou pouvons-nous simplement les attribuer à la communauté ? De cette façon, une voix n'est pas privilégiée par rapport à une autre.

I am getting anxiety. I can remember back as an undergrad...YOU MUST CITE YOUR QUOTES...OMG! What would my professor tell me to do? What do you think, dear reader?

Je commence à m'inquiéter. Je me souviens qu'à l'époque où j'étais étudiante au baccalauréat...VOUS DEVEZ CITER VOS SOURCES...AH BON SANG ! Qu'est-ce que ma professeure ou mon professeur me dirait de faire ? Qu'en pensez-vous ?

But how do we validate the individual within the group? We cannot constantly identify only with the group. Belonging also means that my self—my individuality—my uniqueness—is important within a group. Maybe we

Mais comment valider l'individu au sein du groupe ? Nous ne pouvons pas nous identifier constamment comme groupe. L'appartenance signifie également que ma personne—mon individualité—mon unicité—est importante au sein d'un groupe. Peut-être devrions-nous

should find a way to signify the individual within the group by using quotes.

trouver un moyen de reconnaître l'individu au sein du groupe en utilisant des guillemets.

Umm, well, it is in print, and there is no name. But there are quotes. So I guess you know what we choose to do in this case.

Umm, eh bien, c'est imprimé et il n'y a pas de nom, mais il y a des guillemets. Je suppose donc que vous savez ce que nous avons choisi de faire dans ce cas.

Dear reader, would you make the same decision? Is it worth it to ignore academic training and culture not to make someone else invisible?

Chère lectrice, cher lecteur, prendriez-vous la même décision ? Vaut-il la peine d'ignorer la formation académique et la culture pour ne pas rendre quelqu'un d'autre invisible ?

Every time we make someone visible, we make someone else invisible...

Chaque fois que nous rendons quelqu'un visible, nous rendons quelqu'un d'autre invisible...

Robyn,

Yes.

What about this? What do you think? Does including French AND English columns create more of a sense of belonging?

I...wonder...have we created more of a divide? What values are constructing the separation between the two columns? Is it one or the other?

I was wondering if this was artificial. I would read the French column and you would read the English column. How is that belonging? Is just having ACCESS to language enough? Is just having access to mathematics enough?

You know we wrote this in English? Right?

Yes...Oui...

Qu'est-ce que cela signifie pour notre sentiment d'appartenance ? Puis-je avoir un sentiment d'appartenance tout en excluant les autres ? S'agit-il vraiment d'un sentiment d'appartenance ? Est-il possible de ne pas exclure les autres dans toutes les situations ?

Ok...how about this?

Reader...Reader? Are you still there? *Êtes-vous toujours là ?* We feel we need to invite you back in again? Do we? Have our borders shifted, to exclude you? *Nos frontières se sont-elles déplacées pour vous exclure ?* Is there space in our (we mean you too) circle of concern for entry, exit, and re-entry, then exiting again, and then returning when you want to, or rather IF you want to? Are our borders permeable, and is it your choice as to where you want to be in relation to the community? (inside, outside or ON the border?) *Nos frontières sont-elles perméables et est-ce à vous de décider où vous voulez vous situer par rapport à la communauté ? (à l'intérieur, à l'extérieur ou SUR la frontière ?)*

Lecteur...Lecteur ? *Êtes-vous toujours là ?* Are you still there? Nous pensons qu'il est nécessaire de vous inviter à nouveau. Est-ce le cas ? Nos frontières se sont-elles déplacées pour vous exclure ? *Have our borders shifted, to exclude you?* Y a-t-il de la place dans notre (nous voulons aussi dire votre) cercle de préoccupation pour l'entrée, la sortie et la réentrée, puis la nouvelle sortie, puis le retour quand vous le voulez, ou plutôt SI vous le voulez ? Nos frontières sont-elles perméables, et est-ce à vous de choisir où vous voulez vous situer par rapport à la communauté ? (à l'intérieur, à l'extérieur ou SUR la frontière ?) *Are our borders permeable, and is it your choice as to where you want to be in relation to the community? (inside, outside or ON the border?)*

But we double the space if we do this (Sorry Jennifer, our dear editor of the proceedings!). So almost there—But not yet.

I think we should get back to the mathematics community.

Oui... but we did not leave the mathematics community.

What do you mean?

Well at CMESG the question of English and French and how we communicate is tied to our sense of belonging in our larger (mathematics) community. So you see, mathematics is a defining feature of our mathematics community. BUT so are other things depending on where we are and what context we are in...

Like French and English...or like being a mathematics person versus a mathematics education person...or being a Western Canada mathematics education person versus being an Eastern Canada mathematics education person...we all belong to multiple mathematics (education) communities that converge in the community we (currently) (want to) find ourselves in, in that moment.

*"A barrier is likely required/wanted in communities for communications' sake
(so we all know what we are talking about,
along with the responsibilities (and skills required)).
Hopefully this is a permeable barrier that welcomes and values a collection of diverse ideas ...
maybe instead of a single center,
the mathematics (education) community could be a circle with many **nodes**.
These nodes all belong to the circle and they represent different ideas or values,
and people gravitate towards these nodes based on their beliefs (used loosely here)."*

Manon,

Yes,

We need to share what happened!

What do you mean?

You know

I think so...do you mean at the end? The wonderful freeing moment when the circle was reconstructed?

YES!!

Do you have something in mind for the way that we can talk about the reconstruction?

Oh I did not think about that...how do we semantically convey the pivotal moment?

MomentS you mean?

We're running out of room...Let's just show everyone. We can explain later.

Remember that circle of belonging from the beginning? Scroll back up, we will wait for you.

Are you back? Well...

When we reasoned borders are important for belonging (even though they can also exclude), instead of destroying the circle altogether, we thought of ways to RECONSTRUCT the circle! Our circle became a spiral.



Figure 4.

Yes...we got shivers too...how *mathematical* and how full of *belonging*....YES they can go together!!



Figure 5.

The change from circle to spiral meant that the borders could shift. With a circle you are in or out. With a spiral you can be out and in, you can be at the beginning in the entryway or in the center, you have a choice of where to be within the spiral, you can step right into the center or you can step into the periphery.

If you follow the spiral (the usual path) the journey is longer, but if you step over borders the journey is shorter. People can choose different paths. Some people may never get to the center and that is okay. You can journey out of the spiral AND you can always come back again!

We asked our community members...

What does it mean to belong?

"I believe belonging is a two way street: to accept, and be accepted."

What do we gain from belonging?

"I think we gain self-recognition."

"...as well as a community."

Que signifie l'appartenance ?

« Je crois que l'appartenance, c'est évoluer avec les autres, tout en restant fidèle à soi-même. »

Can you endlessly belong?

"Belonging is not endless. Both individuals and communities change."

L'appartenance peut-elle vous être imposée ?

« L'appartenance ne peut pas être imposée, car l'appartenance n'est pas un statut, mais un sentiment qui dépend de la reconnaissance des membres et de soi. »

What does it mean to belong?

"I believe belonging is to feel seen and accepted."

What do we gain from belonging?

"I think we gain security and safety."

Que nous apporte l'appartenance ?

« Je pense que l'appartenance nous apporte la pluralité et l'unicité. »

Can you endlessly belong, or is there a limit to how long you can belong for or how many things you can belong to?

"I think belonging is fluid, and a safe community will keep welcoming you back."

Do you need permission to belong?

"I believe you need acceptance from both yourself and the community to belong."

Alors, que pensez-vous ? So what do you think?

Que signifie faire partie d'une communauté mathématique (didactique) ?

What does it mean to belong in a mathematics (education) community?

REFERENCES

- Global Oneness Project. (2023). *Sawubona*. Global oneness project.
<https://www.globalonenessproject.org/library/interviews/sawubona>
- Othering & Belonging Institute. (2023). *The circle of human concern: Video + curriculum*. Othering & Belonging Institute. <https://belonging.berkeley.edu/circle-human-concern-video-curriculum>
- Stodd, J. (2021, June 2). *14 Questions about belonging*. Julian Stodd's learning blog.
<https://julianstodd.wordpress.com/2021/06/02/14-questions-about-belonging/>

AN UNDERGRADUATE CURRICULUM BASED ON MATHEMATICAL REASONING SKILLS (WOULDN'T IT BE AWESOME?)

Vincent Bouchard, *University of Alberta*
Asia Matthews, *Quest University*

PARTICIPANTS

Becca Carter
Egan Chernoff
Sean Chorney
Michelle Davidson
Francis Duah
Shannon Ezzat

Susan Gerofsky
Steven Khan
Jenny Lawson
Paul Lehmkuhl
Ralph Mason
Derek Posnikoff

Bernardo Sousa
Asmita Sodhi
Peter Taylor
Simon Tse
Kitty Yan
Rina Zazkis

INTRODUCTION

Teaching should focus less on the product of mathematical thought and more on the process of mathematical thinking. (Skemp, 1971)

In this Working Group, we proposed to turn undergraduate mathematics education inside-out. Undergraduate mathematics curricula are usually designed in terms of topics, while skills are considered as secondary outcomes which students hopefully acquire while learning a given topic. We propose, instead, to construct a full Major in Mathematics program based on skills, independent of context, as primary learning outcomes.

The goals of this Working Group include the following:

- Become familiar with the proposed program structure and learning outcomes;
- Map knowledge content to proposed courses;
- Discuss course activities and assessment;
- Produce a 'marketing' document explaining the program to students, colleagues, and administrators.

We identify the following core mathematical skills that we want students to acquire through the curriculum:

1. Abstraction
2. Communication and Collaboration
3. Computation
4. Ethics, Equity, Diversity and Inclusion (or perhaps Mathematical Responsibility?)
5. Problem solving
6. Proof and Formalism
7. Research

The proposed program is built around a suite of courses based on these mathematical processes, with context-independent learning objectives (those are provided in the Appendix). We envision the major program to be made up of approximately 18 courses, 15 of which are core courses (300-level or lower) centred around these skills, with the remaining 400-level courses such as specialisations in particular skills or perhaps topics.

We recognize that many stakeholders will be firm that there is core knowledge that needs to be covered (which is context-based, such as various topics in calculus and linear algebra). Content can be mapped to the appropriate courses, while keeping in mind that the knowledge would not be taught solely as requisite content knowledge, but rather as context to put in practice the appropriate skills. For instance, vectors could be introduced as an example of abstraction; differential equations as computation and problem solving.

With this program we would teach students how to do mathematics, instead of teaching them how mathematics was done by others. Would it not be awesome?

MOTIVATION / VALUES

What we really care about, as educators and mathematicians, are the core skills of mathematical reasoning. Employers agree: they are looking for people who can solve problems, use abstraction, collaborate and communicate scientifically. Yet what we do in contemporary mathematics education does not align with our values; we teach content over skills. Our assertion is that if we taught the *process of doing mathematics* rather than the mathematical product, this would also help address the common student feedback that undergraduate mathematics education is disconnected from the ‘real world.’ That is, the question “when will I ever need this” would become moot.

SKILLS VS KNOWLEDGE

What do we mean by “mathematical reasoning skills”? Skills are the ability and capacity to carry out processes and be able to use one’s knowledge in a responsible way to achieve a goal. Skills are part of a holistic concept of *competency*, involving knowledge, skills, attitudes and values which are developed interdependently. Competencies can help students thrive in our world and shape a better future. Employers advertise the mobilisation of knowledge, skills, attitudes, and values to meet complex demands as professionally valuable (Organisation for Economic Co-operation and Development [OECD], 2019).

Knowledge has been the main focus of education systems over the past 100 years. In particular, mathematics undergraduate education has mainly focused on disciplinary knowledge—subject-specific concepts and detailed content—with interdisciplinary, epistemic and procedural knowledge addressed in passing.

While knowledge is important in mathematics education and a key component of competencies, we suggest that it should not be the primary focus of mathematics education. As mentioned in the opening quote, we propose to teach the process of doing mathematics rather than the mathematical product: the focus should be on skills instead of knowledge. This is not a new idea, but it is an important one; while the mathematics education community certainly has acknowledged the importance of emphasising skills, changes in mathematics education are lagging behind other disciplines (see, for example, an emphasis on skills in Computer Science).

The OECD Learning Compass 2030 distinguishes between three different types of skills (OECD, 2019):

- *Cognitive and meta-cognitive skills*, which include critical thinking, creative thinking, learning-to-learn and self-regulation.
- *Social and emotional skills*, which include empathy, self-efficacy, responsibility and collaboration.
- *Practical and physical skills*, which include using new information and communication technology devices.

The proposed curriculum focuses mainly on cognitive and metacognitive skills but also includes social and emotional skills (e.g., via Communication and Collaboration, Research, Problem solving, and Ethics, Equity, Diversity and Inclusion) and practical and physical skills (e.g., via Computation).

WORKING GROUP DAILY OUTLINE

The daily goals for the Working Group were as follows :

1. Mapping core knowledge to skills-based courses;
2. Assessing skills;
3. Designing strategies for implementation.

In this contribution we summarise the discussions and results of the Working Group.

DAY ONE

GETTING FAMILIAR WITH THE PROPOSED CURRICULUM

We started by familiarising ourselves with the proposed skills-based curriculum and the learning objectives provided in the Appendix. Unsurprisingly for CMSEG, a lively discussion immediately followed, with a wide spectrum of reactions, from “this is awesome” to “we are already doing that” to “I don’t know if this is a good idea” to “how on Earth will we ever implement such a utopian scenario.”

An interesting point was raised: Is the word *mathematical skills* appropriate? In mathematics education, mathematical skills are often understood in the sense of “compute the square root of a positive integer,” rather than as context-independent skills such as “Construct a mathematical definition which encompasses common characteristics of a set of distinct objects.” Perhaps the words mathematical processes, or experiences, or competencies may be more appropriate? In the end, what we mean to convey is the notion of mathematical virtues from Su (2017).

In the same vein, it was suggested that the seven themes should probably be actionable: perhaps “Abstracting” would be more appropriate than “Abstraction,” and so on.

MAPPING CORE KNOWLEDGE TO SKILLS-BASED CURRICULUM

An immediate counter-argument to a skills-based undergraduate curriculum is that it is “too vague.” There is some core knowledge that must be covered in any mathematics curriculum (e.g., derivatives); how do we ensure that this is the case with a skills-based curriculum? To answer this question, we proposed to identify core mathematical topics that must be covered in an undergraduate mathematics curriculum and map them to skills-based courses. Instructors would then be given the context-independent learning objectives of a skills-based course with an associated list of core knowledge that must be covered in the course as examples of the mathematical skill. It is important to point out here that the idea is *not* to simply reshuffle the components of the mathematics curriculum to align them with skills instead of topics. Instead, the identified core knowledge should be taught as *examples of a mathematical process*: e.g., vectors may be taught as an example of abstraction (“Construct a mathematical definition which encompasses common characteristics of a set of distinct objects, removing explicit dependence on the specific objects.”), rather than as a standalone definition. We teach the process, not the product.

A list of core topics grouped around standard mathematical subjects (such as calculus, linear algebra, etc.) was provided to working group participants. In small groups, participants picked a theme and mapped the core topics to the seven skills-based themes. In the process, they were asked to think about how these topics would be introduced and taught in the context of the corresponding skills-based courses. The results of the activity were recorded on posters, one of which is shown in Figure 1. In the end, the main result of this activity was to get all of us thinking deeper about what a skills-based curriculum could really look like in practice and whether it would be beneficial.

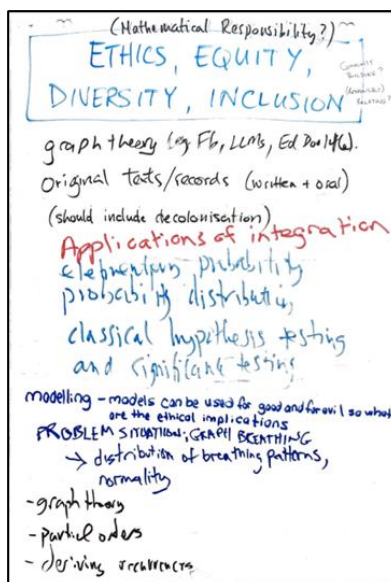


Figure 1. Results of Day 1 activity.

DAY TWO

CONNECTING TO THE REAL-WORLD

Students often report that current mathematics education is not “connected to the real world.” This is highlighted in the recent report from the Global Strategy Group (GSG, 2023); while this report focuses on parents’ perception of K12 mathematics education, similar feedback is usually received at the undergraduate level. Would a skills-based mathematics curriculum address this perceived disconnection?

In small groups, participants were asked to discuss the following questions:

- Why would a skills-based curriculum be valuable, from the perspective of students, employers, and academics?
- How would it prepare students differently from how we currently teach mathematics at the undergraduate level?
- Would students who complete a skills-based mathematics curriculum be well prepared to tackle ‘big issues’ in society, such as the role of artificial intelligence, climate change, the rise of authoritarianism and extremism, the spread of disinformation, and the mental health crisis?

The general consensus was that, indeed, by focusing on skills instead of topics, a mathematics undergraduate curriculum would prepare students better for the job market, whether it is in the private sector, public sector, or academic. It would also enable students to be in a better position to make informed and positive contributions to society.

It is interesting to remark at this point that the overall mood in the Working Group started shifting a bit. Enthusiasm and support for the idea became rather general, with some of the initial reluctance fading away. After all, we are proposing a radical transformation of undergraduate mathematics education. As (one could say that) human beings are hardwired to naturally resist changes, initial hesitancy and scepticism is expected.

ASSESSMENT FRAMEWORK

An important part of any curriculum is the assessment framework. Assessing whether students meet the learning objectives of skills-based courses may be challenging. We discussed what type of assessments and assessment framework may be appropriate in this context.

A general shift towards Grading for Growth (or mastery-based assessments) (Clark & Talbert, 2023) and Universal Design for Learning (CAST, 2023) was proposed. Such frameworks appear to be more appropriate for assessing mathematical skills. Other approaches may also be useful, such as project-based assessments and peer-to-peer feedback. Tutorial-based approaches (Oxford-style) would also work well in this context but may not be scalable. It was also mentioned that oral and written exams could still be used, if designed appropriately.

In small groups participants then picked one of the seven themes and discussed how they would assess whether students meet the learning objectives in this course. What format and type of assessments would be appropriate? What would a passing and failing grade mean in this course? An example of the result of this activity is shown in Figure 2.

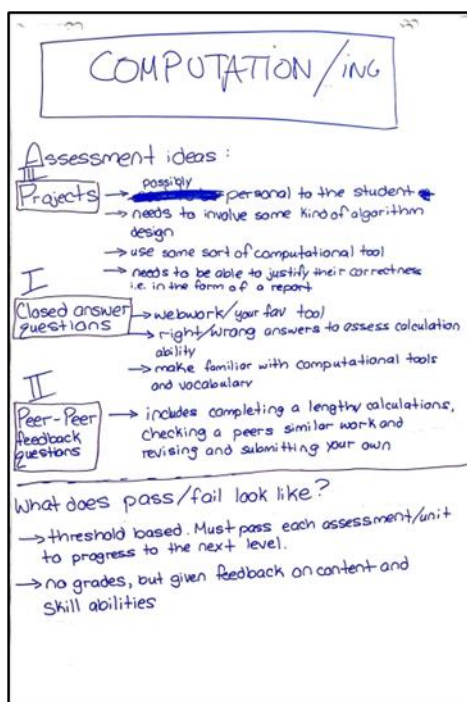


Figure 2. Result of assessment activity.

For most groups it was particularly difficult to pinpoint what a passing grade would mean precisely. This is not entirely surprising: even in standard courses, we, as instructors, have a vague understanding of what passing the course means, but it is rarely spelled out explicitly. Would it not be nice for students to know exactly what they are expected to be able to do to pass a course?

DAY THREE

STRATEGIES FOR IMPLEMENTATION

It is certainly fun and engaging to discuss utopian ideas, but we must also find ways to turn these ideas into reality. Implementation is a stumbling block that was already mentioned in the first day of the Working Group: how can we convince academic institutions and find funding to support the implementation of skills-based courses?

The idea of skills-based courses could be implemented at various levels:

1. in a given course;
2. in a given sequence of courses (such as service courses);
3. as a whole undergraduate program in mathematics.

We focused the discussion on finding strategies for implementation at levels 2 and 3, which appear to be the most challenging.

The Working Group raised a few interesting points:

- Service courses, such as the mathematics sequence for Engineering students, or mathematics courses for future school teachers, may be a natural place to implement such a radical change in the short term. Indeed, in many universities other faculties and departments are dissatisfied with our current mathematics service courses. It may be easier to convince them to move to a skills-based framework, given that what they are mostly looking for in mathematics service courses are mathematical reasoning skills rather than specific content. In fact, computing science is already moving in this direction with the creation of data science programs. They may be supportive of a similar proposal in mathematics.
- Some of the textbooks for ‘teacher-training’ mathematics courses are already organised around skills. It was mentioned that it is somewhat difficult to teach in this way; instructors often return to the familiar content-based approach.
- For a level 3 implementation, convincing administration may be easier than convincing colleagues, although this is speculative. Nevertheless, it was proposed that it may be better to start off on a smaller scale with like-minded people: perhaps a special ‘flagship’ program in a large university? Or a separate academic endeavour?
- Instead of doing a radical full-scale level 3 implementation, a gradual implementation strategy may also be used, with various courses being created one-by-one. “Proof and formalism” courses already exist in many institutions, as well as “Problem solving” courses. Perhaps one could introduce “Abstraction” courses, and so on and so forth, introducing the various components of the program one-by-one? In this respect, to convince departments to open a new course, a possibility is to offer to teach it ‘for free’ initially, as a pilot project. If it is successful, it is then much easier to convince the administration to offer it on a running basis.

Following a query by some participants, ChatGPT produced a clear outline of steps that should be taken to do a level two implementation for service courses for Engineering students; it is reproduced in Figure 3 in graphical form.

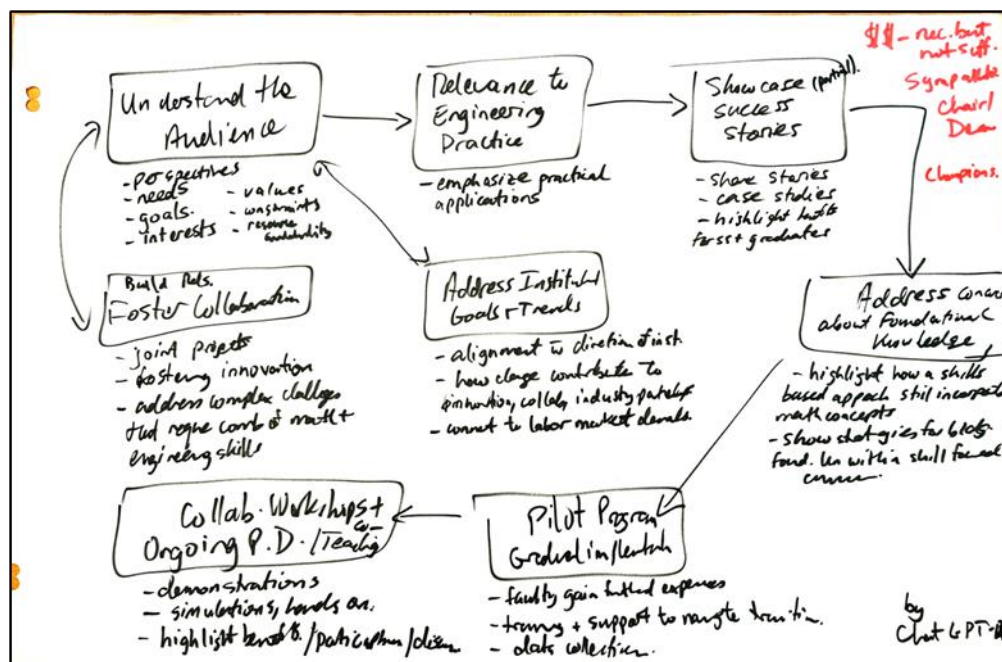


Figure 3.

MARKETING

If we are going to turn this idea into reality, we need to convince the various stakeholders (students, academics, administration, employers) that a skills-based curriculum (or skills-based courses) is beneficial. How can we advertise the project?

Participants were asked to craft a short ‘marketing statement’ that could be distributed to the various stakeholders explaining the ideas behind the project and how it would better prepare students both for academic studies and for the job market. For example, one group produced the following statement:

Imagine a mathematics program that would serve students, mathematics faculty and the communities around us more richly. In our view that would be a program that blends mathematical competencies and competencies for learning mathematics. By competencies we mean abstraction, communication & collaboration, computation, ethics & EDI, problem solving, proof & formalism, and research.

We expect that students will be able to build their capacity as mathematical learners, since this program will make explicit, and provide opportunities to develop, coach and practise the competencies of a mathematician. Students that do this will be more prepared to incorporate mathematics into their work with different communities.

For a mathematics faculty as a group, we envision a scenario where students are more prepared for first and higher level mathematics courses and will face a lower rate of attrition from our courses. This will lead to higher levels of students taking the course, leading to more resource allocation and more pleasurable teaching experiences for faculty.

FINAL WORDS

In the end, the proposal is radical and may ruffle some feathers. But its implementation, whether it is at a course or at a program level, could have significant positive impact, not only for the mathematics community but for the broader society. The potential is enormous. It is certainly worth thinking about, even if it may appear utopian at first sight. As Eduardo Galeano once said:

Utopia is on the horizon. I move two steps closer; it moves two steps further away. I walk another ten steps and the horizon runs ten steps further away. As much as I may walk, I'll never reach it. So what's the point of utopia? The point is this: to keep walking. Galeano (n.d.)

REFERENCES

- Cantor, G. (n.d.). *Georg Cantor*. BrainyQuote. https://www.brainyquote.com/quotes/georg_cantor_212898
- CAST. (2023). *Universal Design for Learning*. CAST. <https://www.cast.org/impact/universal-design-for-learning-udl>
- Clark, D., & Talbert, R. (2023). *Grading for growth: A guide to alternative grading practices that promote authentic learning and student engagement in higher education*. Stylus Publishing. <https://styluspub.presswarehouse.com/browse/book/9781642673814/Grading-for-Growth>
- Galeano, E. (n.d.). *Eduardo Galeano. Quotable quote*. Goodreads. <https://www.goodreads.com/quotes/369797-utopia-is-on-the-horizon-i-move-two-steps-closer>
- Global Strategy Group. (2023). The need to make math more relevant and engaging for K-12 Students [slide deck]. <https://usprogram.gatesfoundation.org/-/media/files/gates-analysis-deck-f040723-website.pdf>
- Organisation for Economic Co-operation and Development. (2019). *Future of education and skills 2030 concept note*. OECD. https://www.oecd.org/education/2030-project/teaching-and-learning/learning/skills/Skills_for_2030_concept_note.pdf
- Skemp, R. R (1971). *The psychology of learning mathematics*. Penguin, Harmondsworth.
- Su, F. E. (2017). Mathematics for human flourishing. *The American Mathematical Monthly*, 124(6), 483–493.

APPENDIX

In this appendix we propose sample context-independent learning objectives for courses based on the seven themes. Those are meant to be a starting point for discussion; they should certainly be refined and improved.

ABSTRACTION

1. **Filter information and identify connections.** Identify, articulate and describe common characteristics of a set of distinct objects. Recognize and remove extraneous information.
2. **Conceptualise, define and give meaning.** Construct a mathematical definition which encompasses common characteristics of a set of distinct objects, removing explicit dependence on the specific objects.
3. **Characterise and expand.** Generate new instances of an abstract definition. Explore the limitations of the definition by identifying extreme cases and non-examples.
4. **Generalise, explore and play.** Given an abstract concept, play around with the concept, explore possible generalisations, observe specific properties and generalise them into conjectures.
5. **Use axiomatic systems.** Deduce simple results using only axioms. Construct axiomatic systems. Explore the meaning of truth in the context of axiomatic systems (any statement can be given a truth value unambiguously) and compare it with real-world argumentation.
6. **Formalise.** Describe the necessity for rigour. Use appropriate mathematical notation.
7. **Explore connections using abstraction as a tool.** Use abstraction to relate conjectures, theorems and methods of proofs in one area to those in another related area of mathematics.
8. **Investigate philosophical aspects of abstraction in mathematics.** Discuss the perception of beauty and aesthetics in mathematics. Contrast the permanence of abstract concepts with the impermanence of real-world objects. Discuss the freedom of playing around and exploring mathematical concepts, abstracted away from real-world objects. (“The essence of mathematics lies entirely in its freedom” - Cantor)

COMMUNICATION AND COLLABORATION

1. **Evaluate information.** Understand and extract meaningful information from the mathematical literature, such as mathematics textbooks. Analyse a variety of mathematical and scientific writing ([OECD global competencies](#)).
2. **Formulate arguments.** Understand the difference between inductive and deductive reasoning. Determine what makes a solution logically consistent, and put it into practice ([OECD global competencies](#)).
3. **Explain complex situations and problems.** Practise creating representative structures (variables, notation, figures, graphs, tables) to communicate mathematical concepts. Determine what makes a mathematics/science presentation engaging and effective and put it in practice. Practise the various conventions of mathematical writing, including formal and popular. Use Overleaf, make podcasts, etc. to communicate mathematical concepts effectively. Rhetoric: Learn about sophisms, logical fallacies (intellectual self-defence), how they are used in common discourse, and identify them in newspapers etc. Practise the seven c’s of communication: clear, correct, complete, concrete, concise, considered and courteous. Communicate using and connecting evidence, identifying biases and gaps in information and managing conflicting arguments ([OECD global competencies](#)).
4. **Understand differences in communication,** recognising the importance of socially-appropriate communication conventions and adapting communication to the demands of diverse cultural contexts ([OECD global competencies](#)). Mentor other students at different levels of learning (and learn to be mentored) and give real-time, dynamic feedback. Learn some history of mathematical communication and collaboration, written and pictorial, e.g., letters, collaboration in different languages, art, Japanese Sangaku.
5. **Evaluate actions and consequences** by identifying and comparing different courses of action and weighing these actions against one another on the basis of short- and long-term consequences ([OECD global competencies](#)). Analyse the benefits and detriments of collaboration vs competition. Practise academic discourse: learn how to make progress in collaboration via questioning and reasoning (how questioning others can be intimidating, but it’s not about being right or wrong, but about making progress and discovering the truth). Distinguish the emotional component of self-worth and correctness from questioning in order to discover the truth.
6. **Incorporate the mathematical virtues of power, justice, freedom, community, love.** ([Su, Flourishing](#))

COMPUTATION

1. **Develop computational literacy.** Understand common techniques and algorithms for computation; a basis of techniques opens doors for a person to be able to play with more nuanced ideas. Understand common computational technology basics, such as Wolfram Alpha, Python, Maxima. Apply knowledge of techniques by carrying out computations with technology.

2. **Understand and apply decomposition and abstraction/generalisation.** Break down a problem or challenge into small, manageable parts and focus only on the most important information and elements of the problem, ignoring irrelevant details or unnecessary details.
3. **Engage in pattern recognition/data and information visualisation.** Sift through information to find similar problems, which makes it easier to organise data, which in turn can help with problem solving.
4. **Practice and evaluate algorithm design.** Use step-by-step solutions that can be followed to solve a problem. Test and debug to ensure that solutions remain fit for purpose. Know the algorithms and computations so that when using a computer to solve problems, you can trouble-shoot and verify that it's doing what you want it to do.
5. **Justified correctness.** Be able to carry out a lengthy calculation correctly. This involves (1) Resiliency, (2) Continuous justification, (3) Scepticism, (4) Consistency checks, (5) estimation.
6. **Evaluate actions and consequences** by identifying and comparing different courses of action and weighing these actions against one another on the basis of short- and long-term consequences ([OECD global competencies](#)).

ETHICS, EQUITY, DIVERSITY AND INCLUSION

1. **Understand ethical implications of mathematical approaches.** Build a collection of examples where assumptions made in mathematical approaches and quantitative reasoning can have real consequences.
2. **Evaluate ethical choices in communication** in presenting information, such as statistics.
3. **Analyse EDI issues in mathematics education.** Analyse racist, sexist, and classist practices in teaching, teaching materials, student interactions, and on an institutional level.
4. **Understand historical bias in mathematics.** Understand situations of gender, race, and class bias in the creation and dissemination of mathematics.
5. **Analyse and evaluate mathematics as democratic knowledge.** Discuss mathematics as open-access and as advancement of knowledge, not of individual importance.
6. **Appreciate how social responsibility relies on quantitative reasoning.** Understand how mathematics-rhetoric-logic is important for a well-functioning society and for social change.
7. **Incorporate the mathematical virtues of power, justice, freedom, struggle.** ([Su, Flourishing](#))

PROBLEM SOLVING

1. **Understand and carry out various problem-solving frameworks,** both corporate and mathematical (e.g., Mason's Entry, Attack Review phases; Polya's Understand the Problem, Devise a Plan, Carry Out the Plan, Look Back)
2. **Solve problems with specific given techniques.**
3. **Solve problems where no technique is obvious.** Incorporate problem-identification, research, creativity, analysis, decision-making, digital-thinking, collaboration, adaptation, and communication in carrying out the comprehensive activity of solving a problem.
4. **Approach problems using structured methods.** Clarify the specifics of a problem statement, gather observations and make choices/assumptions when the set-up is not clear, break cognitive fixedness ([SIAM Math Modeling](#)). Recognize when the problem is too hard and figure out how to make it simpler, and then maybe see that scaling up is not always feasible. Try examples and extreme cases. Generate creative ideas for solutions. As a mathematician does when confronted with a problem, find literature which can inspire you. Identify appropriate mathematical strategies.
5. **Apply metacognitive reflection to the problem solving process.** Appreciate the importance of trying, failing, and trying again. Understand how to try something, anything, and not freeze.
6. **Communicate a solution.** Present the solution to a problem in context, with reference to assumptions, and for a specified audience.
7. **Compare similarities and differences of problems.** Find problems that are the same but also recognise problems that are similar and figure out where the differences in PS may lie.
8. **Practise empathy.** Analyse and apply both human-centred and environmentally responsible approaches to design techniques, such as user research, user experience, and sustainability ([Harvard Design Thinking and Innovation](#))
9. **Incorporate the mathematical virtue of flourishing.** ([Su, Flourishing](#))

PROOF AND FORMALISM

1. **Understand the concept of mathematical proof and its importance.** Describe what a mathematical proof is as an inferential argument that establishes logical certainty. Explain why proofs are important in mathematics.
2. **Communicate mathematics clearly.** Describe why using proper notation and formalism is important in mathematics. Communicate mathematics, including definitions, theorems and proofs in a clear and precise way. Employ the notation, format and completeness conventions of the mathematics community.
3. **Become fluent with various methods of proof.** Recognize and distinguish between different types of proof (direct proof, proof by case enumeration, proof by contraposition, proof by contradiction, proof by induction, etc.). Outline the logical structure of each.
4. **Learn how to construct mathematical statements and prove them.** Construct a mathematical statement. Determine its truth value. Select an appropriate method of proof and use it to first sketch a proof and then write a clear and complete proof.
5. **Highlight the logical structure of a proof.** Given a mathematical proof written in natural language, identify the assumptions and conclusion, any lemmas used, the method of proof and the main logical steps of the proof. Connect with the idea of a formal proof in symbolic logic.
6. **Use good style in proof writing.** Study how proof writing can be simplified and embellished by establishing appropriate definitions and proving intermediate lemmas.
7. **Use the power of visualisation.** Learn how to establish pictorial representations that can be used to visualise and simplify formal mathematical arguments.
8. **Translate between mathematical arguments and everyday language.** Convert non-formal arguments into formal mathematical writing. Explain formal mathematical statements and proof in non-formal, everyday language.
9. **Recognize fallacies.** Recognize logically invalid arguments, both in a mathematical context and in everyday language.

(Some of those are inspired by the course-level learning objectives of [MATH 220 at UBC](#).)

RESEARCH

1. **Learn how to approach a research question.** Become at ease with working on a problem that has no known solution. Learn how to “try things”: if a problem has no obvious line of solution, try various methods of solutions until you find one that works. Become fluent with the idea of reducing a problem to something that you already know how to solve.
2. **Acquire targeted knowledge.** While working on a research question, determine when new knowledge is required, and learn how to acquire focused knowledge without getting lost in the vast literature. Become at ease with making progress in a certain direction without having a full understanding of all its ramifications and foundations.
3. **Test with examples and toy models.** Learn the importance of using simple test examples or toy models to test the main steps of a formal abstract argument in a concrete setting. Use consistency checks.
4. **Establish intuition.** Devise methods to “guess” whether a mathematical statement may be true or not, to guide lines of attacks for proving the truth value of the statement. Use examples to find intuition and learn how to search for appropriate examples that probe extreme cases and may lead to counter-examples.
5. **Find relevant statements in the mathematical literature.** Scan and search the mathematical literature. Identify appropriate sources. Establish methods for finding relevant mathematical results in the vast literature.
6. **Extract information from a mathematical paper.** Read a mathematical paper. Extract meaningful information from a paper even without fully understanding everything. Learn how to read a paper in multiple steps; by establishing a general understanding of the paper with a first read, and then digging deeper in the mathematical statements and proofs with each re-read.
7. **Collaborate.** Work in a research group setting and in a mentor-trainee relationship. Develop high-quality interactions with faculty members and research mentors. Become confident asking and answering questions, highlighting steps of an argument that may be shaky, and proposing future avenues of research.
8. **Search for the truth.** Understand that research in mathematics is about finding whether a mathematical statement is true or not, and not about arguing that one’s opinion is better than another. Learn how to stay humble but confident. Be able to accept making a mistake without taking it personally.

9. **Incorporate interdisciplinarity.** Incorporate interdisciplinarity and/or multi-department research. Connect mathematical results with other fields. Apply mathematical research to science, engineering, computing science, business and economics, etc.

GAMES FOR MATHEMATICAL LEARNING

JEUX POUR L'APPRENTISSAGE DES MATHÉMATIQUES

Sabrina Héroux, *Université du Québec à Montréal*
Janelle McFeetors, *University of Alberta*

PARTICIPANTS

Nat Banting
Amenda Chow
Wendy Forbes

Caroline Lajoie
Marieme Ngom
Susan Oesterle

David Reid
Pat Sargent

INTRODUCTION

We would like to suggest that out of *play* as a key feature of educative experiences in mathematics (Bishop, 1988; Francis, 2019; Su, 2020), *games* provide an authentic, rich, and compelling context for students' mathematical learning. Games have long been recommended as a way for students to develop a meaningful understanding of mathematical ideas before they move toward abstractions (e.g., Diénès, 1971, Kamii & DeClark, 1985; Skemp, 1993). More recent scholarship has investigated noticing mathematical activity during game play in primary class (Héroux, 2023), students' enactment of mathematical processes/competencies such as reasoning (McFeetors & Palfy, 2018) and problem solving (Pinter, 2010), and engagement in mathematical practices through redesigning games (Kim et al., 2021).

As diverse as the purposes of incorporating games in classrooms for mathematical learning, so are the types of games used. Drawing on the work Sabrina and Janelle have been doing in elementary and secondary classrooms, we made three types of games available to colleagues during the working group. The first type of games are instructional games (e.g., Bofferding & Hoffman, 2019; Bragg, 2012; Elofsson et al., 2016), such as a *Memory Game on Ten's Complement*, *Fraction Dominoes*, etc. These are games that are often created by teachers, and sometimes educational resource companies, for the purpose of developing mathematical concepts or developing procedural fluency. The second type of games are ancient games (e.g., McCoy et al., 2007; McFeetors & Mason, 2009; Reeves & Gleichowski, 2006/2007, Zaslavsky, 1998), such as *Oware/Mancala*, *Hnefatafl*, *Jiǎn-shízi/Nim*, *Mū Tōrere*, etc. These are games that arose in cultural contexts and provide opportunities for students to see themselves in their mathematics learning. A third type of games are commercial games (e.g., Fonstad, 2016; Jaques et al., 2019; Maida & Maida, 2011; McFeetors & Palfy, 2017; Reid 2002), such as *Blokus*, *Farkle*, *Gobblet Gobblers*, *Mastermind*, *Qwirkle*, *Sumoku*, etc. While these games are designed primarily for recreation, carefully curated (specifically selected and thoughtfully incorporated) games have possibilities for experiencing mathematical content that is embedded in the game mechanics. At the same time, commercial games are educationally valuable beyond reaching prescribed curricular content outcomes (Higgins & McFeetors, 2019) and occasion students' engagement in processes critical to learning mathematics, such as spatial reasoning, problem solving, logical reasoning, and communication, to name a few. An appendix includes a list of the many games that could be incorporated into mathematics classrooms.

In this report, we share our experiences in the Working Group: the playfulness of participants as they engaged in games, ideas raised in discussions as we reflected on and interpreted our playfulness, and some considerations needed for future inquiries and implementation with teachers. For each day, we planned with a focus in mind (exploring mathematical games, students' mathematical experiences in/through game play, and teachers' considerations for incorporating mathematical game play) that framed our collective experiences.

DAY 1: EXPLORING GAMES FOR LEARNING MATHEMATICS

On the first day, we wanted to explore the experience of playing games, especially games that Sabrina and Janelle had used with students in mathematics class. We had an opportunity to browse through the 50+ games laid out (see the Appendix for a longer list). So, we invited the participants to play a game with the intention of becoming familiar with it—there were many new games to learn!



Figure 1. Wendy, Sabrina (not in the picture), Susan and David playing *Cloud City*.



Figure 2. Amenda and Pam playing *The Game*.



Figure 3. Marieme, Caroline, and Janelle played *Santorini*. We missed taking a picture!

WHAT IS A WELL-PLAYED GAME? HOW MIGHT WE HAVE PLAYED WELL TOGETHER?

We found some grounding for our first discussion about our experience of playing by turning to Bernie de Koven's (2013) thinking around playing games. Interestingly, de Koven began his career as an elementary school teacher, then with children developed a curriculum called the Interplay Games Catalog, and became renowned as a game designer and game/play scholar ("Bernie de Koven," n.d.). In his landmark book, *The well-played game: A player's philosophy* (1978/2013), he writes passionately about a wide range of elements that contributes to playing a variety of types of games and what it means to play them *well*. We shared with the group

If we are going to find a well-played game together, we are going to have to arrive at some common understanding of what it is we are looking for. The most logical way to go about it is by playing together. Since we are playing together, we will have something in common. When we find the game that we can play well together, we'll all know what it is. (de Koven, 2013, p. 1)

Any victory, now that we know what it is that we want to create together, is shared. No matter who wins a game, if we have played well together, we have accomplished what we set out to do. ...Our success in the search for the well-played game can only be measured in terms of how well we have been able to play together. ...It is not measured by the score, it is not measured by the game, it is measured by those of us who are playing it. (de Koven, 2013, p. 5)

We could be playing for education stakes—to improve skills, to widen our access to knowledge. But, as crucial as it is to our survival in the real world to play for such goals, we must first acknowledge our community of and our intention of playing well together. We are playing to learn. We are playing to learn because that particular challenge intrigues us. We begin playing and end playing with the knowledge that we are already worthy and good and wise. What we win is the opportunity to play with that knowledge, and, in so doing, to play well, and, in so playing, to discover that there is yet more for us to play with. (de Koven, 2013, p. 128)

Perhaps it is not surprising that collaboration came up as a shared experience in our opening game play. The *Cloud City* group remarked that their early play was not very strategic as they focused on following the rules but was collaborative in discussing across the table possible tile and walkway placements. As each player has their own individual game board, this collaborative discussion is not an expected part of gameplay. Several turns into the game, however, the balance between collaboration and strategy may have changed! The theme of collaboration came up again in Day 2 when Amenda observed how the competing pair at her table ended up working together to find all the matching pairs in a *Memory-style Ten's Complement* game. On Day 3, Nat commented that the *Fraction Domino's* game became collaborative when they let go of the provided rules and "started tinkering...we felt the right to make changes" and they did this by asking, "What if this was allowed" or setting for themselves other explorations like to "fill all the space on the table." This playfulness arose in the midst of collaboration.

Working group members also selected games that were new to them, so we were *playing to learn* how to simply engage with the game at first. Susan reflected that when they were figuring out the rules for *Cloud City*, it "feels strange...to live with discomfort" and connected this to helping "foster in students the perspective that it's okay to ask questions". In carefully curating games, one of Janelle's criteria is to select games that have relatively bounded (few, straight-forward) rule sets so that students can immediately get started. For example, in *Santorini*, a player moves a worker and then builds a building each turn. Wendy connected this to problem tasks often given in mathematics classrooms that have a "low floor and high ceiling". Adapted to a game context, David described the simplicity (rules or turns are easy to understand) with depth (game is difficult to master).

Overall, the reflection on our first experience of playing well together emphasized the importance of problem solving. In the process of problem solving, Wendy pointed out that players could develop efficiencies. We could understand these efficiencies as strategies that are created by moving from specific instances of moves toward generalized approaches that are successful. Caroline observed that when the group playing *Santorini* increased from two to three players, it resulted in shifting how we played on the common game board. Adapting emerging strategies from the first game to the second game prompted solving a new problem with a more crowded board with six pawns instead of four. The primacy of problem solving arose again as we moved our discussion to what counts as a game.

QU'EST-CE QU'UN JEU ?

En nous appuyant sur nos expériences de jeu pendant la première moitié de la matinée et en réfléchissant à ces expériences, nous nous sommes posé la question suivante : « Qu'est-ce qu'un jeu ? » Après un début enthousiaste et

après avoir joué plus longtemps que prévu, nous avons discuté des caractéristiques des jeux. Notre groupe de travail a soulevé :

L'apparence—Les jeux commerciaux sont plus attrayants en raison de leur apparence par rapport aux jeux « maison ». La boîte et les pièces elles-mêmes contribuent au plaisir esthétique et à l'engagement/l'interaction.

Le but—Avec un jeu, le but est de jouer. Avec les jeux commerciaux, le but n'est pas d'apprendre contrairement aux jeux « maison » qui ressemblent plus à des exercices ou de la pratique comme ce qui se fait dans une classe.

Le plaisir—Liée à la résolution de problèmes, le plaisir semble être évoqué avec le jeu. Le plaisir dépend de la perspective du joueur. Pour un expert qui a une stratégie optimale, le cadre n'est peut-être pas un jeu. Il semble qu'un jeu soit quelque chose qui ne soit pas résolu. En classe, un défi pourrait être de trouver des jeux dans lesquels un joueur expérimenté ne gagne pas tout le temps.

La fiction—Lorsque vous êtes un joueur, vous « devenez » parfois un personnage au fil des tours. Avec certains jeux comme *Serpents et échelles*, il y a une illusion d'agir.

L'objectif—Différent du but lié au fait de jouer, un jeu contient une sorte de condition de victoire. Selon le contexte du jeu, cela crée un aspect de compétition entre les gagnants et les perdants (bien que, dans certains jeux collaboratifs comme *Forbidden Island*, les joueurs s'associent pour jouer contre le jeu).

Le hasard—Certains jeux comportent plus de hasard ou de prise de décision que d'autres. Cela implique que vous pouvez parfois maîtriser complètement les jeux. Le jeu demeure amusant tant que vous ne l'avez pas résolu.

Nous ne sommes pas parvenus à un consensus sur les caractéristiques d'un jeu dans ce groupe de travail. Tel que mentionné par Héroux (2023), les jeux ne sont pas tenus d'avoir toutes les caractéristiques pour être retenus. Sabrina a partagé une liste de caractéristiques développées à travers sa recherche, comprenant des aspects fonctionnels comme les règles, la mécanique, le résultat, les joueurs et les pièces. Notre discussion a fait ressortir l'importance d'un grand nombre de caractéristiques. Nous avons délibérément laissé cette question ouverte à d'éventuelles discussions.

QUE DIT VOTRE PROGRAMME D'ÉTUDES AU SUJET DE JOUER ET/OU DES JEUX ?

Puisque les participants de notre groupe de travail provenaient de différentes provinces canadiennes (Colombie-Britannique, Alberta, Ontario et Québec) et de différents pays (Norvège et États-Unis), nous avons pu avoir les perspectives de différents programmes de formation relativement à l'apprentissage des élèves. Nous avons pris le temps d'explorer et de discuter des diverses manières où jouer et/ou les jeux sont explicitement mentionnés dans les différents programmes de formation.

Colombie-Britannique : Le programme de mathématiques de la Colombie-Britannique se trouve [ici](#). Jouer est mentionné comme compétence curriculaire liée à « understanding and solving » et à « making connections » dans toutes les classes comme approche générale (voir pp. 7 et 10 pour des exemples). De la 6^e à la 12^e année, on s'attend à ce que les élèves « use logic and patterns to solve puzzles and play games » (voir l'exemple de la p. 41) comme compétence du programme. Jouer soutiendrait également l'élaboration de multiples stratégies de résolution de problèmes et de communication.

Alberta : Le programme de mathématiques de l'Alberta comprend trois documents selon les niveaux : de la maternelle à la 6^e année se trouve [ici](#), de la 7^e à la 9^e année se trouve [ici](#), et de la 10^e à la 12^e année se trouve [ici](#). À la maternelle, on s'attend à ce que les enfants reconnaissent les régularités en jouant (p. 12). Dans l'introduction pour les élèves de la 10^e à la 12^e année, on reconnaît que « students best experience change to their understanding of mathematical concepts as a result of mathematical play » (p. 8). De plus, dans quatre des huit cours du secondaire, un résultat d'apprentissage attendu est « analyze puzzles and games that involve spatial reasoning, using problem-solving strategies » (voir l'exemple de la p. 26).

Ontario : Le programme de mathématiques de l'Ontario pour les élèves de la 1^{re} à la 8^e année se trouve [ici](#) et pour les élèves de la 9^e à la 12^e année [ici](#). Le programme ne mentionne pas explicitement le jeu de la 1^{re} à la 8^e année, mais

mentionne les jeux comme ressources d'apprentissage pour les apprenants de langue anglaise (p. 16). Pour le secondaire, les jeux sont mentionnés dans deux cours de 12^e année qui sont liés à la détermination des probabilités.

Québec : Le programme de mathématiques du Québec est séparé en quatre documents selon les niveaux : la maternelle se trouve [ici](#), la 1^{re} à la 6^e année du primaire se trouvent [ici](#), 1^{re} et 2^e année du secondaire [ici](#) et la 3^e à 5^e année du secondaire [ici](#). Le programme de maternelle encourage le jeu et les jeux sont utilisés pour apprendre et résoudre des problèmes. Par le jeu et par leurs activités spontanées, les enfants s'expriment, expérimentent, construisent leurs apprentissages, structurent leurs pensées et développent leur vision du monde. Ils apprennent à être eux-mêmes, à interagir avec les autres et à résoudre des problèmes. Ils développent leur imagination et leur créativité. L'activité spontanée et le jeu sont leurs façons de maîtriser la réalité, ce qui justifie de donner au jeu une place centrale dans l'éducation préscolaire et d'organiser l'espace et le temps en conséquence. Au niveau secondaire, les jeux et les puzzles sont mentionnés explicitement pour les probabilités et le raisonnement spatial.

États-Unis (Tronc commun) : Le Tronc commun (en quelque sorte représentatif des États-Unis) se trouve [ici](#). Il n'y a pas de référence explicite à l'utilisation du jeu dans l'apprentissage des mathématiques. Les jeux sont mentionnés dans le calcul d'un « expected payoff for a game of chance » (p. 83) à l'école secondaire. Dans notre groupe, Marieme a noté que la théorie des jeux est un lien mathématique fort avec des jeux comme dans *Santorini* qui ne contiennent pas d'élément de hasard.

Norvège : Le programme norvégien comprend des compétences de base en résolution de problèmes et en raisonnement qui sont souvent développées et utilisées dans le jeu. Les jeux sont explicitement mentionnés de la 2^e à la 4^e année. Par exemple, en 3^e année, on s'attend à ce que les élèves suivent les règles et à ce qu'ils en inventent. Nous avons appris que le mot « jeu » en norvégien est un verbe, ce qui a mené à une conversation intéressante sur la façon dont le langage est utilisé pour exprimer une gamme de jeux, de façons de jouer et d'activités ludiques.

Parmi les jeux auxquels nous avons joué le premier jour, la plupart des idées mathématiques impliquaient des processus ou des compétences de raisonnement, de la visualisation ou du raisonnement spatial, de la communication et même de la résolution de problèmes. Nous pourrions établir des liens entre ces façons de mettre en œuvre des idées mathématiques et les idées plus larges de compétences que l'on retrouve dans les programmes. En travaillant avec les enseignants en classe, nous pourrions être bien guidés par la suggestion de David « to educate about the ways games can address [what is] already in the curriculum, like competencies and skills for which we can identify suitable games » plutôt que de se concentrer sur la façon d'inclure des références explicites aux jeux dans le programme.

DAY 2: STUDENTS' EXPERIENCES

On the second day, our focus was on students' experiences of mathematical learning through game play. We planned for two important aspects: coming to understand what learning may be available for students and how teachers might notice the learning-in-action. As with the first day, we began playfully with a variety of games. To focus attention, we invited working group members to play with a different colleague, try a different game, and predict what mathematical thinking may be occasioned before they started playing.



Figure 4. Marieme and Susan playing *Otrio*.



Figure 5. Marieme and Susan playing *Tic Tac Toe*.



Figure 6. Caroline, David, and Janelle (not in the picture) playing *Set*.



Figure 7. Caroline and David playing *Six*.



Figure 8. Sabrina, Pam, Wendy, and Amenda playing *Nbr 9*.

Unsurprisingly, we all stretched the game play time out longer than planned. With games that take varying lengths of time to play, two of the small groups were able to try two different shorter games out while one team immersed themselves in *Nbr 9*. Adding to the range of games within our collective experience allowed a fuller discussion about the mathematical aspects available to students while playing these games.

QUE PEUVENT APPRENDRE LES ÉLÈVES PAR/AVEC LE GAMEPLAY EN CLASSE DE MATHÉMATIQUES ?

Papert (1981) est connu pour ses travaux offrant une conceptualisation spécifique de ce que peuvent représenter l'ordinateur et la programmation (à travers le langage de programmation Logo et sa Tortue). Pour Papert, l'ordinateur a le potentiel de créer des « micro-mondes » (tels que Logo) où l'élève peut se plonger, apprenant des mathématiques en en faisant à travers la construction de choses intéressantes pour lui. Il parle d'« objet-avec-lequel-penser », donnant l'exemple d'un ensemble de la tortue Logo et des engrenages qui, dans son adolescence, lui a permis de développer une compréhension très intime et puissante de l'algèbre.

Pour Papert (ibid), il les *objets-avec-lesquels-penser* seraient porteurs de certaines activités qui émergent à travers nos interactions avec ces objets. L'*objet-avec-lequel-penser* est quelque chose de manipulable, tangible ou non, par lequel les idées se manifestent et prennent parfois forme de manière particulière. Les jeux mathématiques semblent de très bons candidats à ce titre.

Papert caractérise ces *objets-avec-lesquels-penser* en parlant de la façon dont ils permettent l'expérimentation autour des idées mathématiques et favorisent l'intuition. Papert se préoccupe de donner aux enfants des expériences mathématiques sans direction prédéterminée. D'autre part, il explique que l'*objet-avec-lesquels-penser* ne peut être réduit à la connaissance qu'il met potentiellement en jeu, ouvrant plutôt à d'autres connaissances (p. ex. aux mathématiques), et qu'il est également en relation avec l'expérience en dehors du *micromonde*.

Le concept d'*objet-avec-lesquels-penser* nous invite à regarder les façons de faire et de penser qui émergent lors de l'utilisation d'un jeu par les élèves. L'aspect ludique du jeu en classe place l'activité mathématique en arrière-plan, mais elle est néanmoins présente. D'après nos deux premières journées de jeux jusqu'à présent, et en réfléchissant à l'ensemble du groupe de travail, nous avons commencé à identifier certaines des possibilités mathématiques. Certains aspects mathématiques comprennent un contenu ou des compétences spécifiques, tandis que d'autres aspects comprennent des compétences importantes pour les idées et l'apprentissage mathématiques des élèves.

Nombres et opérations : Il y a eu une petite inclusion des nombres et d'opérations dans les jeux auxquels nous avons joué. Dans *The Game*, les joueurs ont reconnu les chiffres et ont eu recours à l'ordre croissant et décroissant. *Nmbr 9* et *Cloud City* incluent de l'arithmétique pour le calcul du résultat final. En fait, *Nmbr 9* a même permis une utilisation de la distributivité dans le calcul du résultat final. Cependant, l'utilisation des nombres et des opérations n'était pas le but principal du jeu (contrairement à des jeux comme *Sumoku*).

Suites et régularités : Set s'est appuyé sur l'analyse d'attributs explicitement. La reconnaissance et la distinction entre les attributs se trouvent dans différents volets de programmes de formation, mais elle précède souvent les suites dans les objectifs d'apprentissage.

Mesure : Des idées comme la superficie, la hauteur et la longueur sont devenues importantes—pas seulement superficiellement—pour bien jouer dans des jeux comme *Nmbr 9*, *Tic Tac Toe*, *Cloud City*, et *Santorini*. Par exemple, dans *Nmbr 9*, les joueurs devaient être stratégiques pour construire suffisamment de surface sans trous sur leur couche inférieure, tout en ne gaspillant pas de tuiles en créant une trop grande surface. Sacrifier la surface pour la hauteur était une stratégie suggérée. À *Santorini*, les joueurs ont remarqué que les mouvements en diagonale de leur pion, plutôt qu'une série de mouvements horizontaux et verticaux, permettant plus rapidement à travers le plateau de jeu de bloquer un adversaire.

Théorie du jeu et combinatoire : En jouant à *Set*, les joueurs ont reconnu que devoir trouver un groupe de trois cartes, en respectant les contraintes imposées par les règles, les a amenés à visualiser un arbre de possibilités du jeu. Ce même type de visualisation en arbre de possibilités d'un jeu a aidé les joueurs à prendre des décisions sur les mouvements dans d'autres jeux comme *Santorini*. C'était très intéressant pour *Otrio* et *Tic Tac Toe* d'être identifiés comme des jeux qu'un ordinateur pourrait apprendre à jouer et maîtriser—peut-être un bon candidat pour AlphaZero ou MuZero ! La combinatoire pourrait être utilisée pour analyser des jeux comme *Otrio* ou *Tic Tac Toe* qui ont des informations parfaites.

Probabilités : Les jeux auxquels nous avons joué avaient un caractère aléatoire différent allant de l'absence de hasard (p. ex. *Tic Tac Toe*, *Otrio*, *Six*, et *Santorini*) à un hasard limité (p. ex. les cartes retournées dans *Nmbr 9* ou *Set*, les tuiles disponibles dans *Cloud City*). L'expérience du hasard pourrait conduire à penser et à comprendre les

probabilités. Cependant, il est important de souligner que la prévisibilité—dans le contexte des jeux, savoir ou ne pas savoir ce qu’un adversaire fera et qui changerait le jeu—est différente du caractère aléatoire introduit en raison de la conception du jeu.

Résolution de problèmes : Un lien a été établi avec la façon dont les joueurs sont engagés dans la résolution de problèmes tout en jouant à un jeu, en particulier lorsqu’ils développent plusieurs approches ou stratégies. En fait, le coup d’un joueur pourrait être considéré comme « posant un problème » que l’adversaire doit (re) résoudre au tour suivant. Les joueurs font également l’expérience de la résolution collective de problèmes lorsqu’ils s’entraident pour faire des mouvements en découvrant comment jouer à un nouveau jeu.

Raisonnement : Il y a eu plusieurs exemples de raisonnement tout au long des jeux. Souvent, les joueurs anticipaient les mouvements futurs en construisant des *déclarations si...alors* pour évaluer le geste qu’ils allaient faire. Les joueurs ont *conjecturé* sur les stratégies possibles, puis les ont par la suite *testées*. Dans le jeu *Six*, les joueurs ont utilisé une *inférence* pour déterminer les configurations de pièces de jeu ayant mené à une configuration gagnante pour indiquer comment parvenir à la victoire. Lorsqu’une situation se présentait dans *Set* où il n’y avait pas de jeu de cartes sur la table, les joueurs *justifiaient* mutuellement pourquoi il fallait ajouter d’autres cartes à la table—et cet acte de *justification* semblait mathématique. Après avoir terminé un match dans certains jeux, les joueurs *analysaient* le jeu après pour comprendre une stratégie utilisée par un adversaire ou une série de mouvements qui ont conduit à une victoire/perte. Avec une approche similaire des mouvements, cela a parfois donné lieu à une *généralisation*, ou une stratégie qui serait souvent efficace. De plus, David a mentionné que l’*argumentation* est un processus social et que ce ne sont pas tous les raisonnements fondés sur la pensée individuelle qui comptent comme *argumentation*.

Raisonnement spatial : Des jeux comme *Otrio*, *Tic Tac Toe*, *Santorini*, *Cloud City*, *Nmbr 9* et ceux qui utilisent l’espace tridimensionnel pour construire vers le haut, ont rendu la mise en œuvre du raisonnement spatial plus évidente pour les joueurs. En termes de composante du raisonnement spatial, les joueurs *visualisaient* souvent un mouvement possible pendant un tour. Un exemple spécifique de raisonnement spatial s’est produit lorsque les joueurs ont essayé d’*optimiser* l’empilage/l’ajustement des pièces pour créer des couches dans le jeu *Nmbr 9*.

Nous avons brièvement discuté des résultats des travaux de Héroux (2023) sur les différentes postures qu’un élève peut prendre en jouant. Selon Brousseau (2002), l’élève qui joue à un jeu mathématique dans une salle de classe peut être un « joueur ». L’élève cherche à travers le jeu mathématique un plaisir qui n’est pas nécessairement défini par les règles du jeu. Il donne l’exemple d’un enfant qui perd volontairement pour empêcher son adversaire de se décourager et d’arrêter de jouer. Le côté ludique est très important pour l’élève qui ne veut pas arrêter de jouer. Dans cette posture, l’élève se laisse emporter par le côté ludique du jeu mathématique et applique plus ou moins des concepts, des processus et des raisonnements mathématiques. L’élève peut aussi être « actant » en jouant à un jeu mathématique. Il cherche à gagner en suivant les règles (Brousseau, 2002). Par exemple, certains élèves vont volontairement perdre en suivant les règles et d’autres sont déçus quand ils découvrent une stratégie optimale pour toujours gagner, réduisant ainsi le plaisir de jouer au jeu. Les élèves demanderont des explications sur le jeu, mais il y a une demande sous-jacente de précision sur les concepts, les processus et le raisonnement mathématique. En jouant, nous remarquons que l’interaction de l’élève concerne le jeu comme dans la posture du joueur, mais il semble y avoir une certaine activité mathématique. En jouant à un jeu mathématique, l’élève peut devenir un « apprenant ». Typiquement, ce type d’élève cherche des alternatives et tente de changer son répertoire pour une nouvelle action (Brousseau, 2002). Un élève peut devoir utiliser ses connaissances préalables pour gagner. Par exemple, il peut demander à l’enseignant de lui parler d’un nouveau processus. L’apprenant choisit de s’engager dans le jeu pour gagner. Enfin, pour Brousseau (2002), un élève peut adopter une posture d’« élève » en cherchant auprès de l’enseignant ou auprès d’un autre élève des connaissances qui lui manquent pour gagner. Par exemple, un élève peut utiliser une aide externe pour obtenir la stratégie gagnante sans effort plutôt que de la chercher comme il le ferait dans une posture d’« apprenant ». Quelle que soit la posture de l’élève, nous pouvons voir une activité mathématique lorsqu’il joue à un jeu mathématique dans une salle de classe (Héroux, 2023). Certaines postures (p. ex. joueur, acteur) comportent moins d’activités mathématiques que d’autres (p. ex. apprenant, élève).

In reflecting on students’ opportunities to learn through game play, mathematical topics and competencies were not the only object of learning raised. Wendy acknowledged that “games in the classrooms is a powerful way to increase student engagement.” Two intertwined ideas were challenges for future consideration. First, that such experiences of being mathematical during game play may “allow the students to see themselves as mathematical thinkers/problem solvers” (Susan’s reflection). Pam elaborated by adding possibilities for an impact on students’ “perceptions of their

mathematical abilities, attitudes towards mathematics, ...and perceptions of who does math.” Second, that students could become metacognitively aware as they “see the mathematics/understand/make connections” (Susan’s reflection) while impacting their “perceptions of mathematics itself...and how math is ‘done’” (Pam’s reflection). Janelle’s research with elementary school students confirms that students are capable of discussing how game play contributes to their productive disposition (McFeetors et al., 2021) and are capable of identifying components of reasoning with examples from game play (McFeetors et al., 2024). At the same time, one idea that came out of our discussion was the importance for the teacher to play the game in advance to draw out the possible mathematical thinking of the students.

HOW DO WE NOTICE MATHEMATICAL THINKING/LEARNING DURING GAMEPLAY?

A challenge for teachers and researchers is how to notice the mathematical thinking and learning that students engage in while they are playing games. Mason (2002) offers an approach to noticing in mathematics classrooms, where “to notice is to make a distinction, to create foreground and background, to distinguish some ‘thing’ from its surroundings” (p. 33) and is “a collection of practices both for living in, and hence learning from, experience, and for informing future practice” (p. 29). We can draw into our awareness, for noticing, the ways in which students are being and thinking mathematically during their game play. As we begin to foreground these actions, the next aspect of noticing is to *mark* as “attaching connections so that what is marked can come to mind later without the need for outside triggers” (p. 61). *Recording* follows in which we are “making a brief-but-vivid note of some incident” (p. 34) that is effortful. Working with a record, we can then construct an *account* related to the incident that incorporates both *accounting-of* (description) and *accounting-for* (understanding). A final aspect of noticing we discussed was *wisening* as “probing details of a collection of accounts, seeking what underlies apparent similarity, being explicit about positive, negative, and interesting features of specific acts” (p. 42).

To experience what it might be like to notice students’ mathematical thinking through game play, we invited our working group members to participate in the first few phases of noticing. We split the group into two groups of four. While two participants played the *Ten’s Complement Memory Game*, the two other participants observed. We then exchanged the roles and played again.

Le *Jeu de mémoire du complément* est un jeu fait main (Héroux, 2015) basé sur le classique *Jeu de mémoire*. Vingt-cinq cartes sont placées face cachée. Si les joueurs veulent jouer plus longtemps, ils peuvent ajouter des cartes face cachée de la pioche au début ou pendant la partie. À son tour, chaque joueur retourne une première carte et doit ensuite retourner une deuxième carte à associer à la première pour former un total de 10. Si le total est de 10, le joueur conserve la paire en retirant les cartes de la table et peut ensuite jouer à nouveau. Si le total n’est pas de 10, le joueur retourne les deux cartes face cachée et c’est au tour de l’adversaire. Il n’y a pas de deuxième chance. Le but de ce jeu est de faire autant de paires que possible. Le jeu se termine quand il n’y a plus de paires à faire. Le gagnant est le joueur qui a accumulé le plus de paires.



Figure 9. Le Jeu de mémoire du complément de 10 au début.

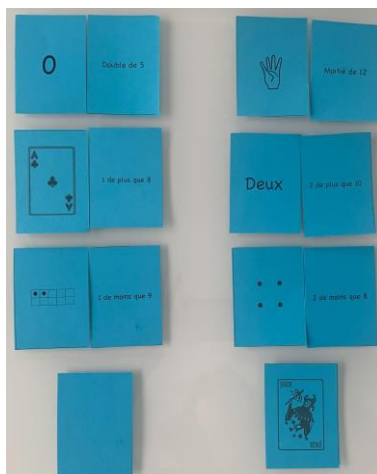


Figure 10. Différentes paires de cartes de complément de 10 avec des représentations différentes, ainsi que les deux jokers.

Voici quelques brèves idées soulevées dans la discussion de notre groupe après observation.

Les stratégies des joueurs pour dévoiler les cartes : Les participants ont comparé les différentes stratégies pour choisir les cartes à retourner. Certains étaient plus systématiques, comme Caroline, tournant les cartes en rangées une après l'autre, ce qui permettait de s'en souvenir plus facilement et d'autres tournaient plus librement les cartes, comme Susan, ce qui rendait plus difficile pour l'adversaire (et pour eux-mêmes) de s'en souvenir.

Les stratégies des joueurs pour se souvenir de la valeur des cartes : Selon les représentations, certains participants se souviennent de la valeur des cartes, mais avec les « représentations écrites », les participants se souviennent plutôt du nombre nécessaire pour faire 10. Cela va confondre un des participants qui tournera la mauvaise carte en pensant qu'elle valait deux quand elle valait huit et qu'il en a fallu plutôt deux pour en faire dix.

Les stratégies des joueurs pour se souvenir de l'emplacement des cartes : Certains participants, comme Sabrina, fermaient les yeux pour se souvenir de l'emplacement des cartes tournées. D'autres balayaient les cartes avec leurs yeux pour se souvenir des cartes dans chaque emplacement. Au cours d'une partie, un participant a même mentionné se souvenir avoir vu une carte d'une certaine valeur nécessaire pour en faire dix, mais malheureusement il ignorait son emplacement.

Les joueurs touchent les cartes : Certains participants touchaient plus les cartes que d'autres et il semble que le fait de toucher les cartes aidait à se souvenir de leurs valeurs. Les participants ont également mentionné qu'ils étaient plus susceptibles de se souvenir des cartes qu'ils ont tournées même s'ils observaient et portaient attention lorsque leur adversaire jouait.

Les interactions des joueurs avec leur adversaire : Certains joueurs étaient plus verbaux que d'autres. On a entendu : « Oh, tu peux faire une paire » d'un participant à son adversaire. Le participant a alors cherché plus soigneusement parmi les cartes qui avaient été révélées précédemment au lieu de tourner rapidement une autre et il a réussi à faire une paire. Un participant mentionne que sa carte est « inutile » lorsqu'il la retourne et qu'il ne peut pas faire de paire à son tour en ouvrant la porte à son adversaire. À un moment donné, les participants semblaient avoir perdu de vue le nombre de paires qu'ils avaient faites pour gagner et ils ont même commencé à jouer ensemble. Un participant a même mentionné le fait qu'il n'y avait plus de paires possibles à faire selon les cartes restantes.

Le niveau de difficulté du jeu : Pour les mathématiciens et les enseignants de mathématiques, ce jeu est facile quant à l'activité mathématique qu'il mobilise. Nous n'avons pas argumenté si une paire de cartes est un complément de dix ou non. Il n'y avait pas de confrontation quant à la valeur des cartes. Nous pensons que cela pourrait être un problème pour les élèves de première année. Certains participants ont fait des erreurs en retournant la deuxième carte, mais ils ont dit que c'était une erreur de mémoire plutôt que mathématique. Certains participants avaient oublié la possibilité

de faire une soustraction pour en faire dix (par exemple, $12 - 2 = 10$) faisant qu'ils ne complétaient pas certaines paires possibles, ce qui profitait à leur adversaire.

Nous nous sommes alors demandé si nous devrions jouer à ce jeu en sous-groupes comme nous l'avons fait ou avec toute la classe. Cela a préparé le terrain pour la dernière journée en considérant comment les enseignants pourraient intégrer le jeu dans leur classe de mathématiques comme expérience éducative.

DAY 3: TEACHERS' EXPERIENCES

Extending the previous days of examining the mathematical thinking available in the games, especially as it relates to students' experiences of doing math, we shifted to exploring what it means for teachers to incorporate game play in their mathematics classrooms as an educative experience for students. We started by playing different games. We played with the intention of a teacher's perspective, including their intentions, possible challenges, approaches to incorporating games, and the support teachers and students might need.



Figure 11. Pam, David, and Janelle playing *Quartex*.



Figure 12. Caroline and Susan playing *Aqualin*.

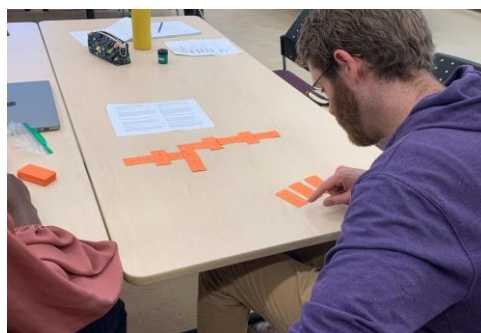


Figure 13. Marieme and Nat playing *Fraction Dominoes*.

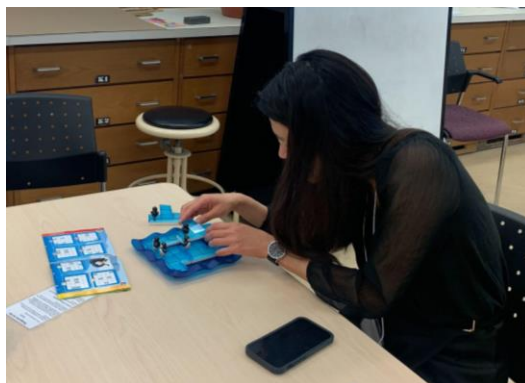


Figure 14. Amenda and Sabrina (not in the picture) playing *Penguins on Ice*.

As we shifted our discussion toward how teachers might use games as a pedagogical tool, we were introduced to some ideas from Elizabeth Ellsworth (2005) that Janelle and her colleague, Marc Higgins, have applied to commercial games. Ellsworth works with pedagogically non-prescriptive texts (e.g., architecture) to examine how they might provoke deep, meaningful learning. She describes pedagogical pivots as places of learning that act “as the vehicles through which we come to know differently” (p. 1). Higgins and McFeetors (2019) agree with “game authors and scholars [who] underscore that games have long been both constructed and understood as a medium which is at odds with teaching and learning in linear and formalized ways and that rather follow children’s playful ways of knowing (Begy, 2017; de Koven, 2013; Faidutti, 2013)” (p. 92, citations in original). The challenge for teachers is to both consider how games might be considered pedagogical pivots and consider how they might incorporate games pedagogically into their classrooms.

We offered two quotes for consideration from Ellsworth (2005):

[Pedagogy] must create a relationship to the outside, to others, to the world, to history, and to the already thought in a way that keeps the future of what we make of that relation and what we might think there open and undecided. (p. 54)

Teachers, understood as being in the making themselves, would necessarily have to create places of learning in embodied terms and in ways that depart from the dominant perception of learning as the acquisition of knowledge driven by cognitive functions. (p. 28)

In recognizing that games may be a pedagogical pivot where “the mathematics is performatively co-constructed in the moment, through action, rather than being prescribed by the designer—making a predetermined, singular cognitive destination an impossibility when playing to learn” (Higgins & McFeetors, 2019, p. 106), we engaged in the following discussion.

WHAT MIGHT TEACHERS CONSIDER WHEN THEY WANT TO INCORPORATE GAME PLAY FOR MATHEMATICAL LEARNING?

Reflecting on this last game time, we have highlighted things to consider when teaching with mathematical games. We agreed that mathematical games can be used in other moments than a ‘Fun Friday.’ We offer here a collection of the ideas from the working group discussion and participants’ subsequent reflections.

Selection of games: The choice of game matters in sponsoring students’ mathematical learning. Susan suggested that selecting games that have limited rules may provide an opportunity for all students to enter the game. Like the selection of problems, Caroline noted that it can be challenging for teachers to evaluate and then select appropriate games. She also noticed that when there was some randomness within the play, it allows less strategic players to play and experience self-efficacy or success in their play against more experienced players.

Curricular connections: Depending on the game, the curricular connections may differ. David offered three ways to think about using games and connecting to the curriculum: (1) developing higher level competencies or mathematical processes; (2) practicing mathematical skills to become more fluent; and (3) learning mathematical content described in curricular outcomes. While the games we had played together fell into the first two categories, David designed a Base N Guessing Game that teaches students how to add and subtract in different bases and is intended to teach

something new. Additionally, Nat mentioned that rather than looking for milestones in their students linked to the curriculum, games could be used as an emerging path instead.

Intentionality: More than just using a game, Caroline pointed out that the value depends on what the teacher does with the game in the class. Sabrina built on this and offered that teachers can bring out, or emphasize, the mathematics within the play of the game. Nat cautioned about the temptation to often want to pre-teach the skills incorporated in a game. A parallel example Caroline offered was when manipulatives are used in mathematics class and then they are put away in the interest of doing mathematics. Rather, games (and manipulatives and good problems) can be used to *do* mathematics itself. Teachers might choose to incorporate games because they offer a good connection to problem solving as we have a need and a way to engage in the problems that games present.

Teaching moves: When implementing, there are particular pedagogical moves that teachers could both craft and implement to use games as a pedagogical pivot. Nat suggested that offering hints and extensions may elicit richer mathematical learning. In this way, teacher moves could propel the learning through games forward for students. David, in discussing argumentation as a social process, noted that teachers and students could engage in argumentation. In fact, as students challenge each other, it could help to develop a classroom environment of question-asking and expectation to support claims made. This could be supported by having pairs of students play as a team against another pair, as it “seems an important and natural format for using games to create opportunities for communication and reasoning.” Using tactfulness (knowing when or when not to), teachers could use “pausing” the game as a teacher moves to ask a student to anticipate the next several moves in the game. This often encourages analysis of the board and conjectures about possible moves. We briefly discussed how teachers could introduce games, as Janelle suggested three ways she has seen this done effectively: showing a video that explains game components and rules and then having a discussion; starting with teaching a small group in a centre; and having a teacher and student demonstrate a game to the class.

Across grades: Rather than relegating play to early childhood mathematics, Nat reminded us that Dan Finkel talks about how mathematicians play. We could encourage students of all ages to see how play can be mathematical. Post-secondary instructors in our working group shared how they are or could incorporate games even into their classes. Susan identified her *Math for Liberal Arts* and *Math for Elementary Teachers* as courses to incorporate games, while Amenda reflected “how important it is to incorporate ‘active learning’ (e.g., games) into university-level math classes.” We may be able to disrupt or challenge traditional views of mathematics and instructional approaches by incorporating games as pedagogical tools.

Preparing to use games: Wendy mentioned the importance of “careful planning...to provide students with a rich, meaningful mathematics experience” and “for in-service and pre-service teachers to engage in workshops similar to the working group where they can play games and determine how they could use this strategy in their practice. The workshop could be centered on developing tasks that include games that they have played.” Some participants commented that teachers need time to play the game and master it before implementing it in a classroom. Deep familiarity with the game provides teachers the opportunity for teachers to “ask the ‘right’ questions at the ‘right’ time [an allusion to Eleanor Duckworth]” (Susan) and to anticipate the strategies students may develop through their play.

Nous avons brièvement discuté des résultats de la recherche de Héroux (2023) sur les différentes postures qu'un enseignant peut prendre lors d'une séance de jeu mathématique. La posture de l'enseignant peut changer pendant une séance de jeu à différents moments. Il peut parfois être plus ludique (p. ex. joueur ou maître de jeu) et parfois plus mathématique (p. ex. pédagogique ou éducatif). Pour De Grandmont (1995), l'intention première d'un enseignant peut être le plaisir que l'on trouve dans le côté fictif d'un jeu mathématique. Sous cet angle, l'enseignant adopte une posture de « joueur ». Avec cette posture, l'enseignant peut agir comme un adversaire jouant au jeu mathématique contre un élève ou toute la classe. Nous avons aussi vu que l'enseignant peut être un joueur, mais ce joueur n'est pas tout à fait comme les autres. Tout en jouant, les enseignants utilisent des concepts mathématiques, des processus et des raisonnements comme tous les autres joueurs, mais avec des intentions particulières et une mise en évidence pour attirer l'attention des élèves sur son activité mathématique. En même temps, il n'est pas impossible qu'il se laisse emporter par le côté ludique.

Une posture de « maître de jeu » est un peu différente. Marinova (2016), parle de la façon dont l'enseignant peut se donner la tâche principale d'assurer le bon déroulement du jeu. Par exemple, l'enseignant peut être une sorte de « guide » lorsqu'il explique aux élèves les règles et les actions des jeux mathématiques. L'enseignant peut être un

« partenaire de jeu » en jouant une partie au cours de laquelle il explique certaines règles et actions tout en répondant aux questions des élèves sur certaines situations à éclaircir. L'enseignant qui adopte cette posture peut également être un « arbitre » qui intervient pour faire respecter les règles du jeu et apporter des éclaircissements sur les procédures du jeu ou lorsque des doutes de tricherie sont présents. La posture du maître de jeu est très axée sur l'aspect ludique, mais un peu moins que la posture du joueur puisque l'enseignant ne participe pas directement au jeu en étant plutôt observateur de la situation bien que ses interventions se rapportent au jeu. En expliquant les règles d'un jeu, nous avons vu que l'enseignant peut proposer ou travailler avec ses élèves des concepts mathématiques, des processus et des raisonnements. Comme être un maître de jeu signifie aussi être arbitre à l'occasion, l'enseignant intervient sur le travail mathématique des élèves afin de les valider ou de les corriger.

Selon De Grandmont (1995), l'enseignant peut adopter une posture « pédagogique » lorsqu'il fait référence à des concepts et à des processus pour sensibiliser les élèves à son activité mathématique. L'intention de l'enseignant est maintenant associée à l'apprentissage de nouvelles choses. Dans cette perspective, selon De Grandmont (ibid), la valeur éducative devrait rester imperceptible pour l'élève même si elle est dans l'esprit de l'enseignant. Par exemple, l'enseignant adopte une posture pédagogique, quand il quitte le jeu pour faire des rappels ou des parenthèses sur des aspects mathématiques. Sa position par rapport à l'activité mathématique est alors très claire et (on peut penser) pas si différente de ce qu'elle serait en dehors du contexte du jeu.

Enfin, De Grandmont (ibid) souligne que l'enseignant peut utiliser le jeu comme prétexte. Cette posture, liée à la précédente, est présentée comme « éducative ». Dans ce contexte, le jeu est orienté vers une réponse unique et il devient un exercice répétitif plus lassant puisqu'il est orienté vers un but à atteindre en perdant la caractéristique du plaisir intrinsèque. L'enseignant qui adopte une posture éducative observe le comportement et le fonctionnement des élèves pendant le jeu. Sous un angle éducatif, le jeu offre à l'enseignante un moyen de vérifier et de tester les compétences de ses élèves et le niveau d'acquisition des concepts et des procédures enseignés. La posture éducative diffère de la posture pédagogique dans les intentions de vérification des connaissances des élèves par l'enseignant. Nous pouvons également reconnaître une posture éducative et des liens avec l'activité mathématique lorsque l'enseignant choisit, par exemple, de ne pas intervenir ou de ne pas appliquer une règle du jeu afin de promouvoir le partage du processus ou du raisonnement.

Cette présentation des différentes postures des enseignants a également été l'occasion d'aborder différentes observations des interventions possibles de l'enseignant lors d'une séance de jeu mathématique en classe de nos travaux (Héroux, 2023). Nous avons noté que l'enseignant peut : ne pas être conscient en jouant de la présence d'un concept mathématique, intervenir spontanément, avoir préparé des interventions pour les élèves sur le concept mathématique ou vouloir laisser les élèves chercher un concept mathématique. L'utilisation de jeux mathématiques comprend une partie de spontanéité dans laquelle il faut être attentif aux éléments mathématiques, afin de garder l'équilibre entre le côté ludique et mathématique du jeu. Les interventions pour aborder les processus mathématiques sont parfois préparées par l'enseignant, parfois émergentes et même parfois préparées pour une non-intervention afin que les élèves les découvrent et que le rappel se fasse par le jeu. Les différentes façons d'aborder les processus mathématiques sont principalement choisies par l'enseignant avant la session de jeu selon ses intentions. Nous avons également remarqué que le raisonnement mathématique est généralement présent, mais peu ou pas mis en évidence lors des séances de jeux par l'enseignant. Avec le raisonnement mathématique, jouer en classe devient une occasion d'aborder un aspect mathématique qui est peut-être moins abordé dans les interventions de l'enseignant puisqu'il interroge peu ou pas les élèves. Le partage des stratégies au moment de la rétroaction est si naturel qu'il semble tout à fait possible d'éclairer le raisonnement mathématique lors du jeu.

FURTHER AREAS TO EXPLORE

Looking back over the three days, our working group generated suggestions for further areas of inquiry, possibilities for incorporating games in mathematics class, and issues that may arise in considering the use of games as a pedagogical tool.

We acknowledge that the types of games that Sabrina and Janelle use in their research and in classrooms with students tend to be card and board games. However, these are not the only types of games that could involve mathematical thinking. Amenda reflected that, “there are many other types of games—some outside like sports games, skipping, hopscotch, etc. and some using computers like video games or Nintendo. I argue that all require math to varying

degrees.” Susan suggested “extending our discussion to include video games, game apps, and puzzles online” to engage with the question “Is there something different that they offer?” We are looking forward to future discussions that incorporate a broader range of types of games and what we might learn from the various game contexts.

Perhaps in synchrony with Ellsworth’s (2005) notion of non-prescriptive pedagogical tools, using games offered a learning context that counters what students might encounter regularly and what teachers may read in curricular documents. David’s reflection on not knowing effective moves after one game of *Cloud City* bears this out, in that it is “perhaps a useful contrast to the way school math is often presented, that you solve the problem and it is over. You never need to think about it again. But every time you play a game it is an opportunity to learn a little bit more.” How might we encourage teachers and students to live suspended in a moment of play and in a moment of learning? As well, rather than a rush to prescribe learning through/with games in curriculums, we could work with teachers to look for ways that games can address curricular expectations. Many agreed that working with teachers to first understand a game well through sustained play and then developing intentional, thoughtful pedagogical approaches would support students’ learning.

Playfulness comes about not just because of the game play, but perhaps in synchrony with de Koven’s (2013) notion of a well-played game, that together we find ways to play, and the game is only an initial invitation to play. Nat mentioned that “there is a tension between the games (as boxed) and the game (as played). The rules, pieces, goal etc. all speak to a specific experience, but I noticed that there were moments when the game (as boxed) fell into the background in favour of some sort of lived game. We threw out the rules...in these moments, it felt the most playful.” This sense was related to how “the game had become collaborative...we started tinkering...we felt right to make the changes.” How might we encourage the students and teachers we work with to approach all of their mathematical doing in a playful way?

Noticing could be an act of a teacher in a mathematics classroom, but our experience around it also opened up ideas about how researchers might notice students’ mathematical thinking and learning in game play. Susan suggested three observable phenomena—physical (e.g., gesture, eye movement, posture), verbal, and affective (e.g., emotional response)—that also raised a wonder about how “thinking may be hidden...[so] how to elicit” that thinking from students would be a worthy endeavour for researchers. Caroline offered that another avenue of research would be to investigate what teachers do to incorporate games in meaningful ways by asking questions such as, « Comment l’enseignant a-t-il choisi le ou les jeux pour la séance ? À quel point est-il familier avec ce jeu ? Comment présente-t-il le ou les jeux à ses élèves ? Comment organise-t-il le tout (p. ex. pour un jeu qui se joue à deux, décider de faire jouer les élèves deux contre/avec deux) ? Quelles sont ses intentions (en lien avec le jeu lui-même, mais avec la manière dont il choisit de l’utiliser en classe) ? Que remarque-t-il lorsqu’il se promène dans la classe pendant que ses élèves jouent ? Quelles questions pose-t-il à ses élèves ? Vers quoi sont orientées ces questions (savoirs mathématiques mobilisés, visualisation, stratégie, anticipation, justification, argumentation...) ? Comment conclut-il la séance ? Quelles sont les suites qu’il entrevoit ? ... » This is a much less emphasized area of research and the field would benefit from attending to a teacher’s pedagogical approach with games.

WHAT QUESTIONS REMAIN?

- Of the skills that students learn and develop during game play, which are most explicitly useful/transferrable when learning non formal math topics from the curriculum?
- How can we support teachers choosing games appropriately and using them?
- How does participation measure against conceptual development and mathematics competence?

SUMMARY

One of the strengths of CMESG are conversations with colleagues which cause us to think more carefully and intentionally about an area of mathematics education that we have been working on with colleagues, teachers, and students! Our discussions were rich within the Working Group and challenged us to think about (1) the nature of mathematical learning and mathematical experiences students could have when playing carefully selected games; (2) how we might support teachers to plan intentionally for mathematical learning through incorporating games in class and how we might notice; and (3) how we might attend to and notice students’ mathematical engagement while they play and engage them in reflective conversations about their mathematical learning.

Additionally, throughout the conference, we had the opportunity to talk with colleagues who participated in other Working Groups about the games they play and how these games sponsor mathematical thinking. We are grateful for two new suggestions that we look forward to trying soon: *The Shipwreck Arcana* and *EvenQuads*.



Figure 15. The *Shipwreck Arcana* and *Even Quads* games.

How do you summarize the experiences and conversations of such an interactive working group? Our colleagues quickly suggested: play! So, during the closing ceremony, we invited everyone to play a quick round of *Set*. Each day a new collection is posted, so as you read this report, we invite you to play a round of *Set*! There was a buzz in the room as experienced *Set* players gave hints to novice players. Notice colleagues pointing at the screen to deictically gesture their selection of three cards for a set, others leaning in to keep their identification of a set secretive from a near-by pair, or other colleagues animatedly justifying their selection of a set.

We hope we have given you a glimpse of what we have done in our Working Group and piqued your curiosity. As is the true nature of a working group, we do not pretend to have explored all of the possibilities of students learning mathematics through game play. What about sudoku, puzzles, video games, apps, outdoor wide games, ...? We hope that you now see that games are legitimate to do mathematics from kindergarten to university.



Figure 16. Conference participants at the closing ceremony play *Set*.

REFERENCES

- Alberta Education. (2007/2016). *Mathematics: Kindergarten to grade 9*.
https://education.alberta.ca/media/3115252/2016_k_to_9_math_pos.pdf
- Alberta Education. (2008). *Mathematics: Grades 10 to 12*.
<https://education.alberta.ca/media/564028/math10to12.pdf>

- Alberta Education. (2023). *Mathematics*. <https://curriculum.learnalberta.ca/curriculum/en/s/mat>
- Begy, J. (2017). Board games and the construction of cultural memory. *Games and Culture*, 12(7–8), 718–738. <https://doi.org/10.1177/1555412015600066>
- “Bernie de Koven.” (n.d.). *Play and Playground Encyclopedia*. <https://www.pgpedia.com/d/bernie-dekoven>
- Bishop, A. J. (1988). Mathematics education in its cultural context. *Educational Studies in Mathematics*, 19(2), 179–191. <https://doi.org/10.1007/bf00751231>
- Bofferding, L., & Hoffman, A. (2019). Children’s integer understanding and the effects of linear board games: A look at two measures. *Journal of Mathematical Behavior*, 56, 1–20. <https://doi.org/10.1016/j.jmathb.2019.100721>
- Bragg, L. A. (2012). Testing the effectiveness of mathematical games as a pedagogical tool for children’s learning. *International Journal of Science and Mathematics Education*, 10, 1445–1467. <https://doi.org/10.1007/s10763-012-9349-9>
- British Columbia Ministry of Education. (2016/2018). *Mathematics*. <https://curriculum.gov.bc.ca/curriculum/mathematics>
- Brousseau, G. (2002). Les doubles jeux de l’enseignement des mathématiques. *Revue du Centre de Recherches en Éducation de l’Université de Saint Étienne*, (22–23), 85–155.
- Common Core State Standards Initiative. (2010). *Mathematics*. <https://www.thecorestandards.org/Math/>
- De Grandmont, N. (1995). Pédagogie du jeu : du normal au déficient. *Logiques*.
- de Koven, B. (2013). *The well-played game: A player’s philosophy*. The MIT Press.
- Dièness, Z. P. (1971). An example of the passage from the concrete to the manipulation of a formal system. *Educational Studies in Mathematics*, 3(3–4), 337–352. <https://doi.org/10.1007/BF00302302>
- Ellsworth, E. (2005). *Place of learning: Media, architecture, pedagogy*. Routledge.
- Elofsson, J., Gustafson, S., Samuelsson, J., & Traff, U. (2016). Playing number board games supports 5-year-old children’s early mathematical development. *Journal of Mathematical Behavior*, 43, 134–147. <https://doi.org/10.1016/j.jmathb.2016.07.003>
- Fonstad, C. (2016). *A narrative inquiry into curriculum-making experiences with games and puzzles in a mathematics class* [Unpublished master’s thesis]. University of Saskatchewan.
- Francis, K. (2019). Play and mathematics. *Journal of the Canadian Association for Curriculum Studies*, 17(1), 75–89.
- Héroux, S. (2023). *Étude exploratoire de l’activité mathématique lors de séances des jeux en classe du primaire* [thèse de doctorat]. Université du Québec à Montréal.
- Higgins, M., & McFeetors, P. J. (2019). Board games as play-full pedagogical pivots for STEM teaching and learning. *Journal of the Canadian Association for Curriculum Studies*, 17(1), 90–110.
- Jaques, S., Kim, B., Shyleyko-Kostas, A., & Takeuchi, M. A. (2019). “I just won against myself!”: Fostering early numeracy through boardgame play and redesign. *Early Childhood Education*, 46(1), 22–29.
- Kamii, C., & DeClark, G. (1985). *Young children reinvent arithmetic: Implications of Piaget’s theory*. Teachers College Press.
- Kim, B., Bastani, R., & Takeuchi, M. A. (2021). Embodied mathematical practices in (re)designing board games in a linguistically diverse classroom. *Pedagogies: An International Journal*, 18(2), 1–22. <https://doi.org/10.1080/1554480x.2021.2013232>
- Maida, M., & Maida, P. (2011). Problem solving around the corner. *Mathematics Teaching in the Middle School*, 16(8), 466–472.

- Marinova, K. (2016). Les jeux de règles et les apprentissages mathématiques. Dans K. Marinova et D. Biron (Dir.), *Mathématiques ludiques pour les enfants de 4 à 8 ans* (pp. 157–203). Presses de l'Université du Québec.
- Mason, J. (2002). *Researching your own practice: The discipline of noticing*. Routledge Falmer.
- McCoy, L., Buckner, S., & Munley, J. (2007). Probability games from diverse cultures. *Mathematics Teaching in the Middle School*, 12(7), 394–402.
- McFeetors, P. J., Jin, Q., & Ahmed, N. (2024, May). *Students' awareness of logical and spatial reasoning* [presentation]. Canadian Society for the Study of Education 2024 Annual Conference, Montreal, PQ.
- McFeetors, P. J., Marynowski, R., & Candler, A. (2021). Generative unit assessment: Authenticity in mathematics classroom assessment practices. *Education Sciences*, 11(7), article 366. <https://doi.org/10.3390/educsci11070366>
- McFeetors, P. J., & Mason, R. T. (2009). Learning deductive reasoning with games of logic. *Mathematics Teacher*, 103(4), 284–290.
- McFeetors, P. J., & Palfy, K. (2017). We're in math class playing games, not playing games in math class. *Mathematics Teaching in the Middle School*, 22(9), 534–544.
- McFeetors, P. J., & Palfy, K. (2018). Educative experiences in a games context: Supporting emerging reasoning in elementary school mathematics. *Journal of Mathematical Behavior*, 50, 103–125. <https://doi.org/10.1016/j.jmathb.2018.02.003>
- Ontario Ministry of Education. (2020). *Mathematics [grades K-6]*. King's Printer for Ontario. <https://www.dcp.edu.gov.on.ca/en/curriculum/elementary-mathematics>
- Ontario Ministry of Education. (2005, 2007, 2021). *Mathematics [grades 9-12]*. King's Printer for Ontario. <https://www.dcp.edu.gov.on.ca/en/curriculum/elementary-mathematics>
- Papert, S. (1981). *Mindstorms: Children, computers, and powerful ideas*. Basic Books.
- Ministère de l'Éducation du Québec. (2023). *Programme cycle de l'éducation préscolaire. Québec : l'auteur*. https://www.education.gouv.qc.ca/fileadmin/site_web/documents/education/jeunes/pfeq/Programme-cycle-prescolaire.pdf
- Ministère de l'Éducation du Québec. (2007). *Programme de formation de l'école québécoise : enseignement secondaire, deuxième cycle, mathématiques*. https://www.education.gouv.qc.ca/fileadmin/site_web/documents/education/jeunes/pfeq/PFEQ_mathematique-secondaire-deuxieme-cycle.pdf
- Ministère de l'Éducation du Québec. (2006). *Programme de formation de l'école québécoise : enseignement secondaire, premier cycle, mathématiques*. https://www.education.gouv.qc.ca/fileadmin/site_web/documents/education/jeunes/pfeq/PFEQ_mathematique-premier-cycle-secondaire.pdf
- Ministère de l'Éducation du Québec. (2001). *Programme de formation de l'école québécoise : éducation préscolaire et enseignement primaire*. <https://www.education.gouv.qc.ca/enseignants/pfeq/primaire/domaine-de-la-mathematique-de-la-science-et-de-la-technologie/mathematique>
- Pinter, K. (2010). Creating games from mathematical problems. *Primus*, 21(1), 73–90. <https://doi.org/10.1080/10511970902889919>
- Reeves, C., & Gleichowski, R. (2006/2007). Engaging contexts for the game of NIM. *Mathematics Teaching in the Middle School*, 12(5), 251–255.
- Reid, D. A. (2002). Describing reasoning in early elementary school mathematics. *Teaching Children Mathematics*, 9(4), 234–237.
- Skemp, R. (1993). *SAIL through mathematics: Structured activities for intelligent learning, an elementary school resource book*. EEC.

Su, F. (2020). *Mathematics for human flourishing*. Yale University Press.

Zaslavsky, C. (1998). *Math games and activities from around the world*. Chicago Review Press.

APPENDIX

INSTRUCTIONAL GAMES

Here is a non exhaustive suggestion list from Thomas Rajotte and Sabrina Héroux's book, *Le jeu en classe de mathématiques : engager activement les élèves et favoriser leur apprentissage*, published in 2021 with Éditions Chenelière.

Addition or multiplication pyramid	Happy families	Oh no! Not 20!
Capture the squares	I have... Who has ...?	Roll for 1 \$
Cover this figure	Matador	Spoon
Cross the sums or product	Math Bingo	Ten's complement memory game
Decimal number golf	Noggle	Three for me
Fraction dominoes	Objectif : Closer to 100!	

ANCIENT GAMES

Alquerque	Mū Tōrere	Senet
Hnefatafl	Nine Men's Morris/Merels	Shut the box/Referme les boîtes
Jiǎn-shízi/Nim	Oware/Mancala	Weiqi/Go
Mehen	Royal Game of Ur	

COMMERCIAL GAMES

7 Wonders	Guess Who?	Racoon
Archelino	Hex	River Crossing
Architecto	Hide & Seek	River Crossing Junior
Battle Sheep	Hive	Rummikub
Battleship	Ingenious	Rush Hour
Blokus	IQ Link	Rush Hour Junior
Blokus 3D	Kanoodle	Santorini
Blueprints	Kingdomino (series)	Sequence Numbers
Bohnanza	Kulami	Set
Camelot Junior	Labyrinth	Set Dice
Camouflage	Leaping Lizards	Settlers of Catan (series)
Carcassonne	Little Red Riding Hood	shapeOmetry
Castle Logix	Lost Cities	Shōbu
Chess	Make 7	Six
Chickyboom	Mancala	Skip-bo
Chocolate Fix	Mastermind	Sleeping Queens
Chroma Cube	Matching Madness	Smart Car 5x5
Chromino	Monopoly	Snakes & Ladders
Cir*Kis	My City	Splendor
Cloud City	NIM	Spot It!
Colorku	Noah's Ark	Squarrels
Crazy Campers	Nowhere to Go	Stone Age
Cribbage	One Up	Sumoku
Curious George Discovery Beach	Othello	Swish, Swish Jr
Da Vinci's Challenge	Outfoxed	Tangramino
Day or Night	Pack & Stack	Temple Run
Debockle	Paris Connection	Three Little Pigs
Doodle Dice	Pass the Pigs	Tic Stac Toe
Dragonwood	Patchwork	Tic Tac Toe
Dutch Blitz	Penguins on Ice	Ticket to Ride (series)
Dvonn	Pentago	Tiny Towns
Element	Perudo	TopThis!
Equilibrio	Phase 10	Ubongo
Even Steven's Odd	Pylos	Uno

CMESG/GCEDM Proceedings 2023 • Working Group Report

Farkle
For Sale
Freeze Frame
Go Nuts!
Gobbler Gobblers

Q-bitz
Quartex
Quatro
Qwirkle
Rack-o

Woolly Bully
Zendo
Zoo Logic

WHERE'S THE MATH? INQUIRING INTO EARLY YEARS MATHEMATICS CURRICULUM

OÙ SONT LES MATHÉMATIQUES? ENQUÊTE SUR LE PROGRAMME DE MATHÉMATIQUES DE LA PETITE ENFANCE

Lynn McGarvey, *University of Alberta*
Evan Throop-Robinson, *St. Francis Xavier University*

PARTICIPANTS

Ann Anderson
Geneviève Barabé
Athar Firouzian
Canan Güneş

Wes Maciejewski
Josh Markle
Ayse Pinar Sen
Elena Polotskaia

Annie Savard
Chad Williams

-----Français en suivent.-----

INTRODUCTION: OUR QUESTIONS

For the last few decades, the number of children in early childhood settings has continued to increase, and research has emphasized the impact of early learning on children's cognitive, social and emotional growth. As a result, mandatory early learning frameworks have sprung up in countries and jurisdictions around the world. Many of these documents include at least some attention to subject-matter curricula. We wondered, "Where's the Math?" in these curriculum frameworks. This is an interesting question given the historical tension between mathematics and early childhood. That is, common beliefs are that early mathematics should only focus on number skill development; should only be taught through play; should be integrated into all subjects rather than be stand alone; should be introduced implicitly to avoid anxiety; should emphasize concrete manipulatives and avoid abstraction; and should not take precedence over language and literacy (Balfanz, 1999; Clements & Sarama, 2018; Lee & Ginsburg, 2009). All of these 'shoulds' may leave little room for the mathematics.

In this Working Group, we explored the many questions that arise when considering mathematics for young children. That is, what do we believe about early learners and how they come to know mathematics in the world? What are the similarities and differences in mathematical activity for early learners across curricula? What does it mean to do mathematics for a 2-year-old? Or a Kindergarten student? What are the starting points in developing mathematics curricula for early learners? How do we navigate the tensions between a play-based and integrated approach to mathematics, and the possibilities of working on specific and precise mathematical concepts with these learners? Are number concepts the dominant content? If so, what are the missing, non-existent, and "hard to find" pieces? What guidance might we provide to educators, researchers and policymakers with regard to early curriculum frameworks? These questions, and many others were raised within our Working Group.

Over our three days together, we addressed the following three themes and questions within our Working Group:

1. Foundational Philosophy of Early Mathematics Learners. What do we believe about early learners?
2. Early Learning Frameworks and Curricula. Where and what is the mathematics?
3. Next steps and possible guidelines. What statement can we make and/or guidance can we give about early learning of mathematics in Canada?

Below we summarize our Working Group's inquiry and discussion into early years mathematics learning and curriculum based on each theme and question posed above.

PART 1: FOUNDATIONAL PHILOSOPHY OF EARLY MATHEMATICS LEARNERS. WHAT DO WE BELIEVE ABOUT EARLY LEARNERS?

ACTIVITY 1: GOOGLE IMAGE SEARCH

Our initial activity was to do a [Google image search](#) with the keywords: early math learner. The results of image searches often reveal the common assumptions, metaphors, and beliefs that internet users have of any topic. In our discussion we noted that there was a predominance of number related activities (e.g., counting, numeral recognition, simple equations), along with some measurement, 3D building with blocks, and constructing and deconstructing 2D shapes. Also noted was the extensive use of coloured blocks and other physical objects, unstructured play activities with those objects, and activity centres in school-like settings. One group asked Chat GPT about early math learners and the key points identified included identifying the developmental stages of numeracy concepts (e.g., counting, quantity, addition, subtraction, problem solving); emphasizing conceptual understanding over memorization, valuing play and play-based activities (e.g., games, puzzles), and including multiple visual representations (e.g., manipulatives, ten frames) for counting and computation. Other groups noted general resources that emphasized how to help children love math, reduce their anxiety, and engage them in activities that indirectly included mathematical content.

Through the Google image search activity, Working Group members identified both strengths and concerns regarding current beliefs about early mathematics learners that appear in the images, resources, and media involving how young children learn, what they are capable of learning, and experiences that support mathematics learning.

Using a Jamboard, Working Group members documented their response to the question: "What do we believe about early learners?" (Figure 1).



Figure 1. Jamboard summary of beliefs about early learners.

Working Group participants used many adjectives to describe early learners (e.g., capable, curious, creative, and imaginative). Repeated by many was the belief that early learners tend to like challenges and are capable of complex mathematical understanding. Participants generally believe that early learners can do high level mathematics and require many relevant experiences to develop complex mathematical concepts. Participants noted that early learners are explorers who need opportunities to share, discuss and question their understandings. The Working Group agreed that early learners bring unexpected and unusual perspectives to problems, tend to trust in their own strategies and can learn far beyond what adults believe they are capable of learning.

PART 2: WHERE AND WHAT IS THE MATHEMATICS?

ACTIVITY 2: MAKING THINKING VISIBLE

We began Day 2 with an activity and data arising from a Number Talks project with Grade 2 students as they make their thinking visible (Throop-Robinson et al., 2023). To begin, we asked the members of the Working Group to experience the same mental math process as the students in Grade 2. We asked our group to close their eyes and imagine the question $27 + 36$ and how they might solve it. As we took up possible solution strategies, we continued to ask: How do we make mental math strategies visible for learners? Our focus on making sense of students' ideas and recording their strategies visually caused us to ask continually: Is there evidence of quantity in the representation as students describe their process? We asked the group to consider the shift from paraphrasing students' ideas with words to "para-imaging" (B. Davis, personal communication, January 28, 2023) their ideas with pictures to show quantity. Para-imaging preserved the students' strategies visually and made their thinking visible for the whole class to see.

After engaging in the activity ourselves, we shared three audio recordings of students who took part in the Grade 2 classroom Number Talk. These recordings showed how two students solved the addition question, $27 + 36$, using different strategies while a third student recognized the connection between the two approaches.

Audio Clip #1: Maddy's Strategy

To solve $27 + 36$, Maddy chose to bring together the tens she sees in her mind as ten rods to make 50. She then takes 4 units away from the 7 and adds them to the 6 units in the second number to make another 10 rod. This gives her 6 tens or 60 to which she adds the remaining 3 units to make 63 (Figure 2).

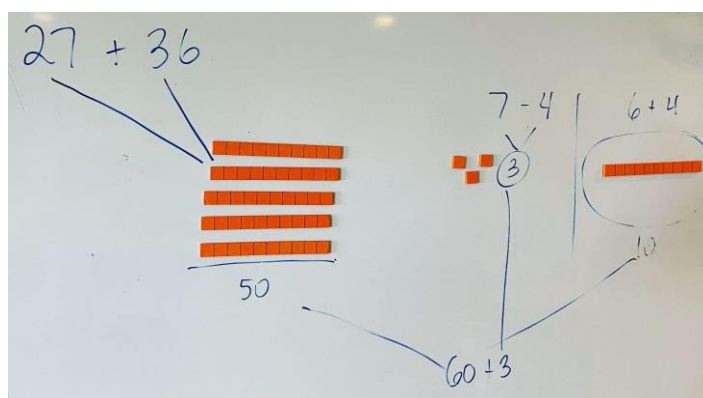


Figure 2. Maddy's mental math strategy made visible for the class.

Audio Clip #2: Yuvein's Strategy

Yuvein also approached the equation by decomposing the numbers by place value. He visually combines the 20 and 30 to make 50. He then combines the 7 units and the 6 units using a counting up strategy to make another 10 and then continues to make 13. He adds the 13 to the 50 mentally to make 63 (Figure 3).

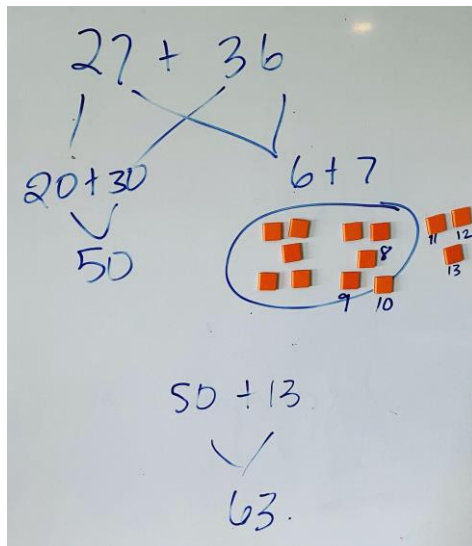


Figure 3. Yuvein's mental math strategy made visible for the class.

Audio clip #3: Abigail's connection

In a follow-up discussion with the class, Abigail noticed a connection between the two strategies as she observed the visual representations on the whiteboard to see which one was most like her thinking. Abigail observed that, instead of taking 4 away from 7 leaving 3 as Yuvein had done, she “took the 3 from the 6 and just added it to the 7 to make 10. That’s all.”

Working Group members discussed the value of making the students’ thinking visible for them to return to and compare with their own strategies. They discussed their own strategies for solving the mental math problem and appreciated how the young learners thought of different strategies. Working Group members noticed how listening to the other students’ ideas helped them understand their own processes more clearly and make connections between the processes to confirm their own mental math strategies.

ACTIVITY 3: EARLY CHILDHOOD ROUTINES

Early childhood classrooms are steeped in regular routines that support a predictable structure and establish classroom norms while promoting social, emotional, and cognitive growth. Routines include coming to and leaving school, taking attendance, play time, circle time, outdoor play, clean up time, and so on (Figure 4). Given that these routines encompass the majority of the time in an early learning classroom, we wondered where the mathematics might be in the day-to-day activities of a preschool classroom.

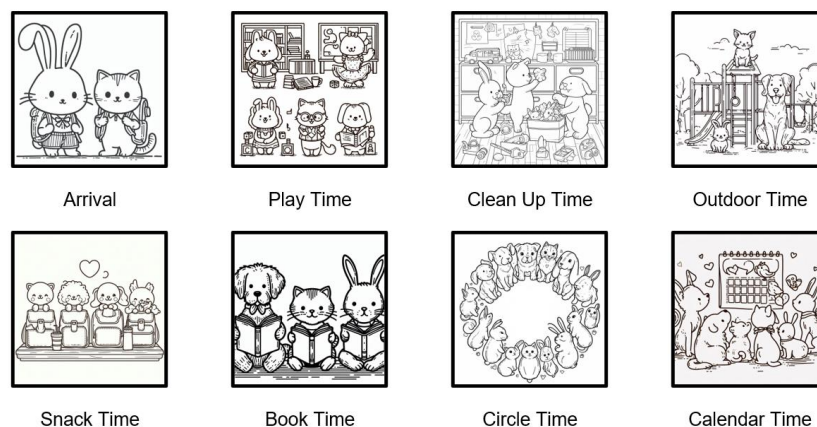


Figure 4. Sample schedule of routines in a preschool classroom. (AI images generated by MS Bing Image Creator).

In small groups we examined video segments of two preschool classrooms engaged in a range of routines from two YouTube videos: *Mrs Kilmer's 4 year old class picking up after playtime, snacks and learning the letter V* (OurSaviorMarlette, 2010); and *Morning Circle at Preschool* (Pennington, 2014). The routines examined in these two videos included the following:

- Cleaning up and lining up
- Snack time
- Book time
- Circle time
- Morning circle and calendar

Working Group members were asked to record the opportunities for mathematical thinking and learning within their assigned routine. Afterwards, we shared and described these opportunities with the whole group. While ‘mathematics’ was not specific to any routine, there were potentially many opportunities for the children to engage in mathematics during these routines including early number concepts (e.g., counting and comparing quantities, number recognition, ordinal numbers), geometry and spatial awareness (e.g., sorting and classifying 2D and 3D shapes), measurement (e.g., size comparisons, ordering), pattern recognition and creation, data analysis (e.g., data collection, frequency comparisons), mathematical vocabulary, and so on. However, while there was potential, language and literacy seemed to take precedence, and many opportunities for mathematics were not explicitly realized during these spaces.

ACTIVITY 4: ANALYSIS OF EARLY LEARNING CURRICULA AND FRAMEWORKS

In 2014, the Council of Ministers of Education in Canada (CMEC), developed the *CMEC Early Learning and Development Framework*. The purpose of the framework was to provide a “pan-Canadian vision for early learning” based on a common set of understandings and values in early childhood development (p. 4). As part of the commitment to high quality learning experiences in the early years, the framework emphasizes a “play-based” approach to learning. There is no reference to specific subject-area curriculum except in the principle, “Learning through play capitalizes on children’s natural curiosity and exuberance” (p.14). This principle outlines that play is the foundational source for skill development in literacy and numeracy, as well as other competencies such as problem solving.

One of the tensions we experienced when reviewing such a framework was that while there are many opportunities for children to engage in mathematical thinking in play, as was noted in the classroom routines analyzed previously, without specific attention to mathematics, it was possible that the mathematics would not be explicitly addressed in the same way that literacy skills often are, such as, “Learning the letter V” in the video above (OurSaviorMarlette, 2010). We contrasted the principles of the CMEC Framework to the many misconceptions that may be held by early childhood educators about mathematics learning for young children (Lee & Ginsburg, 2009):

1. *Young children are not ready for mathematics education.*
2. *Mathematics is for some bright kids with mathematics genes.*
3. *Simple numbers and shapes are enough.*
4. *Language and literacy are more important than mathematics.*
5. *Teachers should provide an enriched physical environment, step back, and let the children play.*
6. *Mathematics should not be taught as stand-alone subject matter.*
7. *Assessment in mathematics is irrelevant when it comes to young children.*
8. *Children learn mathematics only by interacting with concrete objects.*
9. *Computers are inappropriate for the teaching and learning of mathematics.* (p. 38)

In our discussion afterwards we highlighted misconceptions 3, 4, 5, 6, and 8, specifically. That is, without specific attention to mathematics, the focus on numeracy may be superficial and implicit, assumed to occur without adult interaction, and that mental math or mathematical abstractions may be assumed to be beyond the capabilities of young children.

Given the concerns about the implicit attention to mathematics in the framework, we then set out to analyze provincial early learning frameworks in response to the question: Where and what is the mathematics? In small groups, members were assigned a framework and presented their findings based on the following questions:

- What is the mathematics and/or numeracy included?

- What are the assumptions about how children learn mathematics?
- What mathematics is missing?
- Do you note any tensions between the assumptions made about young children and mathematics learning?

Each small group contributed a summary of findings to a shared Google Slides presentation. The highlights to the question, “What is the mathematics and/or numeracy included?” from each of the provincial frameworks are shown in Table 1.

Province/Territory	What and where is the mathematics?
Government of British Columbia, Ministry of Education. (2019). <i>Early learning framework</i> .	<ul style="list-style-type: none"> • Mathematics and number is explicitly referred to alongside other disciplines (e.g., dance, drama, science) within “Pathways for Communication and Literacies.” • Mathematics is referred to as an “expressive language” (p. 80) and numbers are a form of “vocabulary, symbols, and written language” (p. 83). • Mathematics is implicit in Pathways for “Others, Material and World” and part of reference to “multiple disciplines” (p. 76) with a specific example of building 3D structures.
Makovichuk, et al. (2014). <i>Flight. Alberta’s early learning and care framework</i> .	<ul style="list-style-type: none"> • Several references to mathematics, number, and topics such as shapes, and patterns appear throughout the document, usually alongside other disciplines (e.g., literacy, music, science). • Within “multimodal literacies” there is reference to “mathematics uses numbers” and “number games” to support symbol system development (p. 106).
Government of Saskatchewan, Ministry of Education. (2015). <i>Essential learning experiences. Supplement to exploration: Early learning program guide</i> .	<ul style="list-style-type: none"> • Mathematics and science are referred to within the “Intellectual development” domain with “exploring numeracy” as one of the essential experiences within that domain (p. 42). Emphasis is on counting, quantity, symbols, and numbers used in measurement and graphs. • Mathematics is implicit within the “solving problems” essential learning experience (p. 39).
Government of Manitoba. (n.d.). <i>Early returns: Early learning and child care curriculum framework for infant programs</i> .	<ul style="list-style-type: none"> • The document focuses extensively on learning through play. The terms numeracy and math are referred to once each. Both mentions are in relation to play and alongside other disciplines, such as, “provide children with a variety of play choices ... and opportunities to explore music, art, literacy and numeracy” (p. 13).
Best Start Expert Panel on Early Learning. (2007). <i>Early learning for every child today: A framework for early childhood settings</i> . [Ontario]	<ul style="list-style-type: none"> • The framework outlines learning domains from birth to school-age children. There are multiple implicit and explicit references to mathematics throughout. • In the cognition domain for infants and toddlers there is reference to spatial problem-solving (e.g., “stacking blocks”), sorting, and spatial exploration (e.g., “putting things together and taking them apart” (p. 39)). • For preschool/kindergarten, and school-age children, there are multiple explicit references to mathematical skills, including seriating, counting, quantity, shapes, patterns, and so on.
Gouvernement du Québec Ministère de l’Éducation. (2021). <i>Preschool cycle program</i> .	<ul style="list-style-type: none"> • Mathematics appears multiple times alongside other disciplines (e.g., languages, social sciences). • Within the “Cognitive development” emphasis on “thinking skills” there are specific examples for mathematics such as, “explore games involving counting, matching, classifying and logic”; “recognize small quantities”; “identify objects according to their position in space” (p. 50). • Mathematics is implicit in the “Physical and motor development” in reference to exploring concepts of “space” and “time.”

Department of Education and Early Childhood Development. (2019). <i>Navigating the early years: An early childhood learning framework</i> . [Newfoundland/ Labrador]	<ul style="list-style-type: none"> • “Early numeracy” is part of “Language Development” when “children hear the language of mathematics in play,” such as number songs, and comparisons (e.g., bigger and smaller). • Children experience mathematical concepts, such as “logic, comparison, classification, opposites, matching, shape recognition, measurement, and time,” and when using manipulatives (p. 40).
Best Start Panel on Early Learning. (2008). <i>Curriculum framework for early learning and child care</i> . [New Brunswick]	<ul style="list-style-type: none"> • Implicit notions of mathematics occur within “playful exploration and problem solving” whereby children engage in “playful exploration, investigation, and problem solving” (p. 34). They explore the “properties of objects” by noting similarities and differences, as well as creating patterns. • Mathematics, like language and music, is viewed as a “multimodal literacy” with a “sign system” (p. 44).
Government of Nova Scotia, Department of Education and Early Childhood Development. (2019). <i>Early learning curriculum framework and resources</i> .	<ul style="list-style-type: none"> • Reference to mathematics occurs within two learning goals: “Discovery and Invention,” and “Language and Communication.” Mathematics and other discipline specific subjects are implicit in the description of the goals through activities such as “problem-solving, inquiry, experimentation, hypothesizing, researching, and investigating” (p. 51). However, specific examples are provided within the strategies for achieving these goals such as the following: <ul style="list-style-type: none"> ○ “create and use representations to organize, record, and communicate mathematical ideas and concepts” (p. 63). ○ “use language to communicate thinking about quantities to describe attributes of objects and collections, and to explain mathematical ideas” (p. 64).
Flanagan, K. (2011). <i>Early learning framework: Relationships, environments, experiences</i> . [Prince Edward Island]	<ul style="list-style-type: none"> • “Problem solving and numeracy” is an explicit set of skills within the “Exploration and discovery” learning goal. Children are expected to playfully investigate: quantity, number symbols, and other concepts to solve problems that occur in their play. Guidance for supporting these skills are provided with examples of experiences such as “utilize a variety of kinesthetic methods to reproduce number symbols in writing, such as tracing number in sand, finger paint, etc.” (p. 107). • “Symbols and representations” are part of the “Expression and communication” learning goal. Within this section, there is reference to number symbols and using fingers to represent quantity.

Table 1.

The ten early learning frameworks we viewed varied considerably in format, intention, and function. With regard to our primary question, “Where and what is the mathematics?” we noted some similarities across the documents. After the presentations on each provincial early learning framework, we made several observations.

1. The frequency in which the terms associated with mathematics such as mathematics, numeracy, number, and shape, varied considerably across the documents. In many cases, mathematics was mentioned only a handful of times and usually alongside other disciplines such as science, language, or music. There was minimal explicit attention given to mathematical skills or processes.
2. Many of the documents referred to general competencies and processes such as problem solving, experimentation, inquiry, reasoning, and exploration. While there were occasionally examples that referred to mathematical ideas, for the most part, these competencies were not specific to mathematics. In many instances, these processes appeared to invoke ideas in science.
3. When concepts, skills, and processes in mathematics were identified, they were usually relegated to cognitive and communication domains. Learning mathematics did not seem to be relevant to other goals such as well-being and belonging, personal and social responsibility, and physical and motor development. Even within the realm of communication, attention to mathematics was with regard to the number symbols, rather than to expressions of thought or creativity.
4. The early learning frameworks were predominantly prepared by government ministries of family and social services, rather than education.

5. The early learning frameworks subscribed almost exclusively to a play-based philosophy of learning. While descriptions of play-based learning varied, it tended to emphasize activities selected and directed by the children.

This set of observations was somewhat concerning to us as mathematics educators. The implicit nature of mathematics within the documents means that early childhood educators need to have expertise in eliciting mathematical thinking within children's play; yet, may not have sufficient experience, background, or motivation to do so. This concern was evident in the set of early learning classroom routines we examined in Activity 3.

ACTIVITY 5: INTERNATIONAL CURRICULA AND FRAMEWORKS FOR EARLY LEARNING.

The provincial early learning frameworks frequently acknowledged being informed and influenced by the frameworks of other provinces and countries. Two frameworks from other countries referred to repeatedly were *Te Whāriki* (New Zealand Ministry of Education, 2017) and *Belonging, Being and Becoming: The Early Years Learning Framework* (Department of Education and Training (Australia), 2023). Although we did not have sufficient time to do a thorough exploration and analysis, we examined both documents briefly.

Both the New Zealand and Australian frameworks included explicit attention to mathematics skills and processes, particularly within outcomes related to exploration, problem solving, and communication. In the New Zealand curriculum it states, “while mathematics is explicit in communication and exploration, it is also implicit in other strands” (p. 52). An example of an explicit learning outcome within Communication is, “Recognising mathematical symbols and concepts and using them with enjoyment, meaning and purpose” (p. 42) with examples of how these outcomes may be observed by an early childhood educator. Within the mathematics and statistics learning area within the Exploration strand, “Students explore relationships in quantities, space and data and learn to express these relationships in ways that help them to make sense of the world around them” (p. 57). In the Australian early learning framework, mathematical ideas and concepts are noted within the goal that “Children develop a range of skills and processes such as problem solving, inquiry, experimentation, hypothesizing, research and investigating” (p. 38).

The working group discussions in Activities 4 and 5 highlighted the mathematics learning in the early years frameworks, what was missing for early learners in mathematics, and the possible tensions we experienced as mathematics educators. These discussions led to Part 3, and the goal to identify next steps and recommendations.

PART 3: NEXT STEPS AND POSSIBLE GUIDELINES. WHAT STATEMENTS CAN WE MAKE ABOUT EARLY LEARNING FRAMEWORKS IN CANADA?

Through the working group discussion on the final day, we identified five key points with regard to our shared beliefs about early learners in mathematics:

- Children can do and understand complex mathematics.
- They are curious, explorers, creative beings and sense makers.
- They like challenges and want to be challenged.
- They like learning on their own, with peers, and with and from adults.
- There are many opportunities to make mathematics visible and to include mathematics in daily activities/schedule/routines.

From this set of beliefs emerged the following shared understandings and recommendations for future consideration by curriculum developers and curriculum writers for early learning frameworks in Canada.

Through our discussions and explorations we highlighted two sets of understandings and recommendations. First, is to offer a vision of what mathematics is and can be (e.g., qualitative and quantitative mathematical practices) in the early learning environment. However, we recognize that for such a vision to be successfully implemented, it requires specific guidance for the early childhood educator in establishing a rich mathematical environment. Second, attending to mathematics in the early years more explicitly may mean a reconsideration of the emphasis on play-based pedagogy. Such a philosophy may be employed to the exclusion of learning with and from adults. Planning in the early years is of paramount importance, and both children and educators play a role in initiating activities for learning. As such, learning can occur through intentional teaching and teacher prompting and questioning without being akin to school-

like activities. As a working group, we recognized that our understandings and recommendations about and for early learning in Canada were still only a beginning.

INTRODUCTION : NOS QUESTIONS

Au cours des dernières décennies, le nombre d'enfants fréquentant des structures ou centres pour le développement de la petite enfance n'a cessé d'augmenter et la recherche a souligné l'impact de l'apprentissage précoce sur le développement cognitif, social et émotionnel des enfants. En conséquence, des cadres obligatoires d'apprentissage précoce ont vu le jour dans des pays et des juridictions du monde entier. De nombreux documents accordent au moins une certaine attention aux programmes d'enseignement des matières enseignées. Nous nous sommes demandé où étaient les mathématiques dans ces programmes. Cette question est intéressante compte tenu de la tension historique entre les mathématiques et la petite enfance. En effet, les croyances communes veulent que les mathématiques dans la petite enfance doivent se concentrer uniquement sur le développement des compétences numériques, être enseignées par le jeu, être intégrées à d'autres matières plutôt qu'isolées, être introduites implicitement pour éviter l'anxiété, l'accent doit être mis sur la manipulation concrète et éviter l'abstraction, et elles ne doivent pas avoir priorité sur le langage et la littératie (Balfanz, 1999 ; Clements & Sarama, 2018 ; Lee & Ginsburg, 2009). Tous ces « devoirs » peuvent laisser peu de place aux mathématiques.

Dans ce groupe de travail, nous avons exploré les nombreuses questions qui se posent lorsque les mathématiques sont envisagées pour les jeunes enfants. En d'autres termes, quelles croyances avons-nous au sujet des jeunes apprenants et de leurs manières d'apprendre et connaître les mathématiques dans le monde ? Quelles sont les similitudes et les différences entre les activités mathématiques destinées aux jeunes enfants dans les différents programmes scolaires ? Que signifie faire des mathématiques pour un enfant de 2 ans ? Ou pour un élève de maternelle ? Quels sont les points de départ de l'élaboration de programmes d'enseignement des mathématiques pour les élèves en bas âge ? Comment gérer les tensions entre une approche ludique et intégrée des mathématiques et les possibilités de travailler sur des concepts mathématiques spécifiques et précis avec ces apprenants ? Les concepts numériques constituent-ils le contenu dominant ? Dans l'affirmative, quels sont les éléments manquants, inexistantes ou « difficiles à trouver » ? Quelles orientations pourrions-nous donner aux éducateurs, aux chercheurs et aux décideurs politiques en ce qui concerne les cadres de programmes d'éducation pour l'enfance et la petite enfance ? Ces questions, et de nombreuses autres ont été soulevées au sein de notre groupe de travail.

Au cours de nos trois journées de travail, nous avons abordé les trois thèmes et questions suivants au sein de notre groupe de travail :

1. Philosophie fondamentale des apprenants précoces en mathématiques. Quelles sont nos convictions sur les apprenants précoces ?
2. Cadres et programmes d'apprentissage des jeunes enfants. Où sont les mathématiques et quelles sont-elles ?
3. Prochaines étapes et lignes directrices possibles. Quelle déclaration pouvons-nous faire et/ou quels conseils pouvons-nous donner au sujet de l'apprentissage précoce des mathématiques au Canada ?

Nous résumons ci-dessous l'enquête et la discussion de notre groupe de travail sur l'apprentissage des mathématiques et les programmes d'études pour les jeunes enfants en fonction de chaque thème et de chaque question posée ci-dessus.

PARTIE 1 : PHILOSOPHIE FONDAMENTALE DE L'APPRENTISSAGE DES MATHÉMATIQUES CHEZ LES JEUNES ENFANTS. QUE CROYONS-NOUS AU SUJET DES APPRENANTS PRÉCOCES ?

ACTIVITÉ 1 : RECHERCHE D'IMAGES SUR GOOGLE

Notre première activité consistait à effectuer une recherche d'images sur Google avec les mots clés « early math learner » (apprenant précoce en mathématiques). Les résultats des recherches d'images révèlent souvent les hypothèses, métaphores et croyances communes des internautes sur un sujet donné. Au cours de notre discussion, nous avons constaté une prédominance des activités liées aux nombres (par exemple, le comptage, la reconnaissance des chiffres, les équations simples), ainsi que quelques mesures, des constructions en 3D avec des blocs, et la

construction et la déconstruction de formes en 2D. L'utilisation intensive de blocs de couleur et d'autres objets physiques, les activités de jeu non structurées avec ces objets et les centres d'activités dans des environnements de type scolaire ont également été notés. Un groupe a demandé au Chat GPT de se pencher sur les jeunes apprenants en mathématiques. Les points clés identifiés sont les suivants : identifier les stades de développement des concepts de numération (par exemple, compter, quantité, addition, soustraction, résolution de problèmes) ; mettre l'accent sur la compréhension conceptuelle plutôt que sur la mémorisation, valoriser le jeu et les activités basées sur le jeu (par exemple, les jeux, les puzzles) ; et inclure de multiples représentations visuelles (par exemple, le matériel de manipulation, les cadres de dix) pour le comptage et le calcul. D'autres groupes ont mentionné des ressources générales qui mettent l'accent sur la manière d'aider les enfants à aimer les mathématiques, de réduire leur anxiété et de les faire participer à des activités qui incluent indirectement un contenu mathématique.

Grâce à l'activité de recherche d'images sur Google, les membres du groupe de travail ont identifié à la fois des points forts et des préoccupations concernant les croyances actuelles sur les jeunes apprenants en mathématiques. Ces croyances apparaissent dans les images, les ressources et les médias et concernent la manière dont les jeunes enfants apprennent, ce qu'ils sont capables d'apprendre et les expériences qui soutiennent l'apprentissage des mathématiques.

À l'aide d'un Jamboard, les membres du groupe de travail ont documenté leur réponse à la question « Que croyons-nous au sujet des apprenants précoces ? » (figure 1).

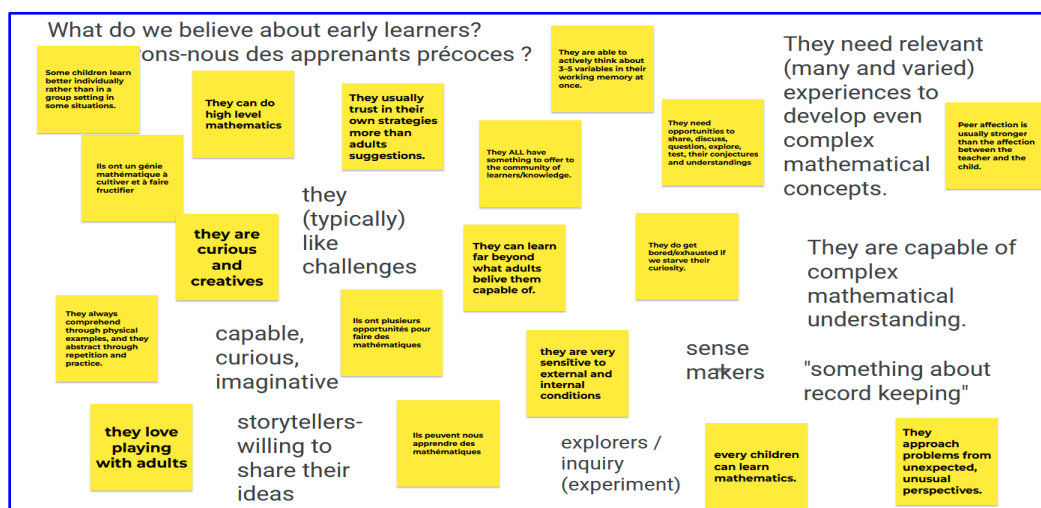


Figure 1. Résumé du Jamboard sur les croyances concernant les apprenants précoces.

Les participants du groupe de travail ont utilisé de nombreux adjectifs pour décrire les apprenants précoces (par exemple, capables, curieux, créatifs et imaginatifs). De nombreux participants ont répété que les élèves en bas âge ont tendance à aimer les défis et sont capables d'une compréhension mathématique complexe. Les participants croient généralement que les élèves en bas âge peuvent faire des mathématiques de haut niveau et qu'ils ont besoin de nombreuses expériences pertinentes pour développer des concepts mathématiques complexes. Les participants ont noté que les apprenants précoces sont des explorateurs qui ont besoin d'occasions de partager, de discuter et de remettre en question leur compréhension. Le groupe de travail a convenu que les élèves en bas âge apportent des perspectives inattendues et inhabituelles aux problèmes, ont tendance à faire confiance à leurs propres stratégies et peuvent apprendre bien au-delà de ce que les adultes pensent qu'ils sont capables d'apprendre.

PARTIE 2 : OÙ ET QUE SONT LES MATHÉMATIQUES ?

ACTIVITÉ 2 : RENDRE LA PENSÉE VISIBLE

Nous avons commencé la deuxième journée par une activité et des données issues du projet Number Talks réalisé avec des élèves de deuxième année visant à rendre visibles leurs raisonnements (Throop-Robinson et al., 2023). Pour commencer, nous avons demandé aux membres du groupe de travail de faire l'expérience du même processus de calcul mental que les élèves de deuxième année. Nous avons demandé à notre groupe de fermer les yeux et d'imaginer

la question $27 + 36$ et la manière dont ils pourraient y répondre. Tout en examinant les stratégies de résolution possibles, nous avons continué de nous demander : comment rendre les stratégies de calcul mental visibles pour les apprenants ? Notre souci de donner un sens aux idées des élèves et d'enregistrer visuellement leurs stratégies nous a amenés à nous poser continuellement les questions suivantes : « Y a-t-il une preuve de quantité dans la représentation de la question ? La représentation des élèves décrivant leur processus fait-elle apparaître une quantité ? » Nous avons demandé au groupe de réfléchir au passage de la paraphrase des idées des élèves avec des mots à la « para-imagerie » (B. Davis, communication personnelle, janvier 28, 2023) de leurs idées avec des images pour visualiser les quantités. La para-imagerie présente les stratégies des élèves sur le plan visuel et rend leur réflexion visible pour l'ensemble de la classe.

Après avoir participé nous-mêmes à l'activité, nous avons partagé trois enregistrements audio d'élèves qui ont pris part à la discussion sur les nombres en classe de 2^e année. Ces enregistrements montrent comment deux élèves ont résolu la question de l'addition $27 + 36$ en utilisant des stratégies différentes, tandis qu'un troisième élève a reconnu le lien entre les deux approches.

Extrait audio n° 1 : la stratégie de Maddy

Pour résoudre $27 + 36$, Maddy a choisi de rassembler les dizaines qu'elle voit dans son esprit sous la forme de barres de 10 pour obtenir 50. Elle retire ensuite 4 unités du 7 et les ajoute aux 6 unités du deuxième nombre pour obtenir une autre barre de 10. Elle obtient ainsi 6 dizaines, soit 60, auxquelles elle ajoute les 3 unités restantes pour obtenir 63 (figure 2).

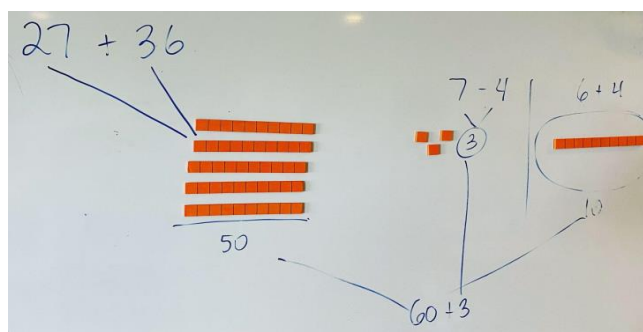


Figure 2. La stratégie de calcul mental de Maddy est visible pour la classe.

Extrait audio n° 2 : La stratégie de Yuvein

Yuvein a également abordé l'équation en décomposant les nombres en fonction de la valeur de position. Il combine visuellement les 20 et les 30 pour obtenir 50. Il combine ensuite les 7 unités et les 6 unités en utilisant une stratégie de comptage pour obtenir 10 unités supplémentaires, puis il continue pour obtenir 13. Il ajoute mentalement le 13 au 50 pour obtenir 63 (figure 3).

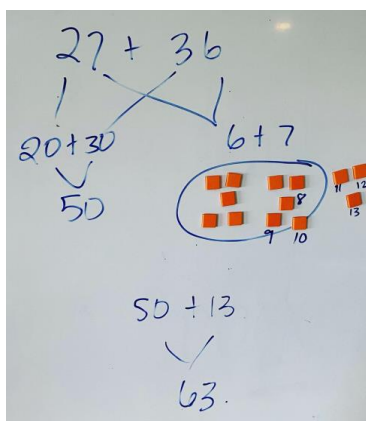


Figure 3. La stratégie de calcul mental de Yuvein rendue visible pour la classe.

Extrait audio n° 3 : Le lien d'Abigail

Lors d'une discussion de suivi avec la classe, Abigail a remarqué un lien entre les deux stratégies en observant les représentations visuelles sur le tableau blanc pour voir laquelle correspondait le plus à son raisonnement. Abigail a observé qu'au lieu d'enlever 4 à 7 en laissant 3 comme l'avait fait Yuvein, elle « a pris le 3 du 6 et l'a juste ajouté au 7 pour faire 10. C'est tout ».

Les membres du groupe de travail ont discuté de l'intérêt de rendre visible le raisonnement des élèves pour qu'ils puissent y revenir et le comparer à leurs propres stratégies. Ils ont discuté de leurs propres stratégies pour résoudre le problème de calcul mental et ont apprécié la façon dont les jeunes apprenants ont pensé à des stratégies différentes. Les membres du groupe de travail ont remarqué que le fait d'écouter les idées des autres élèves les aidait à mieux comprendre leurs propres processus et à établir des liens entre les processus pour confirmer leurs propres stratégies de calcul mental.

ACTIVITÉ 3 : ROUTINES DE LA PETITE ENFANCE

Les classes de la petite enfance sont imprégnées de routines régulières qui soutiennent une structure prévisible et établissent les normes de la classe tout en favorisant le développement social, émotionnel et cognitif. Les routines comprennent l'arrivée et le départ de l'école, la prise des présences, l'heure du jeu, l'heure du cercle, le jeu à l'extérieur, l'heure du nettoyage, etc. (figure 4). Étant donné que ces routines représentent la majeure partie du temps passé dans une classe d'éducation préscolaire, nous nous sommes demandé où se trouvaient les mathématiques dans les activités quotidiennes d'une classe préscolaire.

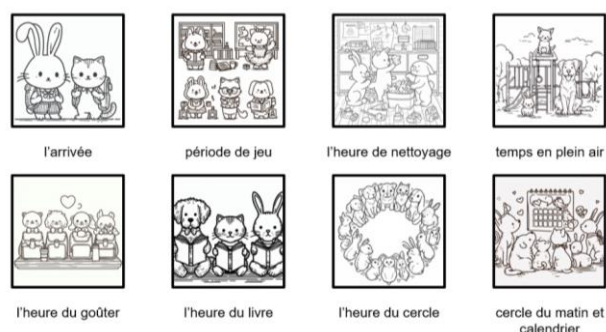


Figure 4. Exemple de routine dans une classe de maternelle (images générées par l'IA MS Bing Image Creator).

En sous-groupes, nous avons examiné des séquences vidéo de deux classes préscolaires engagées dans une série de routine tirée de deux vidéos YouTube : *Mrs. Kilmer's 4-year-old class picking up after playtime, snacks and learning the letter V* (OurSaviorMarlette, 2010); et *Morning Circle at Preschool* (Pennington, 2014). Les routines examinées dans ces deux vidéos sont les suivantes :

- Nettoyage et mise en rang
- L'heure du goûter
- L'heure du livre
- L'heure du cercle
- Cercle du matin et calendrier

Les membres des groupes de travail ont été invités à noter les occasions de réflexion et d'apprentissage mathématiques dans le cadre de la routine qui leur était assignée. Ensuite, nous avons partagé et décrit ces possibilités avec l'ensemble du groupe. Bien que les « mathématiques » ne soient pas spécifiques à chaque activité, les enfants avaient potentiellement de nombreuses occasions de faire des mathématiques au cours de ces activités, notamment les premiers concepts numériques (par exemple, compter et comparer des quantités, reconnaître les nombres, les nombres ordinaux), la géométrie et le sens spatial (par exemple, trier et classer des formes 2D et 3D), la mesure (par exemple, comparer des tailles, ordonner), la reconnaissance et la création de modèles, l'analyse de données (par exemple, la collecte de données, les comparaisons de fréquence), le vocabulaire mathématique, etc. Cependant, bien qu'il y ait eu

potentiel, la langue et l'alphabétisation semblaient avoir la priorité, et de nombreuses opportunités de faire des mathématiques n'ont pas été explicitement exploitées dans ces espaces.

ACTIVITÉ 4 : ANALYSE DES PROGRAMMES ET DES CADRES D'APPRENTISSAGE DES JEUNES ENFANTS

En 2014, le Conseil des Ministres de l'Éducation du Canada (CMEC) a élaboré le *Early Learning and Development Framework*. L'objectif de ce cadre était d'offrir une vision pancanadienne pour l'apprentissage des jeunes enfants fondée sur un ensemble de connaissances et de valeurs communes en matière de développement de la petite enfance (p. 4, traduction de l'auteur). Dans le cadre de l'engagement en faveur d'expériences d'apprentissage de haute qualité au cours des premières années, le cadre met l'accent sur une approche ludique de l'apprentissage. Il n'y a pas de référence à un programme d'études spécifique, sauf dans un principe suivant expliquant que l'apprentissage par le jeu tire parti de la curiosité et de l'exubérance naturelles des enfants (p. 14, traduction de l'auteur). Ce principe souligne que le jeu est la source fondamentale du développement des compétences en matière de lecture, d'écriture, de calcul, ainsi que d'autres compétences telles que la résolution de problèmes.

L'une des tensions que nous avons ressenties lors de l'examen d'un tel cadre était que, bien qu'il y ait de nombreuses occasions pour les enfants de s'engager dans la réflexion mathématique dans le jeu, comme cela a été noté dans les routines de classe analysées précédemment, sans une attention particulière aux mathématiques, il était possible que les mathématiques ne soient pas explicitement abordées de la même manière que les compétences en littératie le sont souvent, par exemple « Apprendre la lettre V » dans cette vidéo (OurSaviorMarlette, 2010). Nous avons opposé les principes du cadre du CMEC aux nombreuses idées fausses que peuvent avoir les éducateurs de la petite enfance sur l'apprentissage des mathématiques pour les jeunes enfants (Lee & Ginsburg, 2009, p.39, traduction de l'auteur) :

1. *Les jeunes enfants ne sont pas prêts pour l'enseignement des mathématiques.*
2. *Les mathématiques sont réservées aux enfants brillants qui ont des gènes mathématiques.*
3. *Des chiffres et des formes simples suffisent.*
4. *Le langage et l'alphabétisation sont plus importants que les mathématiques.*
5. *Les enseignants devraient fournir un environnement physique enrichi, prendre du recul et laisser les enfants jouer.*
6. *Les mathématiques ne doivent pas être enseignées comme une matière à part entière.*
7. *L'évaluation en mathématiques n'est pas pertinente lorsqu'il s'agit de jeunes enfants.*
8. *Les enfants n'apprennent les mathématiques qu'en interagissant avec des objets concrets.*
9. *Les ordinateurs ne sont pas adaptés à l'enseignement et à l'apprentissage des mathématiques.*

Dans la discussion qui a suivi, nous avons mis en évidence les idées fausses 3, 4, 5, 6 et 8, en particulier. En d'autres termes, si aucune attention particulière n'est accordée aux mathématiques, l'accent mis sur la numératie peut être superficiel et implicite, il est censé se produire sans l'interaction d'un adulte, et le calcul mental ou les abstractions mathématiques peuvent être considérés comme dépassant les capacités des jeunes enfants.

Compte tenu des préoccupations concernant l'attention implicite portée aux mathématiques dans le cadre, nous avons ensuite entrepris d'analyser les cadres provinciaux d'apprentissage des jeunes enfants en réponse à la question suivante : Où et que sont les mathématiques ? En sous-groupes, les membres se sont vus attribuer un cadre et ont présenté leurs conclusions en se basant sur les questions suivantes :

- Qu'est-ce qui est inclus dans les mathématiques et/ou la numératie ?
- Quelles sont les hypothèses sur la façon dont les enfants apprennent les mathématiques ?
- Quelles sont les mathématiques manquantes ?
- Constatez-vous des tensions entre les hypothèses émises sur les jeunes enfants et l'apprentissage des mathématiques ?

Chaque sous-groupe a présenté un résumé de ses conclusions dans une présentation Google Slides partagée. Le tableau 1 présente les points saillants des réponses à la question « Quelles sont les mathématiques et/ou les notions de calcul incluses ? »

Province/Territoire	Quelles sont les mathématiques et où se trouvent-elles ?
Government of British Columbia, Ministry of Education. (2019). <i>Early learning framework</i> . traduction de l'auteur	<ul style="list-style-type: none"> Les mathématiques et les nombres sont explicitement mentionnés au même titre que d'autres disciplines (par exemple, la danse, l'art dramatique, les sciences) dans les « Pathways for Communication and Literacies ». Les mathématiques sont considérées comme un langage expressif (p. 80) et les nombres sont une forme de vocabulaire, de symboles et de langage écrit (p. 83). Les mathématiques sont implicites dans les parcours « Others, Material and World » et font partie de la référence aux disciplines multiples (p. 76) avec un exemple spécifique de construction de structures en 3D.
Makovichuk, et al. (2014). <i>Flight. Alberta's early learning and care framework</i> . traduction de l'auteur	<ul style="list-style-type: none"> Plusieurs références aux mathématiques, aux nombres et à des sujets tels que les formes et les modèles apparaissent tout au long du document, généralement aux côtés d'autres disciplines (par exemple, l'alphabétisation, la musique, la science). Dans le cadre de la littératie multimodale, il y a référence aux mathématiques qui utilisent les nombres et aux jeux de nombres pour soutenir le développement du système de symboles (p. 106).
Government of Saskatchewan, Ministry of Education. (2015). <i>Essential learning experiences. Supplement to exploration: Early learning program guide</i> . traduction de l'auteur	<ul style="list-style-type: none"> Les mathématiques et les sciences sont mentionnées dans le domaine du développement intellectuel avec l'exploration de la numératie comme une des expériences essentielles (p. 42). L'accent est mis sur le comptage, les quantités, les symboles et les nombres utilisés dans les mesures et les graphiques. Les mathématiques sont implicites dans l'expérience d'apprentissage de la résolution de problème (p. 39).
Government of Manitoba. (n.d.). <i>Early returns: Early learning and child care curriculum framework for infant programs</i> . traduction de l'auteur	<ul style="list-style-type: none"> Le document met largement l'accent sur l'apprentissage par le jeu. Les termes calcul et mathématiques sont mentionnés une fois chacun. Les deux mentions sont en relation avec le jeu et avec d'autres disciplines, comme offrir aux enfants une variété de choix de jeux...et des occasions d'explorer la musique, l'art, la littératie et la numératie (p. 13).
Best Start Expert Panel on Early Learning. (2007). <i>Early learning for every child today: A framework for early childhood settings</i> . [Ontario] traduction de l'auteur	<ul style="list-style-type: none"> Le cadre décrit les domaines d'apprentissage de la naissance à l'âge scolaire. Il contient de multiples références implicites et explicites aux mathématiques. Pour les nourrissons et les enfants en bas âge, le domaine de la cognition fait référence à la résolution de problèmes spatiaux (par exemple, empiler des blocs), au tri et à l'exploration spatiale (par exemple, assembler et démonter des objets (p. 39). Pour les enfants d'âge préscolaire et scolaire, il existe de nombreuses références explicites aux compétences mathématiques, y compris la sériation, le comptage, la quantité, les formes, les modèles, etc.
Gouvernement du Québec Ministère de l'Éducation. (2021). <i>Preschool cycle program</i> . traduction de l'auteur	<ul style="list-style-type: none"> Les mathématiques apparaissent à plusieurs reprises aux côtés d'autres disciplines (langues, sciences sociales, etc.). Dans la section sur le développement cognitif, qui met l'accent sur les capacités de réflexion, on trouve des exemples spécifiques de mathématiques tels qu'explorer des jeux impliquant le comptage, l'appariement, la classification et la logique ; reconnaître de petites quantités ; identifier des objets en fonction de leur position dans l'espace (p. 50). Les mathématiques sont implicites dans le développement physique et moteur en référence à l'exploration des concepts d'espace et de temps.
Department of Education and Early Childhood Development. (2019). <i>Navigating the early years: An early childhood learning framework</i> . [Newfoundland/ Labrador]	<ul style="list-style-type: none"> La numératie précoce fait partie du développement du langage lorsque les enfants entendent le langage des mathématiques dans le jeu (p.35), par exemple dans les chansons de comptage et les comparaisons (plus grand et plus petit).

traduction de l'auteur	<ul style="list-style-type: none"> Les enfants rencontrent des concepts mathématiques, tels la logique, la comparaison, la classification, les contraires, l'appariement, la reconnaissance des formes, la mesure et le temps (p. 40)
Best Start Panel on Early Learning. (2008). <i>Curriculum framework for early learning and child care</i> . [New Brunswick] traduction de l'auteur	<ul style="list-style-type: none"> Des notions implicites de mathématiques apparaissent dans le cadre de l'exploration ludique et de la résolution de problèmes, où les enfants s'engagent dans l'exploration ludique, la recherche et la résolution de problèmes (p. 34). Ils explorent les propriétés des objets en notant les similitudes et les différences, et en créant des modèles. Les mathématiques, comme le langage et la musique, sont considérées comme une littératie multimodale avec un système de signes (p. 44).
Government of Nova Scotia, Department of Education and Early Childhood Development. (2019). <i>Early learning curriculum framework and resources</i> . traduction de l'auteur	<ul style="list-style-type: none"> Les mathématiques sont évoquées dans le cadre de deux objectifs d'apprentissage : « Discovery and Invention » et « Language and Communication ». Les mathématiques et d'autres disciplines spécifiques sont implicites dans la description des objectifs par le biais d'activités telles que la résolution de problèmes, la recherche, l'expérimentation, l'émission d'hypothèses, la recherche et l'investigation (p. 51). Cependant, des exemples spécifiques sont fournis dans les stratégies permettant d'atteindre ces objectifs, comme les suivants : <ul style="list-style-type: none"> créer et utiliser des représentations pour organiser, enregistrer et communiquer des idées et des concepts mathématiques (p. 63). utiliser le langage pour communiquer sa réflexion sur les quantités, pour décrire les attributs des objets et des collections et pour expliquer les idées mathématiques (p. 64).
Flanagan, K. (2011). <i>Early learning framework: Relationships, environments, experiences</i> . [Prince Edward Island] traduction de l'auteur	<ul style="list-style-type: none"> La résolution de problèmes et la numération constituent un ensemble explicite de compétences dans le cadre de l'objectif d'apprentissage « Exploration and discovery ». On attend des enfants qu'ils étudient de manière ludique les quantités, les symboles numériques et d'autres concepts pour résoudre les problèmes qui se posent dans le cadre de leurs jeux. Des conseils pour soutenir ces compétences sont fournis avec des exemples d'expériences tels qu'utiliser une variété de méthodes kinesthésiques pour reproduire les symboles des nombres par écrit, comme tracer un nombre dans le sable, la peinture au doigt, etc. (p. 107). Les symboles et représentations font partie de l'objectif d'apprentissage « Expression and communication ». Dans cette section, il y a référence aux symboles numériques et à l'utilisation des doigts pour représenter les quantités.

Tableau 1.

Les dix cadres d'apprentissage que nous avons consultés varient considérablement en termes de formats, d'intentions et de fonctions. En ce qui concerne notre question principale « Où et quelles sont les mathématiques ? », nous avons noté certaines similitudes entre les documents. Après les présentations de chaque cadre provincial d'apprentissage des jeunes enfants, nous avons fait plusieurs observations.

1. La fréquence des termes associés aux mathématiques, tels que mathématiques, calcul, nombre et forme, varie considérablement d'un document à l'autre. Dans de nombreux cas, les mathématiques n'ont été mentionnées qu'une poignée de fois et généralement aux côtés d'autres disciplines telles que les sciences, les langues ou la musique. Les compétences ou les processus mathématiques ne font l'objet que d'une attention explicite minimale.
2. De nombreux documents font référence à des compétences et à des processus généraux tels que la résolution de problèmes, l'expérimentation, la recherche, le raisonnement et l'exploration. Bien que certains exemples fassent parfois référence à des idées mathématiques, la plupart du temps, ces compétences ne sont pas spécifiques aux mathématiques. Dans de nombreux cas, ces processus semblaient faire appel à des idées scientifiques.
3. Lorsque des concepts, des compétences et des processus mathématiques ont été identifiés, ils ont généralement été relégués aux domaines de la cognition et de la communication. L'apprentissage des

mathématiques ne semblait pas pertinent par rapport à d'autres objectifs tels que le bien-être et l'appartenance, la responsabilité personnelle et sociale, et le développement physique et moteur. Même dans le domaine de la communication, l'attention portée aux mathématiques concernait les symboles numériques plutôt que l'expression de la pensée ou de la créativité.

4. Les cadres d'apprentissage précoce ont été principalement préparés par les ministères de la famille et des services sociaux, plutôt que par les ministères de l'éducation.
5. Les cadres d'apprentissage précoce souscrivent presque exclusivement à une philosophie d'apprentissage basée sur le jeu. Bien que les descriptions de l'apprentissage par le jeu varient, elles tendent à mettre l'accent sur des activités choisies et dirigées par les enfants.

Cette série d'observations nous a quelque peu inquiétés en tant qu'enseignants en mathématiques. La nature implicite des mathématiques dans les documents signifie que les éducateurs de la petite enfance doivent avoir l'expertise nécessaire pour susciter la réflexion mathématique dans le jeu des enfants ; or, ils n'ont peut-être pas l'expérience, les préalables ou la motivation suffisants pour le faire. Cette préoccupation était évidente dans l'ensemble des routines des salles de classe d'apprentissage précoce que nous avons examinées dans l'activité 3.

ACTIVITÉ 5 : PROGRAMMES ET CADRES INTERNATIONAUX POUR L'APPRENTISSAGE PRÉCOCE

Les cadres provinciaux d'apprentissage précoce ont souvent reconnu avoir été informés et influencés par les cadres d'autres provinces et pays. Deux cadres d'autres pays ont été mentionnés à plusieurs reprises : *Te Whāriki* (Ministère de l'éducation de la Nouvelle-Zélande, 2017) et *Belonging, Being and Becoming: The Early Years Learning Framework* (Ministère de l'éducation et de la formation [Australie], 2023). Bien que nous n'ayons pas eu suffisamment de temps pour effectuer une exploration et une analyse approfondies, nous avons examiné brièvement ces deux documents.

Les cadres néo-zélandais et australien accordent une attention explicite aux compétences et aux processus mathématiques, en particulier dans les résultats liés à l'exploration, à la résolution de problèmes et à la communication. Le programme scolaire néo-zélandais indique que « si les mathématiques sont explicites dans la communication et l'exploration, elles sont également implicites dans d'autres domaines » (p. 52, traduction de l'auteur). Un exemple de résultat d'apprentissage explicite dans le domaine de la communication est « Reconnaître les symboles et les concepts mathématiques et les utiliser avec plaisir, sens et objectif » (p. 42, traduction de l'auteur), avec des exemples de la manière dont ces résultats peuvent être observés par un éducateur de la petite enfance. Dans le domaine d'apprentissage des mathématiques et des statistiques du volet Exploration, « les élèves explorent les relations entre les quantités, l'espace et les données et apprennent à exprimer ces relations de manière à donner un sens au monde qui les entoure » (p. 57, traduction de l'auteur). Dans le cadre australien d'apprentissage précoce, les idées et concepts mathématiques sont mentionnés dans l'objectif suivant : « Les enfants développent une série de compétences et de processus tels que la résolution de problèmes, la recherche, l'expérimentation, l'émission d'hypothèses, la recherche et l'investigation » (p. 38, traduction de l'auteur).

Les discussions des groupes de travail dans le cadre des activités 4 et 5 ont mis en évidence l'apprentissage des mathématiques dans les cadres de référence pour la petite enfance, ce qui manquait aux apprenants précoces en mathématiques et les éventuelles tensions que nous avons rencontrées en tant qu'enseignants de mathématiques. Ces discussions ont conduit à la partie 3 et à l'objectif d'identifier les prochaines étapes et les recommandations.

PARTIE 3 : PROCHAINES ÉTAPES ET LIGNES DIRECTRICES POSSIBLES. QUELS CONSTATS POUVONS-NOUS FAIRE SUR LES CADRES D'APPRENTISSAGE PRÉCOCE AU CANADA ?

Les discussions des groupes de travail le dernier jour ont permis d'identifier cinq points clés concernant nos convictions communes sur les apprenants précoces en mathématiques :

- Les enfants peuvent faire et comprendre des mathématiques complexes.
- Ils sont curieux, explorateurs, créatifs et créateurs de sens.
- Ils aiment les défis et veulent être stimulés.
- Ils aiment apprendre par eux-mêmes, avec leurs pairs et avec des adultes.

- Il existe de nombreuses possibilités pour rendre les mathématiques visibles et pour les inclure dans les activités/programmes/routines quotidiens.

De cet ensemble de convictions sont nées les compréhensions communes et les recommandations suivantes à l'intention des concepteurs et des rédacteurs de programmes d'études pour les cadres d'apprentissage des jeunes enfants au Canada.

Nos discussions et explorations nous ont permis de mettre en évidence deux séries d'idées et de recommandations. La première consiste à proposer une vision de ce que sont et peuvent être les mathématiques (par exemple, les pratiques mathématiques qualitatives et quantitatives) dans l'environnement d'apprentissage précoce. Cependant, nous reconnaissons que pour qu'une telle vision soit mise en œuvre avec succès, il faut que l'éducateur de la petite enfance reçoive des conseils spécifiques pour créer un environnement mathématique riche. Deuxièmement, le fait de s'intéresser de manière plus explicite aux mathématiques dans les premières années de la vie peut impliquer de reconsidérer l'importance accordée à la pédagogie basée sur le jeu. Une telle philosophie peut être employée à l'exclusion de l'apprentissage avec et par les adultes. La planification dans les premières années est d'une importance capitale, et tant les enfants que les éducateurs jouent un rôle dans la mise en place d'activités d'apprentissage. Ainsi, l'apprentissage peut se faire par le biais d'un enseignement intentionnel, de l'incitation et du questionnement de l'enseignant, sans pour autant s'apparenter à des activités de type scolaire. En tant que groupe de travail, nous avons reconnu que notre compréhension et nos recommandations concernant l'apprentissage précoce au Canada n'étaient qu'un début.

REFERENCES

- Balfanz, R. (1999). Why do we teach young children so little mathematics? Some historical considerations. In J. V. Copley (Ed.), *Mathematics in the early years* (pp. 3–10). NAEYC/NCTM.
- Best Start Expert Panel on Early Learning. (2007). *Early learning for every child today: A framework for Ontario early childhood settings*. Ontario Ministry of Education.
- Clements, D., & Sarama, J. (2018). Myths of early math. *Education Sciences*, 8(2), 71. <https://doi.org/10.3390/educsci8020071>
- Council of Ministers of Education, Canada / CMEC Early Childhood Learning and Development Working Group. (2014). *CMEC early learning and development framework*. CMEC. <https://www.cmec.ca/Publications/Lists/Publications/Attachments/327/2014-07-Early-Learning-Framework-EN.pdf>
- Department of Education and Early Childhood Development. (2019). *Navigating the early years: An early childhood learning framework*. Government of Newfoundland/Labrador.
- Department of Education, Employment and Training. (2023). *Belonging, being and becoming: the early years learning framework for Australia*. Council of Australian Governments.
- Early Childhood Research and Development Team. (2008). *Curriculum framework for early learning and child care*. Department of Social Development, Government of New Brunswick.
- Flanagan, K. (2011). *Early learning framework: Relationships, environments, experiences*. Prince Edward Island Department of Education and Early Childhood Development.
- Gouvernement du Québec, Ministère de l'Éducation (2021). *Preschool cycle program*. Author.
- Government of British Columbia, Ministry of Education. (2019). *British Columbia Early learning framework*. Author.
- Government of Manitoba (n.d.). *Early returns: Early learning and child care curriculum framework for infant programs*. Author.
- Government of Nova Scotia, Department of Education and Early Childhood Development. (2019). *Early learning curriculum framework and resources: Nova Scotia's Early Learning Curriculum Framework*. Author.

- Government of Saskatchewan, Ministry of Education (2015). *Essential learning experiences. Supplement to exploration: Early learning program guide*. Author.
- Lee, J.S. & Ginsburg, H.P. (2009). Early childhood teachers' misconceptions about mathematics education for young children in the United States. *Australasian Journal of Early Childhood*, 34(4), 37-45. doi.org/10.1177/183693910903400406
- Makovichuk, L., Hewes, J., Lirette, P., & Thomas, N. (2014). *Flight. Alberta's early learning and care framework*. <https://www.flightframework.ca/>
- New Zealand Ministry of Education. (2017). *Te Whāriki. He whāriki mātauranga mō ngā mokopuna o Aotearoa. Early childhood curriculum*. New Zealand Government.
- OurSaviorMarlette. (2010, Mar. 6). *Mrs Kilmer's 4 year old class picking up after playtime, snacks and learning the letter V* [Video]. YouTube. <https://www.youtube.com/watch?v=GIJh3zCK6Pg>
- Pennington, J. L. [JuJuBeans]. (2014, Jan. 28). *Morning circle at preschool* [Video]. YouTube. https://www.youtube.com/watch?v=PZY-hB2C_Iw
- Throop-Robinson, E., Husband, M., & Little, M. (2023, Spring/Summer). Making mental math strategies visible during a number talk: Challenges and benefits. *Mathematics Matters: A Professional Newsletter for Nova Scotia Mathematics Teachers*, 9-10. <http://mta.nstu.ca/images/mta/MTANewsletterJun2023.pdf>

Topic Sessions

Séances thématiques

HOT DOGS AND MATHEMATICS EDUCATION: A CAREER SANDWICHED BETWEEN RESEARCH AND PRACTICE

Nat Banting

After an invitation to deliver a topic session at CMESG 2023 compelled me to reflect on the influences from both research and practice that had led me to that moment, I eventually arrived at a single, foundational aphorism as particularly potent for designing occasions for, partaking in, and observing mathematical activity in my classroom.

In a topic session designed to be one-half participatory, one-quarter poetic, one-eighth theatrical, one-sixteenth absurd, one-thirty-second provocative, etc., etc., I discussed this guiding dictum, its humble culinary origins, and how my mathematics classroom hinges on my ability to take it seriously.

DISCLAIMER

This topic session was unique in the sense that it involved stage lighting, audio cues, several props, and timed video clips—all facilitated through a fifty-six-stanza poem. Modeled after the high school mathematics classroom where I spend most of my time, it was designed to be ever-changing, unpredictable, and—thanks to the students that I am privileged to teach—enjoyable. In many ways, the session was a performance, and it might remain a ‘had to be there’ moment despite my best efforts to communicate its thesis through this hopelessly static medium of print.

To do so, stanzas of the poem that was used to guide the participants through the session have been placed in juxtaposition and run parallel to one another. The goal is to contrast the (apparently serious) study of quadratic growth with the timeless (and unquestionably serious) debate of whether a hot dog meets the minimum requirements to be considered a sandwich.

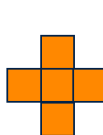
As both debates rage on simultaneously, we arrive eventually at a moment of synergy where it is proposed that the quest to define sandwichness can teach us an important lesson for the teaching and learning of mathematics. Namely; that *the solution to a math problem is an argument*.

TWO PARALLEL PROMPTS

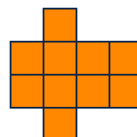
Is a hotdog a sandwich?



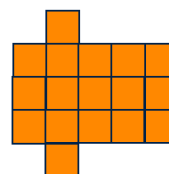
How many orange blocks in the 20th stage?



Stage 1



Stage 2



Stage 3

IS A HOT DOG A SANDWICH?

The root of my whole problem is what a sandwich is depends
On how a sandwich is defined, and how far that view extends.

Can lasagna be a sandwich? Quesadillas are a must!
You see, it soon becomes more nuanced than a filling and some crust.

Can we all agree on classics like the turkey bacon club?
And do we dare bestow the name to just any Subway sub?

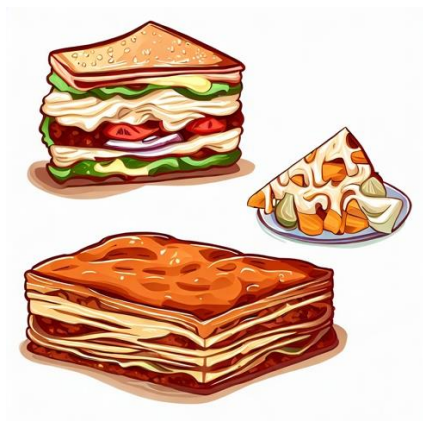


Figure 1.

I've spent years debating options, none of which has cleared the fog,
Of what constitutes a sandwich, take, for instance, the hot dog.

Any starch that holds a filling. That's a sandwich—so I thought,
But the bun remains one piece, and so, for some, that means it's not.

But if the bread can't be one piece, then no wraps would qualify.
And what about the taco? Can a sandwich be deep-fried?

HOW MANY ORANGE BLOCKS IN THE 20TH STAGE?

Now in a common classroom, after all the work is through,
A solution is declared. In this case, 442.

All who got that number, are assumed to know quadratics,
Then the teacher helps the others who cannot do mathematics.

While you may have boxed your number, as all doting students do,
If we learn something from hot dogs, the answer's not 442.

By this I'm not suggesting that's the wrong number of blocks,
But a solution is an argument, not a number—It's the thoughts

That compose a student's process. That's where mathematics thrives,
A student does mathematics when a student justifies.

Four hundred forty-two doesn't show me *how* you know,
You need to show your argument. How'd you see it grow?

Perhaps you sliced it sideways, freed the middle with two cuts?
A rectangle grows with every stage, and the two blocks are untouched.

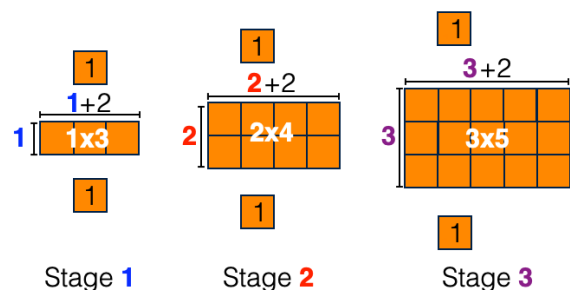


Figure 2.



Figure 3.

In some cases, there's agreement, and in others, cause for war,
And some people you once liked, perhaps, are called
your friends no more.

Can a sandwich have no opening, trapping filling—
cold or warm?
Can something lose its sandwichness if its starch
becomes deformed?

Some say filling must be solid; they reject liquids and
gels,
And some contend, sandwichness depends on how the
dog is held.

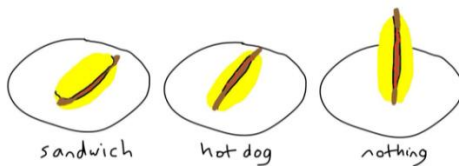


Figure 6.

A POINT OF MERGER

Consider how you acted when you justified your claims.
Your attention tried to validate every sandwich that you named.

Simply stating your position didn't quell any dissent.
You provided your solution in the form of arguments.

I've found that mathematics class lacks this same outlook,
And solutions live as static things—the appendices of books.

Perhaps you saw a perfect square, and cut in this
design?
The square's now paired with sections that both grow
by one each time.

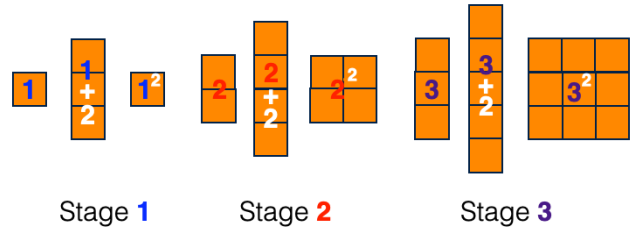


Figure 4.

Perhaps you moved some blocks to complete the
larger square?
Now the square's growth is what matters, and the one
block is just there.

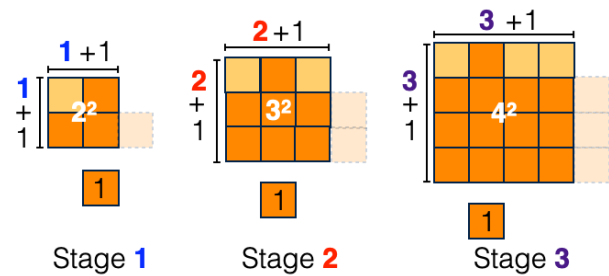


Figure 5.

These answers are not numbers, they persuade with
arguments,
And mathematics is composed of these ways we
maintain sense.

CMESG/GCEDM Proceedings 2023 • Topic Sessions Report

The process is defined by the teacher's set of steps,
And the students are unscathed if they follow each request.

When solutions become endpoints, we've ignored the argument,
But if you call something a sandwich, you had best provide defence.

I now know that student argument is the focus of my job,
A fact I owe, quite earnestly, to time spent with hot dogs.

As they math, with numbers and graphs, my teacher gaze stays keen.
And attention once paid to ends and grades, now better paid to means.

It's the hot dog that reminds us of this necessary switch,
That *solutions are not numbers, but, rather, arguments*.

That's where mathematics lives, not in numbers boxed on tests,
The teacher's job is to foment them and pursue them without rest.

See, mathematics is a process, that's the thesis of my pitch,
So, the next time you sit down for lunch, give thanks to your sandwich.

REPRÉSENTER OU NE PAS REPRÉSENTER—EST-CE LA QUESTION ? DÉVELOPPEMENT DE LA PENSÉE MATHÉMATIQUE À L'ÉCOLE PRIMAIRE.

REPRESENT OR NOT REPRESENT—IS THIS THE QUESTION? MATHEMATICAL THINKING DEVELOPMENT IN ELEMENTARY SCHOOL.

Elena Polotskaia
Université du Québec en Outaouais

ABSTRACT

Representing or using manipulatives while solving word problems is very popular in elementary school. Probably, the great majority of teachers accept that this way, the student will better understand and better learn mathematics. However, many of those teachers would be confused if asked other questions. What should be represented and why, at what moment, by whom, and how this representation can be used to foster mathematical thinking? I would like to share my thoughts and ideas about how the relational paradigm transforms the word, problem solving, into a powerful teaching tool promoting the development of students' mathematical thinking.

RÉSUMÉ

Représenter ou utiliser du matériel de manipulation lors de la résolution de problèmes écrits est très populaire à l'école primaire. Probablement, la grande majorité des enseignants acceptent que, de cette façon, l'élève comprenne et apprenne mieux les mathématiques. Cependant, bon nombre de ces enseignants seraient confus si on leur posait des questions supplémentaires. Que doit-on représenter et pourquoi, à quel moment, comment cette représentation peut-elle être utilisée pour alimenter la pensée mathématique ? J'aimerais partager mes réflexions et mes idées sur la façon dont le paradigme relationnel transforme la résolution de problèmes écrits en un outil pédagogique puissant favorisant le développement de la pensée mathématique des élèves.

INTRODUCTION

La représentation ou l'utilisation de manipulables lors de la résolution de problèmes est très populaire à l'école élémentaire. Probablement, la grande majorité des enseignants acceptent que, de cette manière, l'élève comprenne et apprenne mieux les mathématiques. Cependant, beaucoup de ces enseignants seraient confus s'ils étaient interrogés sur d'autres questions. Que doit-on représenter et pourquoi, à quel moment ? Pourquoi et comment ces représentations peuvent-elles être utilisées pour favoriser la pensée mathématique ?

Le domaine de la recherche en éducation mathématique et d'autres domaines connexes fournissent de nombreuses idées qui peuvent potentiellement clarifier les choses pour les enseignants et les aider à gérer efficacement l'apprentissage des élèves à travers des activités de résolution de problèmes. Il existe tant d'idées merveilleuses dans la littérature que huit pages ne seront pas suffisantes pour mentionner toutes les sources. Malheureusement, dans ce texte, je n'en mentionnerai que quelques-unes. Dans ce qui suit, je voudrais résumer quelques réflexions et idées sur la manière dont le paradigme relationnel peut transformer la résolution de problèmes en un puissant outil pédagogique favorisant le développement de la pensée mathématique des élèves.

HISTOIRE VRAIE N1

Cette histoire se déroule dans une école élémentaire urbaine. En tant que chercheuse, j'observe une leçon consacrée à la résolution de problèmes. L'enseignante utilise un tableau blanc interactif pour présenter de petits problèmes écrits à ses élèves, un à la fois. Les élèves travaillent sur des tableaux blancs effaçables. La tâche est la suivante : résoudre le problème et laissez des traces de votre réflexion.

Voici l'un des problèmes : Anne a 2 ans de plus que son frère. Anne a 6 ans. Quel âge a le frère ?

Il faut quelques minutes aux élèves pour proposer des solutions :

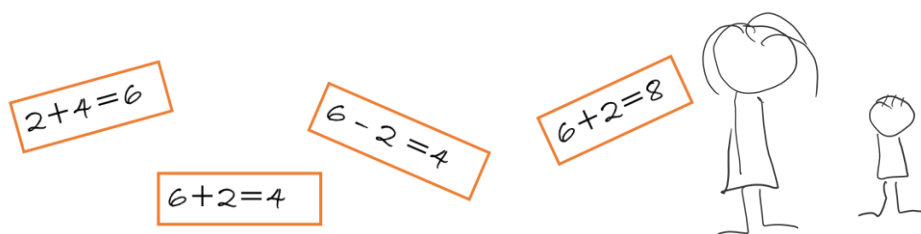


Figure 1. Traces des élèves pour le problème d'Anne.

L'enseignante choisit une solution qui semble correcte : $6 - 2 = 4$. Cependant, elle réalise que de nombreux élèves ne comprennent pas pourquoi la soustraction est utilisée dans cette solution. Elle décide d'expliquer et recommence à lire le problème. En lisant, elle insiste sur l'expression « plus âgée ». À ce moment précis, l'enseignante se rend compte que l'expression « plus âgée » ne correspond pas à l'opération de soustraction. Après un court silence, elle dit : j'ai choisi un mauvais problème.

HISTOIRE VRAIE N2

Curieuse au sujet des nouvelles technologies, j'ai demandé à Chat GPT (v 3.5) d'expliquer le même problème mathématique.

- Elena : Que doit faire l'enseignant pour aider les élèves à comprendre le problème ?
- Chat GPT : Pour résoudre ce problème, nous devons comprendre les informations qui nous sont données. Tout d'abord, nous savons qu'Anne a 2 ans de plus que son frère. Cela signifie que si nous ajoutons 2 ans à l'âge de son frère, nous obtiendrons l'âge d'Anne.
- Chat GPT : Maintenant, nous savons qu'Anne a 6 ans. Si nous soustrayons 2 ans de son âge, nous obtiendrons l'âge de son frère.
- Chat GPT : Pour expliquer cela plus clairement aux élèves, l'enseignant pourrait utiliser des dessins ou des exemples concrets pour illustrer la situation. Par exemple, l'enseignant pourrait dessiner deux personnages, l'un représentant Anne et l'autre représentant son frère, et placer des étiquettes d'âge sur chacun.

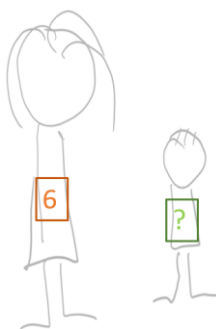


Figure 2. Personnages du problème.

Bien sûr, notre enseignante et Chat GPT—la connaissance collective du monde—sont d'accord sur le fait que pour mieux comprendre un problème mathématique, des représentations visuelles sont indispensables. Cependant, nous pouvons constater que cette « stratégie d'enseignement » à elle seule n'est pas très utile en classe. Ni les mots « plus âgée », ni l'image de deux personnes ne permettent d'expliquer pourquoi la soustraction devrait être utilisée. Probablement, d'autres questions devraient également être abordées :

1. Pourquoi les élèves devraient-ils résoudre des problèmes mathématiques ?
2. Que signifie résoudre un problème ?
3. Quel type de pensée mathématique aide à comprendre un problème ?
4. Que doit-on représenter ?
5. Quel est le rôle et quelle est la place d'une représentation dans le processus d'apprentissage ?

QUELQUES RÉFLEXIONS THÉORIQUES POUR RÉPONDRE AUX 5 QUESTIONS.

OBJECTIF DE RÉOLUTION

Au début du XX^e siècle, le principal objectif de l'apprentissage des mathématiques à l'école était le développement cognitif et l'apprentissage de la pensée (Mukhopadhyay & Greer, 2001). Même si d'autres objectifs ont été formulés depuis, le développement de la pensée mathématique reste le plus important. Ainsi, la réponse à notre première question est la suivante : L'objectif principal de la résolution de problèmes mathématiques dans les salles de classe à l'école primaire est le développement de la pensée mathématique.

Si la réponse numérique n'est pas le but de la résolution, *une solution* recherchée par cet exercice peut être plutôt une pensée spécifique qui aide à résoudre le problème, par exemple une stratégie de calcul, une équation, une manière de résoudre l'équation. De plus, dans l'apprentissage, la résolution d'un problème devrait produire une pensée réutilisable et transférable, donc plutôt une manière de trouver une stratégie de calcul ou une équation. En d'autres termes, le but ultime de résolution de problèmes écrits doit être la construction d'une *pensée transférable*, utile pour résoudre d'autres problèmes plus complexes. Par exemple, Nesher et al. (1982) explique ainsi le niveau 4 du développement de connaissances sur les structures additives :

« Directional (ordered) descriptions (i.e., 'more', 'less') can be handled in a flexible fashion. The arithmetic at this level includes the ability to handle inequality, and the ability to equalize inequality by addition or subtraction. » (p. 382)

On peut confirmer donc que la flexibilité de la pensée sur les structures quantitatives soit l'objectif principal des exercices de résolution. Dans le cas du problème de Anne, une *pensée flexible* doit permettre à l'élève de transformer l'expression « Anne est 2 ans plus âgée que son frère » en une opération de soustraction.

PENSÉE RELATIONNELLE, THÉORIQUE, FLEXIBLE, TRANSFÉRABLE

Pour répondre à la troisième question, Davydov (2008) fait la distinction entre la pensée théorique et la pensée empirique. La pensée empirique utilise des caractéristiques observables pour classer ou décrire l'objet. La pensée théorique examine la structure interne, invisible de l'objet pour en révéler et expliquer le fonctionnement. Dans le cas du problème d'Anne, la pensée théorique serait la compréhension de la structure mathématique de la situation, en d'autres termes, la relation entre l'âge d'Anne, l'âge du frère et la différence donnée—« 2 ans plus âgée ». Plus concrètement, l'apprenant devrait comprendre que l'âge d'Anne est lié à l'âge du frère d'une manière spéciale : la comparaison additive de deux quantités révèle une différence. L'apprenant peut également comprendre que chacune des trois quantités dans cette situation joue un rôle spécifique dans la relation. L'âge d'Anne est la plus grande quantité des deux comparées, l'âge du frère est la plus petite quantité et 2 ans est l'écart entre les deux autres (la différence). Selon Davydov (1982), seule cette sorte de pensée à propos du problème—pensée relationnelle—assure une flexibilité, soutient une résolution réussie, et produit une connaissance transférable.

Davydov (1982) suggère que la relation que nous avons identifiée dans le problème d'Anne soit étudiée et comprise par les apprenants dès le début, avant l'étude formelle des nombres et opérations sur les nombres. Dans nos travaux, mes collègues et moi avons développé davantage cette idée et proposé plusieurs outils pédagogiques pour soutenir la pensée relationnelle des élèves lors de la résolution de problèmes (Polotskaia et al., 2019, 2023, elenapolotskaia.com). Nous avons reformulé l'objectif de la résolution d'un problème écrit comme étant *d'identifier les relations quantitatives impliquées et les utiliser pour élaborer une stratégie de calcul*. Nous appelons cette emphase sur les

relations le « paradigme relationnel dans l'enseignement et l'apprentissage des mathématiques » (Polotskaia et Savard, 2018).

REPRÉSENTATION

Revenant à la question des représentations, il semble logique de proposer que les *dessins*, les *traces de pensée*, etc., devraient représenter la structure mathématique du problème ou un schéma des relations quantitatives impliquées. En représentant la relation, l'apprenant peut exprimer sa compréhension visuellement, la communiquer et la discuter avec les autres, et l'utiliser pour construire une stratégie de calcul (Figure 3).

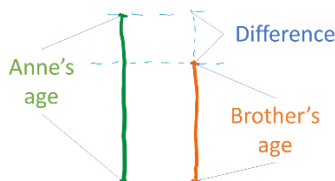


Figure 3. Schéma relationnel du problème d'Anne.

Ce type de représentations offre de nombreuses caractéristiques didactiques utiles. Tout d'abord, on peut représenter une relation entre des quantités connues et inconnues. La construction d'un schéma relationnel ou modélisation relationnelle, ainsi qu'une discussion de tels schémas impliquent l'analyse sémantique du texte, donc une verbalisation de la pensée de l'élève. D'autre part, ce travail offre une possibilité d'intégration de manipulation et de gestes associés au sens de la situation, aux rôles des quantités dans le problème. Par exemple, les barres de pâtes à modeler représentant les âges des deux personnes peuvent être placées de différentes façons pour visualiser la différence ou la somme (si nécessaire). L'apprenant peut glisser son doigt le long des bars pour indiquer, par exemple, un segment représentant l'écart.

Un modèle relationnel construit peut être utilisé pour élaborer une stratégie de résolution. En examinant le schéma (Figure 3), on peut proposer que l'âge d'Anne soit composé de deux parties : la même que l'âge du frère et la différence (2 ans). Ainsi, nous pouvons supprimer la différence pour obtenir l'âge équivalent à celui du frère. Alternativement, nous pouvons voir que l'âge du frère est de 2 ans de moins que celui d'Anne, donc nous pouvons utiliser la soustraction. À cette étape aussi, des gestes significatifs tels que cacher un segment, mettre ensemble, couper en parties égales, etc. peuvent être intégrés au processus de négociation du sens. L'étude de certains travaux (ex. Alibali & Nathan, 2012 ; Cook et al., 2008 ; Duval, 2006 ; Radford, 2011) suggère que la communication du sens par des moyens variés contribue au développement d'une compréhension plus profonde et une connaissance plus durable. Il me semble qu'on peut étendre les idées de Vygotsky (1962) sur le rôle du langage dans le développement de la pensée y ajoutant pour le cas de mathématique l'utilisation de gestes et de modélisation visuelle.

GESTION D'APPRENTISSAGE

Pour répondre à la dernière question, il est important de se rappeler que l'apprentissage passe par l'action. Comme le propose Davydov (2008), apprendre les relations et apprendre à les représenter devrait être une partie intégrale de l'activité de l'apprenant lors de la résolution de problèmes. Avant de penser aux calculs possibles, les élèves devraient analyser la logique du problème et sa structure mathématique de manière explicite.

David et Laura participent à une collecte de fonds pendant
jours pour rénover le terrain de jeux de leur école. David a
amassé de plus que Laura. Ensemble, ils ont accumulé
Quelle somme David a-t-il amassée?

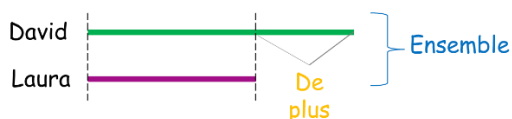


Figure 4. Problème de fonds et son schéma.

Sur la figure ci-dessus (Figure 4), les données numériques sont intentionnellement cachées aux apprenants. Ce geste didactique simple envoie un message important aux étudiants : la logique du problème est plus importante que les nombres. Pour analyser et représenter la logique du problème, le code couleur peut aider à associer de manière significative les informations (non seulement les nombres) du texte à leur représentation schématique.

Des segments ou des barres de pâte à modeler peuvent être utilisés pour représenter des quantités de manière générale, de sorte que l'attention de l'observateur se porte sur la relation entre elles. Cependant, la représentation d'un ensemble d'objets par un segment peut être difficile au début, car elle nécessite une connexion mentale significative entre la *quantité d'objets discrets* et la *quantité continue étendue*. Nous proposons aux apprenants de commencer avec de la pâte à modeler. L'apprenant peut créer quelques boules de pâte à modeler pour représenter des objets puis, les arranger en segment et rouler le segment en une saucisse. La saucisse aide à imaginer n'importe quel nombre d'objets, ou même à imaginer un nombre inconnu d'objets. Inversement, une saucisse peut être coupée en morceaux discrets si nécessaire.



Figure 5.

Les saucisses peuvent être facilement manipulées et placées pour être comparées ou pour former un *total*, reflétant ainsi la relation nécessaire. Le passage du texte à la pâte à modeler et au schéma relationnel, puis aux solutions arithmétiques ou algébriques, est la voie qui peut favoriser le développement de la pensée relationnelle, le type de pensée au cœur de la pensée mathématique (Davydov, 2008 ; Smith & Thomson, 2017). Les représentations schématiques dont j'ai discuté ci-dessus, si elles sont utilisées correctement, permettent l'engagement simultané de plusieurs outils de réflexion tels que l'analyse visuelle, les gestes significatifs et autres manipulations, la verbalisation en langage naturel et les expressions mathématiques. Selon certains chercheurs (Duval, 2006 ; Radford, 2011), le transfert de la pensée d'un « langage » à un autre est ce qui rend la pensée profonde et produit une connaissance durable.

CONCLUSIONS

J'ai commencé avec deux histoires vraies pour formuler 5 questions sur la résolution de problèmes et les représentations à utiliser à l'école primaire. J'ai partagé les réflexions que j'ai développées en me basant sur la littérature de recherche et sur ma propre expérience. En parcourant cet article, j'ai réalisé qu'une question importante n'avait toujours pas été abordée. Pourquoi tant d'enseignants ne sont-ils pas incités à utiliser des activités de résolution de problèmes de manière efficace pour promouvoir la pensée mathématique des élèves, et même la « connaissance mondiale commune » ne peut pas les aider ? Peut-être parce qu'il s'agit de connaissances didactiques théoriques complexes, pas seulement d'une stratégie d'enseignement ou d'un nouveau manuel scolaire ? Peut-être parce que cela prend du temps et des ressources pour apprendre quelque chose de nouveau ? Peut-être parce que nos enseignants sont accablés par une crise scolaire permanente ? Qu'en pensez-vous ?

REFERENCES

- Alibali, M. W., & Nathan, M. J. (2012). Embodiment in mathematics teaching and learning: Evidence from learners' and teachers' gestures. *Journal of the Learning Sciences*, 21(2), 247–286. <https://doi.org/10.1080/10508406.2011.611446>
- Cook, S. W., Mitchell, Z., & Goldin-Meadow, S. (2008). Gesturing makes learning last. *Cognition*, 106(2), 1047–1058. <https://doi.org/10.1016/j.cognition.2007.04.010>
- Davydov, V. V. (1982). Psychological characteristics of the formation of mathematical operations in children. In T. P. Carpenter, J. M. Moser, & T. A. Romberg (Eds.), *Addition and subtraction: Cognitive perspective* (pp. 225–238). Lawrence Erlbaum Associates.

- Davydov, V. V. (2008). *Problems of developmental instruction: A theoretical and experimental psychological study*. Novapublisher.
- Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61(1–2), 103–131. <https://doi.org/10.1007/s10649-006-0400-z>
- Mukhopadhyay, S., & Greer, B. (2001). Modeling with purpose: Mathematics as a critical tool. In B. Atweh, H. Forgasz, & B. Nebres (Eds.), *Sociocultural research on mathematics education: An international perspective* (pp. 295–311). Lawrence Erlbaum Associates.
- Nesher, P., Greeno, J. G., & Riley, M. S. (1982). The development of semantic categories for addition and subtraction. *Educational Studies in Mathematics*, 13, 373–394.
- Polotskaia, E., & Savard, A. (2018). Using the relational paradigm: Effects on pupils' reasoning in solving additive word problems. *Research in Mathematics Education*, 20(1), 70–90. <https://doi.org/10.1080/14794802.2018.1442740>
- Polotskaia, E., Gélinas, M.-S., Gervais, C., & Savard, A. (2023). Représenter pour mieux raisonner. Résolution de problèmes écrits de multiplication et de division. *JFD*.
- Polotskaia, E., Gervais, C., & Savard, A. (2019). Représenter pour mieux raisonner Résolution de problèmes écrits d'addition et de soustraction. *JFD*.
- Radford, L. (2011). Grade 2 students' non-symbolic algebraic thinking. In J. Cai & E. Knuth (Eds.), *Early algebraization* (pp. 303–322). Springer. https://doi.org/10.1007/978-3-642-17735-4_17
- Smith III, J. P. J., & Thompson, P. W. (2017). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput (Ed.), *Algebra in the early grades* (pp. 95–132). Taylor and Francis. <https://doi.org/https://doi.org/10.4324/9781315097435-5>
- Vygotsky, L. (1962). *Thought and language*. MIT Press. <https://resolver.ebscohost.com/Redirect/PRL?EPPackageLocationID=1054.443576.1611394&epcustome rid=s2947694>

WHAT IS AN APPLIED PROBLEM? ... A JOURNEY THROUGH TIME ...

Bernardo Galvão-Sousa
University of Toronto

ABSTRACT

I have been teaching applied math courses for undergraduate students for over 10 years. Over this time, my idea of what an applied math problem is has changed.

On this CMESG meeting, I hosted a topics session where I showcased questions from applied math courses that I taught. The participants were invited to think about these questions and distill the ideas behind these problems. I will also report here on these discussions.

BACKGROUND

The first question that comes to mind when thinking about applied problems and how I thought about them is how can I go back to my course materials over the years and find good indicators of how I thought about applied problems?

Putting on the student hat, I figured that students will focus their learning on what they think will be tested on midterms and exams, so the best indicators should be questions from past tests and exams. These should give an idea of what I thought the students should strive to learn.

Thus the topic session focused primarily on problems that I wrote for my applied classes over the years, the main ideas underlying these problems, and how these have evolved.

The session was meant to be interactive, with the participants thinking about the different problems, distilling what they thought about them and engaging in discussions. For each of the six problems I presented, I will report back on the main topics that arose from the discussion between the participants.

At the beginning of the session, and before I showed any of my questions, I posed a question to the participants:
What are features of an applied problem?

My goal with this question was two-fold:

1. I wanted to know how the participants, mathematics educators across Canada teaching to different students, thought about applied problems, before my interference.
2. I wanted to be able to gauge whether my talk would change how they thought about applied problems.

Given the heterogeneity of the participants, I did not have any expectations.

At the end of the session, I posed the same question again and I will report on the results at the end of this report.

I then posed several questions from tests and exams and posed the following two questions to the participants:

(Q₁) What are students doing while working on this exercise?

(Q₂) What are students learning?

My goal with these two questions was to focus the participants' attention to the point-of-view of the students who needed to solve these problems, instead of the content that the instructor was 'covering.'

EARLY DAYS

This section features problems that I posed to students early in my teaching career. This was a time when I had just switched my mathematical focus from research to teaching and education. This is also when I started to interact more with the education community. There is a (somewhat-)wide range on the type of questions that I asked during this period, so I focused on two that seemed more representative.

PROBLEM #1—BIOLOGIST STUDYING A VIRUS

You are working for a biologist who is studying a new kind of virus that reproduces extremely quickly, with the population following the function

$$p(t) = \int_0^t e^{x^2} dx,$$

where t is in days. The size of each virus is $\frac{1}{10} \mu m^2$.

The biologist wants to make sure he has enough space for this virus to grow for 4 days.

Using the Midpoint rule with $n = 4$, give an approximation of the necessary area.

Figure 1. Biologist studying a virus problem.

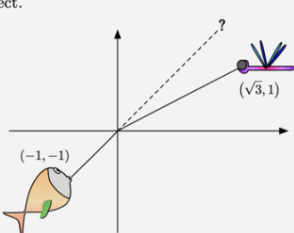
Participants' discussion:

- (Q₁) Students are getting rid of words and figuring out which computations to do.
- (Q₁) Students are extracting information.
- (Q₁) Students are mostly performing computations.
- (Q₂) Students are learning that standard techniques do not always work.

PROBLEM #2—ARCHER FISH

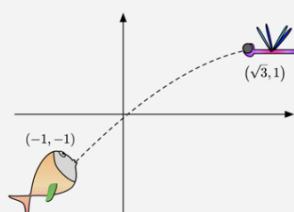
Archer fish hunt by spitting a jet of water at a nearby flying insect.

(a) Assume that the fish's position, as shown, is $(-1, -1)$ and the insect's position is $(\sqrt{3}, 1)$ and the water surface is at $y = 0$. When the fish sees the insect, water refraction changes the angle of the light as in the figure on the right, causing the fish to misjudge the insect's position.



The fish sees the path to the insect as a straight line. What does the fish think is the position of the insect?

(b) The fish hunts the insect by spitting a jet of water in the direction of the dotted line. Ignoring the effects of water and air resistance, but considering gravity, $\vec{a} = (0, -g)$, how fast should the fish spit the water to hit the insect? Assume all distances given are in cm and the insect is not moving.



(c) How much time does the insect have to move out of the way?

Figure 2. Archer fish problem.

Participants' discussion:

- (Q₁) Students are using simplified physics and triangles.
- (Q₁) Students are engaging in a fun game.
- (Q₁) Students are solving a “fake world” problem.
- (Q₂) Students are learning that mathematicians do not connect to the real world.

EXTRA PROBLEM—TAYLOR, THE CAT

A cat named Taylor is planning his escape from a dog called Maclaurin.

His plan is to wait until the dog almost reaches his position (2,0), and then jump away quickly to avoid the dog.

Before deciding where to go, Taylor (the cat) takes one good look at the dog and calculates a few derivatives of the function

$f(x)$ = path the dog intends to take.

- (a) Assume that the cat calculates the following

$$f(0) = 0$$

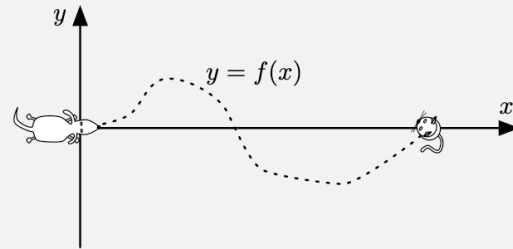
$$f''(0) = -1$$

$$f'(0) = \frac{5}{6}$$

$$f'''(0) = \frac{1}{4}$$

Give an expression that approximates the path $f(x)$ of the dog using all this information.

- (b) Using the calculations from (a), help Taylor estimate the derivative $f'(2)$.
- (c) As the dog approaches the cat, from the cat's perspective is the dog moving to the right (towards positive y) or to the left (towards negative y)?



(the dotted path is not accurate; it's just an example)

Figure 3. Taylor the cat problem.

THOUGHTS

It should be clear that these problems are closed ended. They are artificial applied problems (“fake world” problems) that are specifically designed so that students can apply some tools or techniques in the course under the guise of beginning an applied problem.

Like some participants shared, students were learning that mathematics is disconnected from the real world. This was worrisome, since one of the goals I had in mind was to get the students to recognize that mathematics is a very powerful tool to study the real world.

MIDDLE AGES

After the first few years, I started dabbling in new ideas, learning what other educators did and published. I wanted to move away from traditional problems and ideas and explore better ways to teach, so during this time, you could see the widest range of questions.

PROBLEM #3—COMPUTER KEYBOARD

Consider a computer keyboard.

The following are some properties of keys:

- (P_1) Keys can only move vertically.
- (P_2) Each key has a spring to make the key return to its original position after being pressed (Hooke's Law: "the force is proportional to the extension").
- (P_3) Each key must also include some damping, so that it doesn't keep oscillating back and forth once pressed.
- (P_4) A typical letter key is 15mm×15mm and when pressed has a maximum displacement of 0.5mm.
- (P_5) Keys last 50 million presses on average.
- (P_6) On average, a man exerts the force of 50 N with one finger on a key.
- (P_7) On average, a woman exerts the force of 35 N with one finger on a key.
- (a) Find a model for the position $y(t)$ of one key using only positive constants. Justify the terms you included.
- (b) What would be the initial condition(s) to figure out what happens after pressing a key? Justify your answer.

Figure 4. Computer keyboard problem.

Participants' discussion:

- (Q_1) Students are extracting data from the problem.
- (Q_1) Students are processing lots of information and figuring out where it fits.
- (Q_1) Students are not doing computations.
- (Q_2) Students are learning about modelling / this model.

PROBLEM #4—CHIRPING BIRD

A bird is chirping to find a mate. Unfortunately it is standing next to a cave which echoes its chirps. Consider the following:

- (P_1) The bird chirps once every minute;
- (P_2) The maximum volume the bird can chirp is M dB;
- (P_3) If it hears a chirp, then it chirps at a volume proportional to the volume of the chirp it heard times the difference between the maximum volume it is capable and the volume heard with constant $A \frac{1}{dB}$;
- (P_3) The cavern echoes sounds with volume $\frac{1}{2}$ of the original volume (ie., if the bird chirps at 10 dB, the echo is 5 dB);
- (P_4) The echo reaches the bird in less than one minute.
- (a) Find a difference equation for v_k = the volume of the k^{th} chirp in dB.
- (b) What is the volume of the first chirp that will make all the following chirps have the same volume?
- (c) Find all the values of the constant A such that it is possible for the bird to chirp in the manner described in (b).



Figure 5. Chirping bird problem.

Participants' discussion:

- (Q₁) Students are taking a set of instructions and writing a set of equations (model)—note that all information is relevant.
- (Q₁) Students are doing some (not many) computations.
- (Q₂) Students are learning about making choices when modelling.
- (Q₂) Students are learning about equilibria.

EXTRA PROBLEM—DRAG RACE CAR

In this question, we will try to find a simplified model for the velocity and amount of fuel in a drag race car.

Let

- $v(t)$ be the velocity of the car in m/s;
- $f(t)$ be the amount of fuel left in the car in L.

Also assume the following properties:

(P₁) If the car has no fuel, then its velocity decreases proportionally to its current velocity with constant 2 s^{-1} ;

(P₂) In the absence of air resistance, the car will accelerate proportionally to the amount of fuel left with constant $1 \text{ m} / (\text{L s}^2)$;

(P₃) The amount of fuel decreases proportionally to the velocity of the car with constant $2 \text{ L} / \text{m}$;

(a) Write a system of linear first-order ODEs for $\vec{c}(t) = \begin{bmatrix} v(t) \\ f(t) \end{bmatrix}$ that describes this interaction between the car's velocity and the amount of fuel. Justify your answer.

(b) If the car starts with a full tank of $T \text{ L}$, give initial conditions for \vec{c} . Justify your answer.




Figure 6. Drag race car problem.

THOUGHTS

As the participants noted, there is a lot less focus on computations, and a lot more on the modelling process. The problems are less textbook style and closer to real world problems.

I want to also point out that on the first problem (Figure 4), students chose to focus on different paths: some students focused on the key release; whereas, some students focused on the keypress, which in retrospective also makes sense since information about the pressure people exert on the key is given.

This open-ended type of problems with multiple different paths was something that I started looking for in problems to give students.

RECENTLY

This most recent stage of my teaching path came after I had a research study leave and so I had more time to reflect on what I would like students to learn from applied courses.

It marks a big leap in my thinking about applied problems.

PROBLEM #5—WHY ARE TREES SO TALL?

Why are trees so tall?

Let us create a model for the height of trees in a forest. To simplify our model, let us consider only two types of trees:

- $f(t)$ = average height of fir trees in the forest in metres;
- $s(t)$ = average height of spruce trees in the forest in metres.

We assume the following properties for both types of trees:

- (P_1) It costs energy for the tree to grow tall so if left on their own each type of tree will grow shorter.
 (P_2) If the other type of tree is taller, then it will block sunlight, so this type of trees will grow taller.
 (P_3) Spruce trees have a specific height H that optimizes the loss of leaves from the local fauna and isn't too tall.

We have the following model:

$$f'(t) = \boxed{\begin{array}{l} \text{Rate of change of} \\ \text{height of fir trees} \end{array}} = \underbrace{-\bar{a}f(t)}_{(P_1)} + \underbrace{b(s(t) - f(t))}_{(P_2)}$$

$$s'(t) = \boxed{\begin{array}{l} \text{Rate of change of} \\ \text{height of spruce} \\ \text{trees} \end{array}} = \underbrace{-\alpha s(t)}_{(P_1)} + \underbrace{\beta(f(t) - s(t))}_{(P_2)} \cdot \underbrace{(H - s(t))}_{(P_3)}$$

All constants $\bar{a}, b, \alpha, \beta$ are positive constants that would need to be experimentally estimated.

We can then simplify this system to:

$$\begin{cases} f' = -af + bs \\ s' = -\alpha s + \beta(f - s)(H - s) \end{cases}$$

where $a = \bar{a} + b$.

Your task is to assess this model.

- (a) Explain a property of solutions $(f(t), s(t))$ of this model that matches what you would expect. Explain your calculations and the reasons for all your decisions and assumptions.
- (b) Explain a property of solutions $(f(t), s(t))$ of this model that does not make sense.
- (c) Explain an improvement you could make to the model. Your answer should contain a new improved model as well as an explanation of how to change the modelling process, your calculations, and the reason for your modelling decisions and assumptions.
- (1 bonus point) if your improved model includes a unique improvement and is well justified.

Figure 7. Why are trees so tall?

Participants' discussion:

- (Q₁) Students are working on an open ended real world problem.
 (Q₁) Students are assessing a model / checking the correctness of their work.
 (Q₁) Students are rewarded for checking answers.
 (Q₁) Students are communicating and justifying.
 (Q₂) Students are learning the process of modelling (assessment part).
 (Q₂) Students are learning resiliency and creativity in math.

PROBLEM #6—STUDENT EXPERIMENT

Get up out of your chair! Then report back.

Instructions for mini-experiment:

1. Get out of the room where you are working now: go to a different area of your home or outside.
2. Change something: fill, twist, shake, stir, cut, ... (just moving = changing position is not allowed)
3. Focus on the rate of change of **one** quantity.
4. Reflect on what you've changed and different ways you can quantify and model that change.

Write a report with the following:

- (a) The title of your mini experiment
- (b) Explain how can you model how the change occurred or how you can quantify how the change occurred?
- (c) Sketch a graph of a rate of change related to your experiment as a function of time, where $t=0$ is when you started your experiment. You do not need to include any numbers on your axes – the shape is the important thing.

One of the learning objectives from Week 1 is:

“Give an intuitive argument about why the Fundamental Theorem of Calculus makes sense.”

- (d) Provide an intuitive argument for why the FTC makes sense in this situation for a particular time interval using a copy of your graph to support your argument.

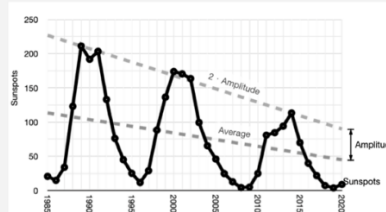
Figure 8. Student experiment.

Participants' discussion:

- (Q₁) Students are having fun!
- (Q₁) Students are communicating.
- (Q₂) Students are learning that class is important.
- (Q₂) Students are learning about experimental modelling.
- (Q₂) Students are learning to focus on qualitative data.
- (Q₂) Students are learning about creativity in math.

EXTRA PROBLEM—SUNSPOTS

Sunspots usually indicate active areas of the sun and can affect communication satellites on Earth orbit, so it is vital to monitor them closely. Below you can find a graph and a table with the number of sunspots per year.



Year	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000	2001	2002	2003	2004	2005
Sunspots	203	133	76	45	25	12	29	88	136	174	170	164	99	65	46
Year	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
Sunspots	25	13	4	5	25	81	85	94	113	70	40	22	7	4	9

Let us try to approximate the number of Sunspots $S(t)$ as a function of time t (in years).

The solar activity seems to be cyclical, although with decreasing amplitude and average.

Let us consider the family of functions: $S(t) = a(t) + b(t) \cdot \sin(2\pi \cdot (t - 1998))$.

- (a) Connect the function/constant on the left with its meaning on the right:

- | | | | |
|--------|---|---|---|
| $a(t)$ | • | • | The vertical shift (average) of the number of Sunspots. |
| $b(t)$ | • | • | The amplitude of the number of Sunspots. |
| c | • | • | The frequency or period of the number of Sunspots. |

- (b) Find an expression for the function $S(t)$ above.

- (c) What is the last full year when your model will make physical sense?

Figure 9. Sunspots problem.

EXTRA PROBLEM—SCIENTIFIC ARTICLE EXPLORATION

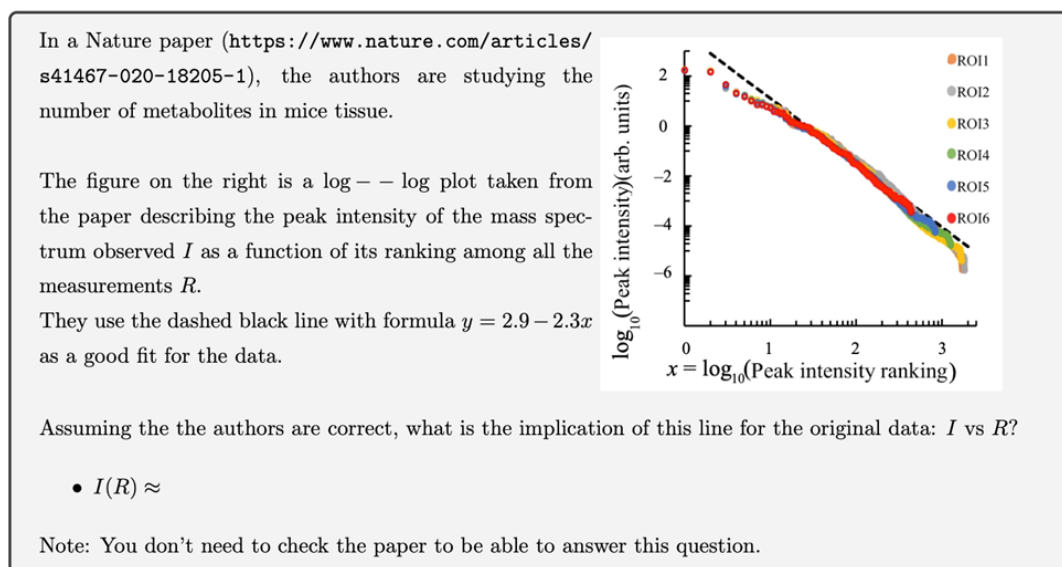


Figure 10. Scientific article exploration.

THOUGHTS

The discussion among the participants for these last problems changed tone dramatically. The problems are much more open ended, giving rise to more student agency and creativity on their solutions. They also feature real world data, taken from scientific articles and messy or collected by the students themselves. The problems focus a lot more on communication and assessment of models and the focus is not on computations but on qualitative properties.

I also want to point out that, in general, students have a lot more difficulty with these kinds of problems, so to be able to write these in tests and exams, the instructor needs to routinely have students tackle similar types of problems in class.

WHAT ARE FEATURES OF AN APPLIED PROBLEM?

I posed this question at the beginning of the session and again at the end. Here are the comments from the participants in Figure 11.

Before	After
<ul style="list-style-type: none"> • They involve applications to the real world • They use a math tool in a context other than where it was defined • Include different ways of using a result/tool (not proofs) • They have numbers/data 	<ul style="list-style-type: none"> • They start with <u>actual real world</u> situations • Context in some physical quantity or measurement • Should include a known tool in a novel situation • There should be some parameters • No <u>one</u> correct answer • Include feedback and assessment

Figure 11. Participants' comments.

Observe that after, the participants emphasized “actual real world.” The participants realized with the problems shown that there are “real world” and “actual real world” problems/situations.

Also, at the end, there are some new ideas. The participants noticed that having multiple possible correct answers makes for a much better problem, as well as the focus on assessment of a model and communication.

I was happy that the session had an impact on the participants' ideas of what makes an applied problem.

CONCLUSIONS

The evolution of what I think is an applied problem or an applied mathematics course was very dramatic: from the initial traditional closed ended and computational textbook style problems to open ended real problems with messy real data. The idea of fostering creativity and communication in math courses also becomes prevalent in more recent problems.

There were several lessons I learned along the way:

- Obviously fake problems might bring unwanted reactions from the students: I want them to learn that mathematics can study the world, and they are learning that it is a fake tool that can only study fake problems.
- Exploring different types of problems does not always succeed, and that is ok! One example is the Keyboard problem (Figure 4), which did not work out very well the first time I used it. Students were not used to having (lots of) extra information in problems, so they tried to include every bit of information in their model. The path to creating good new interesting problems passes through failing.
- Even first year undergraduate students can study scientific papers. I learned that it is never too early to start studying parts of real scientific articles. One of my goals when teaching an applied course is that by the end, students should feel confident that they can read and understand scientific articles.

My hope is that this session and this document will resonate with the reader, and it will make one rethink on what constitutes a good applied problem and to move away from closed ended textbook style problems to open ended problems based on the actual real world.

ACKNOWLEDGEMENTS

All this evolution of ideas was only possible due to the many, many, many hours of conversations with my colleagues at the University of Toronto, with CMESG and FYMSiC members during meetings, and participants of other workshops and conferences. So I want to thank you all for letting me pick your brains!

I also want to thank my students, who willingly (some did go out of their way to take my courses!!) or unwillingly were my 'guinea pigs!' Sometimes the problems fell flat, and they were the first to let me know. Thank you for all the feedback and for the learning of mathematics.

Thank you also to the participants of the session who were willing to come and think and discuss and share their ideas on my questions.

Finally, thank you to the universe for being an unending source of questions and problems.

TEACHING FINANCIAL NUMERACY IN SCHOOLS: AN OVERVIEW OF MATHEMATICAL PRACTICES AT PLAY

ENSEIGNER LA NUMÉRATIE FINANCIÈRE DANS LES ÉCOLES : UN APERÇU DES PRATIQUES MATHÉMATIQUES EN JEU

Annie Savard
McGill University

-----Français en suivent.-----

Financial Education is present in some recent Canadian Mathematics curricula such as Alberta, British Columbia, Ontario and Québec. However, it is not new that mathematics education plays an important role regarding Financial Education, because mathematics education has a long tradition to teach about money and to use financial contexts in problem solving tasks.

BACKGROUND

Financial education is an international trend around the world. Since the global financial crisis in 2008, the need for financial education is even stronger. While some have argued that the 2008 financial crisis was more a result of a lax regulatory environment and predatory lending practices than a financially illiterate citizenry (Pinto, 2009, 2012; Willis, 2008, 2009), there is general agreement that financial literacy education deserves a place in today's schools (Lusardi et al., 2013; Lusardi et al., 2010). The need is enhanced, among others, by the disempowerment of governments regarding pension plans, the changes, and the tensions of local and international markets regarding globalization and sustainability. In addition to that, the economic changes caused by the Covid-19 pandemic and the increases of financial fraud and financial violence require financial competencies for all. Therefore, citizens need to learn about finance at different ages, at different places including schools and at different periods of their life. Planning about finances should be ahead of covering important needs such as getting an apartment or a house because a lot of money is required at the start.

FRAMING FINANCIAL EDUCATION TO FINANCIAL NUMERACY

Financial Education is also known as financial literacy. Since 2012, the Organization for Economic Cooperation and Development (OECD) has administered a financial literacy assessment via their triennial Program for International Student Assessment (PISA). In 2015, Canadian students coming from some provinces wrote this financial literacy assessment for the first time. OECD (2019) defines financial literacy as

Financial literacy is the knowledge and understanding of financial concepts and risks, and the skills, motivation and confidence to apply such knowledge and understanding in order to make effective decisions across a range of financial contexts, to improve the financial well-being of individuals and society, and to enable participation in economic life. (p. 128)

This definition inspired many countries to put attention on financial literacy. For instance, in Canada, the Financial Consumer Agency of Canada (2021) designed a framework for Canadian citizens. This framework is the core of the national financial literacy strategy, which aims that all Canadian citizens can benefit from an accessible, inclusive,

and effective financial ecosystem. In fact, the vision of this framework is that all Canadian citizens can build a financial resilience in the digital world.

Savard and Cavalcante (2021) preferred to reframe financial literacy as financial education, which provides insights more broadly. In this regard, financial education is a field situated in education rather than an outcome of education. Thus, their definition is not looking for what people can do, rather what this field is about:

Financial Education is the field of teaching and learning the financial dimension of the production and management of resources mediated by financial instruments (currency, models, concepts). The use of financial instruments can also lead to assigning a value to an action (service) or an object (good). (p. 6)

At this end they conceptualized the field, where a special focus is present on financial numeracy, which could be seen as the competence (knowledge, confidence and ability) to use qualitative and quantitative data in financial and economical situations (Savard & Cavalcante, 2021). In fact, numeracy is a social practice on how people used mathematics (Camiot & Jeanotte, 2016; Goos et al., 2019; Yasukawa et al. 2018), while financial numeracy is also about financial praxis. In fact, they claimed that financial numeracy is an intersection between financial education and mathematics education. Three dimensions highlight this intersection. The first dimension, the contextual dimension, is the place where financial concepts and practices are used as a context to teach mathematics. The aims for teachers are to teach and learn mathematics in financial contexts. Savard (2018) demonstrated how a context in a mathematical situation could be changed without changing anything to the relationships in the situation (Savard & Polotskaia, 2017). In the second dimension, the conceptual dimension, mathematics is used to conceptualize financial concepts and practices. Mathematics is a tool to make sense of the situation and develop a critical thinking about it (Savard, 2011). Without mathematics, it is not possible to understand a financial practice (Cavalcante & Savard, 2022). The mathematics present in the third and last dimension, the systemic dimension, raises questions on financial concepts and practices in relation to other systems (beliefs, economical, inequity, political). The complexity of the situation requires that mathematics plays a role of leader in empowering citizens into ethical questions.

SOME EPISTEMOLOGICAL CONSIDERATIONS

Some financial concepts are found in mathematics programs or curricula in different domain areas. For instance, currency or money can be found in arithmetic or in measurement elementary school curricula (Buys & De Moor, 2008; Roegiers, 2013; Savard et al., 2020). Simple and compound interests are often part of secondary school mathematics curricula, under the domain on financial mathematics (Savard & Cavalcante, 2021). In fact, Savard & Cavalcante (2021) argue that financial numeracy is mathematical content. First of all, because money is a unit of measurement of the value of products, which includes good and services, money is a mathematical concept (Roegiers, 2013; Savard et al., 2020). Measuring using money as unit of measurement means assigning a value to something. One problem with money is that the unit of measurement is not the same everywhere. Canada has the Canadian dollar, United States has the American dollar, and Australia has the Australian dollar. Countries have different currencies. Another problem is that measuring with money uses continuous numbers, but they are represented with physical objects such as coins and bills that use discrete numbers (Savard et al., 2020). This epistemological obstacle must be considered carefully.

Money is a mathematical concept, but money has financial functions in our modern society (Hill, 2010): as a store of value, a medium of exchange and a unit of account. Hill (2010) explains that a store of value means that money can be used to transfer purchasing power and value from the present to the future. The second function, as described by Hill (2010), is that money can be used as a medium of exchange, which is a measurable tool that helps facilitate a transaction between parties. This tool must be easily measurable and represent a standardized value to be used as a medium of exchange. According to Roegiers (2013), money is used to measure prices such as purchase price, cost price, selling price, profit and loss. Profit and loss are expressed as a percentage. The third function, as described by Hill (2010), is a unit of account, that “can be used to measure and compare the value of goods and service in relation to one another” (p. 30). The idea of a unit of account is used in the daily lives of consumers when comparing the value of different goods and services, which is considered as representational measurement (Hand, 2016). In fact, unit of account can take two different forms of measurement: representational or pragmatic measurements (Hand, 2016). When measuring, we assign a quantity to a property or an attribute of a given object. The concept of mapping properties of objects from the physical to the numerical world by using units of measurement is what Hand (2016) calls *representational measurement*. Such mapping must conserve the relationships that are observed in the physical world. Therefore, representational measurement translates objects and their relationships to numbers.

According to Hand (2016) *pragmatic measurement* is the practice of constructing the property of an object by describing the way this property must be measured. In other words, it is by being measured that the property comes to existence. Defining the economic value of a product is therefore an activity rooted in pragmatic measurement because it needs to take into consideration many different measurable (and subjective) variables such as the cost of making, the amount or quantity sold, and the place it is sold. A student might estimate the cost of a pencil by extrapolating from other data (strictly mathematical reasoning), recording the price from a sticker (representational measurement), or by contrasting it with its economic value (pragmatic measurement) (Căprioară et al., 2020).

CONCLUDING REMARKS

Teaching financial numeracy in mathematics classrooms is about, among others, the understanding and the use of mathematical practices such as calculating, comparing, modeling, estimating, and assessing values using money. It is about making sense of a situation involving money as a mathematical concept rather than only applying mathematical concepts and processes on money. Thus, numeracy is a key word to make sense of situated mathematical practices used in daily life. At this end, it is time that national curricula embrace this important mathematical domain.

L'éducation financière est présente dans certains récents programmes d'études mathématiques au Canada, comme ceux de l'Alberta, de la Colombie-Britannique, de l'Ontario et du Québec. Toutefois, ce n'est pas nouveau que l'enseignement des mathématiques joue un rôle important dans l'éducation financière, car l'enseignement des mathématiques possède une longue tradition d'enseignements relatifs à l'argent et d'utilisation de contextes financiers dans des tâches de résolution de problèmes.

MISE EN CONTEXTE

L'éducation financière est en tendance partout dans le monde. Depuis la crise financière mondiale de 2008, le besoin d'éducation financière est encore plus grand. Même si certains ont avancé que la crise financière de 2008 était davantage le résultat d'un environnement réglementaire laxiste et de pratiques de prêt prédatrices que d'une population analphabète en matière financière (Pinto, 2009, 2012 ; Willis, 2008, 2009), il existe un consensus sur le fait que l'éducation financière mérite sa place dans les écoles d'aujourd'hui (Lusardi et al., 2013 ; Lusardi et al., 2010). Ce besoin est renforcé, entre autres, par la perte de pouvoir des gouvernements en matière de régimes de retraite, les changements et les tensions des marchés locaux et internationaux concernant la mondialisation et la durabilité. À cela s'ajoutent les changements économiques provoqués par la pandémie de Covid-19 et l'augmentation de la fraude financière et de la violence financière qui exigent des compétences financières pour tous. Les citoyens doivent donc se familiariser avec la finance à différents âges, à différents endroits, notamment à l'école, et à différentes périodes de leur vie. La planification financière doit être anticipée avant de couvrir des besoins importants tels que l'achat d'un appartement ou d'une maison, car il faut beaucoup d'argent au départ.

DE L'ÉDUCATION FINANCIÈRE À LA NUMÉRATIE FINANCIÈRE

L'éducation financière est également connue sous le nom de littératie financière. Depuis 2012, l'Organisation de Coopération et de Développement Économiques (OCDE) organise une évaluation de la littératie financière via son Programme triennal pour le suivi des acquis des étudiants internationaux (PISA). En 2015, des étudiants canadiens venant de certaines provinces ont passé pour la première fois cette évaluation de littératie financière. L'OCDE (2019) définit la littératie financière comme suit :

La littératie financière est la connaissance et la compréhension des concepts et des risques financiers, ainsi que les compétences, la motivation et la confiance nécessaires pour appliquer ces connaissances et cette compréhension afin de prendre des décisions efficaces dans une gamme de contextes financiers, pour améliorer le bien-être financier des individus et de la société, et permettre la participation à la vie économique. (p. 128, notre traduction)

Cette définition a inspiré de nombreux pays à prêter attention à la littératie financière. Par exemple, au Canada, l'Agence de la consommation en matière financière du Canada (2021) a conçu un cadre pour les citoyens canadiens. Ce cadre est au cœur de la stratégie nationale de littératie financière, qui vise à ce que tous les citoyens canadiens

puissent bénéficier d'un écosystème financier accessible, inclusif et efficace. En fait, la vision de ce cadre supporte que tous les citoyens canadiens puissent bâtir une résilience financière dans le monde numérique.

Savard et Cavalcante (2021) ont préféré recadrer la littératie financière comme une éducation financière qui fournit des perspectives plus larges. À cet égard, l'éducation financière est un domaine situé dans l'éducation plutôt qu'un résultat de l'éducation. Ainsi, leur définition ne cherche pas à identifier ce que les gens peuvent faire, mais plutôt définir ce qu'est ce domaine :

L'éducation financière est le domaine de l'enseignement et de l'apprentissage de la dimension financière de la production et de la gestion des ressources médiatisées par les instruments financiers (monnaie, modèles, concepts). Le recours aux instruments financiers peut également conduire à attribuer une valeur à une action (un service) ou à un objet (un bien). » (p. 6, notre traduction)

À cette fin, ils ont conceptualisé le domaine, où un accent particulier est mis sur la numératie financière qui pourrait être comprise comme la compétence (connaissance, confiance et capacité) à utiliser des données qualitatives et quantitatives dans des situations financières et économiques (Savard et Cavalcante, 2021). En fait, la numératie est une pratique sociale relative à la façon dont les gens utilisent les mathématiques (Camiot & Jeanotte, 2016 ; Goos et al., 2019 ; Yasukawa et al., 2018), tandis que la finance est également une pratique financière. Ils affirment que la numératie financière est une intersection entre l'éducation financière et l'enseignement des mathématiques. Trois dimensions mettent en valeur cette intersection. La première dimension, la dimension contextuelle, est le lieu où les concepts et pratiques financières sont utilisés comme contexte pour enseigner les mathématiques. Dans ce contexte, les buts de l'enseignant sont d'enseigner et d'apprendre les mathématiques dans des contextes financiers. Savard (2018) a montré comment le contexte d'une situation mathématique pouvait être modifié sans changer les relations dans la situation (Savard & Polotskaia, 2017). Dans la deuxième dimension, la dimension conceptuelle, les mathématiques sont utilisées pour conceptualiser les concepts et les pratiques financières. Les mathématiques sont un outil pour donner du sens à une situation et développer une pensée critique à son égard (Savard, 2011). Sans mathématiques, il n'est pas possible de comprendre une pratique financière (Cavalcante & Savard, 2022). Les mathématiques présentes dans la troisième et dernière dimension, la dimension systémique, soulèvent des questions sur les concepts et pratiques financières en relation avec d'autres systèmes (croyances, économique, iniquité, politique). La complexité de la situation exige que les mathématiques jouent un rôle de leader en responsabilisant les citoyens sur les questions éthiques.

QUELQUES CONSIDÉRATIONS ÉPISTÉMOLOGIQUES

Certains concepts financiers se retrouvent dans les programmes de mathématiques sous différents domaines. Par exemple, la monnaie ou l'argent peuvent être trouvés dans les programmes d'arithmétique ou de mesure des écoles primaires (Buys & De Moor, 2008 ; Roegiers, 2013 ; Savard et al., 2020). Les intérêts simples et composés peuvent faire partie des programmes de mathématiques du secondaire, dans le domaine des mathématiques financières (Savard & Cavalcante, 2021). En fait, Savard et Cavalcante (ibid) soutiennent que la numératie financière est un contenu mathématique. Tout d'abord, parce que la monnaie est une unité de mesure de la valeur des produits, qui inclut les biens et les services. Par conséquent, la monnaie est un concept mathématique (Roegiers, 2013 ; Savard et al., 2020). Mesurer en utilisant l'argent comme unité de mesure signifie attribuer une valeur à quelque chose. Une difficulté réside dans le fait est que l'unité de mesure n'est pas la même partout, car il y a différentes devises à travers le monde. Le Canada utilise le dollar canadien ; les États-Unis utilisent le dollar américain alors que l'Australie utilise le dollar australien. Les pays ont des monnaies ou devises différentes. Une autre difficulté réside dans le fait que mesurer avec de l'argent se fait avec des nombres continus, mais que ceux-ci sont représentés par des objets physiques tels que des pièces de monnaie et des billets qui font intervenir des nombres discrets (ibid). Cet obstacle épistémologique doit être considéré avec attention.

L'argent est un concept mathématique, mais l'argent a des fonctions financières dans notre société moderne (Hill, 2010) : c'est une réserve de valeur, un moyen d'échange et une unité de compte. Hill (ibid) explique qu'une réserve de valeur signifie que l'argent peut être utilisé pour transférer du pouvoir d'achat et de la valeur du présent vers le futur. La deuxième fonction, telle que décrite par Hill (ibid), est que l'argent peut être utilisé comme moyen d'échange, comme un outil mesurable qui contribue à faciliter une transaction entre les parties. Cet outil doit être facilement mesurable et représenter une valeur standardisée pour être utilisé comme moyen d'échange. Selon Roegiers (2013), la monnaie est utilisée pour mesurer des prix tels que le prix d'achat, le prix de revient, le prix de vente, les profits et les pertes. Les profits et les pertes sont exprimés en pourcentage. La troisième fonction, telle que décrite par Hill

(2010), est une unité de compte qui « peut être utilisée pour mesurer et comparer la valeur des biens et des services les uns par rapport aux autres » (p. 30). L'idée d'unité de compte est utilisée dans la vie quotidienne des consommateurs lorsqu'ils comparent la valeur de différents biens et services, ce qui est considéré comme une mesure représentative (Hand, 2016). En fait, l'unité de compte peut prendre deux formes de mesure différentes : les mesures représentatives ou les mesures pragmatiques (ibid). Lorsque nous mesurons, nous attribuons une quantité à une propriété ou à un attribut d'un objet donné. Le concept de cartographie des propriétés d'objets du monde physique au monde numérique en utilisant des unités de mesure est ce que Hand (ibid) appelle la mesure représentationnelle. Une telle cartographie doit conserver les relations observées dans le monde physique. Par conséquent, la mesure représentationnelle traduit les objets et leurs relations en nombres.

Selon Hand (2016), la mesure pragmatique est la pratique consistant à construire la propriété d'un objet en décrivant la manière dont cette propriété doit être mesurée. En d'autres termes, c'est en étant mesurée que la propriété naît. Définir la valeur économique d'un produit est donc une activité ancrée dans une mesure pragmatique, car elle doit prendre en considération de nombreuses variables mesurables (et subjectives) telles que le coût de fabrication, la quantité vendue et le lieu de vente. Un élève peut estimer le coût d'un crayon en extrapolant à partir d'autres données (raisonnement strictement mathématique), en identifiant le prix sur une étiquette (mesure représentative) ou en le comparant à sa valeur économique (mesure pragmatique) (Căprioară et al., 2020).

CONCLUSION

Enseigner la numératie financière dans les classes de mathématiques concerne, entre autres, la compréhension et l'utilisation de pratiques mathématiques telles que le calcul, la comparaison, la modélisation, l'estimation, l'évaluation de valeurs à l'aide de l'argent. Il s'agit de donner un sens à une situation impliquant l'argent en tant que concept mathématique plutôt que d'appliquer uniquement des concepts et des processus mathématiques à l'argent. Ainsi, la numératie est un mot clé pour donner du sens aux pratiques mathématiques situées utilisées dans la vie quotidienne. À cette fin, il est temps que les programmes nationaux intègrent cet important domaine mathématique.

REFERENCES / RÉFÉRENCES

- Buys, K., & De Moor, E. (2008). Domain description measurement. In M. Van den Heuvel-Panhuizen & K. Buys (Eds.), *Young children learn measurement and geometry* (pp.15–36). Sense Publishers.
- Camiot, C., & Jeanotte, D. (2016). La numératie financière chez les jeunes adultes: une compétence à développer pour une meilleure gestion financière? In A. Adihou, J. Giroux, D. Guillemette, C. Lajoie, & K. Huy (Eds.), *La diversité des mathématiques : dimensions sociopolitiques, culturelles et historiques de la discipline en classe : Actes du Colloque du Groupe de didactiques des mathématiques du Québec 2016* (pp. 55–67). uOttawa.
- Căprioară, D., Savard, A., & Cavalcante, A. (2020). Empowering future citizens in making financial decisions: A study of Elementary School mathematics textbooks from Romania. In D. Flaut, S. Hoskova-Mayerova, C. Ispas, F. Maturo, & C. Flaut (Eds.), *Decision making in social sciences: Between traditions and innovations* (119–134). Springer. Nature Switzerland AG.
- Cavalcante, A., & Savard, A. (2022). Understanding our world in a time of crisis: Mathematics education pedagogy toward financial numeracy. *Journal of Honai Math*, 5(2), 109–126.
- Financial Consumer Agency of Canada (2021). *Make change that counts: National Financial Literacy strategy 2021-2026*. <https://www.canada.ca/en/financial-consumer-agency/programs/financial-literacy/financial-literacy-strategy-2021-2026.html>
- Goos, M., Geiger, V., Dole, S., Forgasz, H., & Bennison, A. (2019). *Numeracy across the curriculum: Research-based strategies for enhancing teaching and learning* (1st ed.). Allen & Unwin.
- Hand, D. J. (2016). Measurement: A very short introduction. *Oxford Academic: Very short introductions online*. ISBN 9780198779568.
- Hill, A. T. (2010). Money matters for the young learner. *Social Studies and the Young Learner*, 22(3), 25–31.

- Lusardi, A., Mitchell, O. S., & Curto, V. (2010). Financial literacy among the young. *Journal of Consumer Affairs*, 44(2), 358–380.
- Lusardi, A., Michaud, P.-C., & Mitchell, O.S. (2013). Optimal financial knowledge and wealth inequality. *National Bureau of Economic Research Working Paper 18669*. <https://www.nber.org/papers/w18669>
- Organization for Economic Cooperation and Development. (2019). *PISA 2018 assessment and analytical framework*. PISA, OECD Publishing. <https://doi.org/10.1787/b25efab8-en>
- Pinto, L. E. (2009). Is financial literacy education the solution to credit crises? *Our Schools/Our Selves*, 19(1), 123–133.
- Pinto, L. E. (2012). The politics of Canadian financial literacy education as moments in the circuit of culture. *Leadership & Policy Quarterly*, 1(1), 36–57.
- Roegiers, X. (2013). *Les mathématiques à l'école primaire. Tome 2* (18–26). De Boeck.
- Savard, A. (2011). Teaching citizenship education through the mathematics course. In B. Sriraman & V. Freiman (Eds), *Interdisciplinarity for the 21st century: Proceedings of the 3rd International Symposium on Mathematics and its Connections to Arts and Sciences (MACAS)* (pp.127–137). Information Age.
- Savard, A. (2018). Teaching mathematics and developing citizenship: How to use contexts to enhance problem-based learning. In A. Kajander, J. Holm, & E. Chernoff (Eds), *Teaching and learning secondary school mathematics: Canadian perspectives in an international context* (pp. 317–324). A volume in B. Sriraman (Series Ed.), *Advances in Mathematics Education*. Springer International Publishing
- Savard, A., & Cavalcante, A. (2021). *Financial numeracy in mathematics education: Research and practice*. Springer Nature.
- Savard, A., Cavalcante, A., Turineck, L.-P., & Javaherpour, A. (2020). Some considerations towards the concept of money in elementary school: Money as measurement. *For the learning of mathematics*, 40(3), 22–25.
- Savard, A., & Polotskaia, E. (2017). Who's wrong? Tasks fostering understanding of mathematical relationships in word problems in elementary students. *ZDM Mathematics Education*, 49(6), 823–833.
- Willis, L. E. (2008). Against financial-literacy education. *Iowa Law Review*, 94(1), 197–285.
- Willis, L. E. (2009). Evidence and ideology in assessing the effectiveness of financial literacy education. *San Diego Law Review*, 46(2), 415–458.
- Yasukawa, K., Rogers, A., Jackson, K., & Street, B. (2018). *Numeracy as a social practice: Global and local perspectives*. Routledge.

PEDAGOGICAL ENCOUNTERS AND MATHEMATICAL DETOURS

Rina Zazkis
Simon Fraser University

PRELUDE

Early in this century (I believe) the executive committee of CMESG decided to establish a tradition of inviting an ‘elderly’ member of the group to give a talk (i.e., a topic session). The idea behind this initiative was to introduce the new CMESG members to the generation of founders/retirees or soon-to-be-retirees. This tradition has been followed sporadically, and I believe that my talk at CMESG in 2023 is an example of its implementation. I joined the Simon Fraser University in 1991. But in 1990 something very important happened: SFU hosted the CMESG meeting; it was the first time I participated at CMESG. Only about five CMESG members acknowledged that their first CMESG was on or before 1990—this definitely puts me in the group of elders. I hope that the tradition of inviting elders continues. In what follows is the summary of my talk.

MY WORK AS A MATHEMATICS TEACHER EDUCATOR

In my work, mathematics and pedagogy are juxtaposed. Because my main affiliation is at the Faculty of Education, my students—who are teachers or prospective teachers—expect that my courses focus on pedagogy. However, my goal is to extend teachers’ personal mathematics and connect school mathematics with mathematics learned at the University. As such, I constantly seek tasks with a pedagogical flavor, which help teachers extend their engagement with mathematics.

But there is a caveat: for such a task to be successful, I need to know more about the prior knowledge of my students. Note that teachers in schools usually have a rather good idea about the expected prior knowledge of their students: when planning tasks for their students, teachers follow prescribed curriculum and have taught the same class of students the day or the week before. For a teacher educator the case is different. My students graduated from different universities, in different countries, and in different decades. Some of their undergraduate mathematics courses were taken years ago and some mathematical concepts appear for them as an echo from the faded past. So, *how can a teacher educator ‘scan’ the group’s knowledge of a mathematical topic in order to plan for, or adjust, subsequent instruction?*

To answer this question, in what follows, I

- describe a task of a pedagogical nature that provides teacher educators with a window into strengths and weaknesses in teachers’ knowledge of a particular mathematical topic;
- provide a brief overview of the main themes that emerged from teachers’ responses to the task, and exemplify several responses that informed our design of follow-up instructional activities; and
- illustrate the follow-up instructional engagements.

ON ROLE PLAYING AND LESSON PLANNING

In teacher education, we are faced with a continuous struggle: How do we support and improve teacher development? Watson and Mason (2005) suggest that “the fundamental issue in working with teachers is to resonate with their experience so that they can *imagine* themselves ‘doing something’ in their own situation” (p. 208, emphasis added).

The phrase “imagining doing something” brings to the fore the idea of ‘role playing.’

Role playing is an unscripted “dramatic technique that encourages participants to improvise behaviors that illustrate expected actions of persons involved in defined situations” (Lowenstein, 2007). In other words, role playing is an experiment in which a person is asked to behave as if he were a particular person in a particular situation.

Many advantages for learners involved in role playing are describe in literature. These include, among others:

- Increased interest
- Increased involvement
- Empathy
- Understanding various perspectives
- Deeper engagement with content
- Increased retention

Several CMESG colleagues successfully implement role playing in their education courses (Lajoie, 2018; Lajoie & Maheux, 2013). However, the main disadvantage of role playing is time and participation logistics. That is, when a small number of students are actively engaged in role playing, others are observers rather than actors. *How is it possible, in a limited amount of time, to give all students an opportunity to participate?* My answer is in imagined role playing, or scriptwriting.

ON LESSON PLANS, LESSON PLAYS AND SCRIPTWRITING

Rather than role playing, I engage students in scriptwriting. I ask my students to write a dialogue between a teacher and her students in which they imagine how an instructional situation may proceed. The tradition of writing scripted dialogues dates back to Socrates (470–399 BC) and Galileo (1564–1642) ... so this was not my original invention. More recently in mathematics and mathematics education, presenting ideas in the form of a dialogue has been featured in the works of Lakatos (1976) and Mason and Watson (2009).

For me personally, the idea of scriptwriting started with my dissatisfaction with ‘lesson plans.’ That is, many prospective teachers in my courses expected to produce and share their lesson plans, so they would start their school year with a bunch of ready plans in support of their preparation for teaching. However, I disliked the idea of a lesson plan from the days that I had to create those in my teacher education program. In a good lesson plan we can find, among others,

- How students are expected to engage in an activity that is appropriately chosen
- How students will use manipulatives or other visual representations
- Objectives that are well-clarified
- Clear organization around main concepts or ideas
- Opportunities for students to share their ideas and reflect on the ideas of classmates
- An evaluation process that is clearly related to the declared objective
- Opportunities for extension
- Clear links to students’ prior knowledge

However, even in the best lesson plan we cannot find

- What student difficulties, errors or misconceptions are expected
- How students are guided to overcome those
- What language is used, introduced, corrected and/or supported
- What specific questions are used to assess, extend or refine students’ understanding
- How different forms of reasoning are treated

The traditional format of a lesson plan does not allow for these issues, which are at the heart of teaching, to emerge. The idea of Lesson Play was developed in order to get a glimpse of how teachers address these issues. Lesson Play refers to presenting a lesson in the form of a script for a conversation between a teacher and her students (Zazkis et al., 2009, 2013). It was later noted that an instructional interaction, or any interaction regarding mathematical concepts,

is not limited to a lesson. As such, I abandoned the notion of a Lesson Play and started to describe the method as scripting or scriptwriting. To reiterate, scriptwriting in my research refers to representing an imaginary conversation between a teacher and her students or among students (Zazkis & Herbst, 2018).

TOWARDS A PROMPT

In the beginning of our work that involved scripting, we asked prospective teachers to write a dialogue on how they imagined a lesson on a particular topic. The results were rather dull—most dialogues involved lengthy lecture parts of a teacher-character, with occasional questions and correct answers from student-characters.

We then asked prospective teachers to write a dialogue for a part of a lesson in which they witnessed a particular student error or misconception. The initial dialogue followed the pattern of (a) teacher asks a question, (b) student-A gives a wrong answer, (c) teacher asks a class for other ideas, (d) student-B gives a correct answer, (e) student-B is praised, and (f) the teacher proceeds to the next question or next explanation. We quickly understood that our participants have a very limited repertoire of misconceptions or potential issues that may arise in teaching. As such, we started to develop ‘prompts’—the beginning of the interaction in which some problematic issue is introduced. Such an issue can involve a student error or misconception, faulty reasoning, a disagreement among students, or a student question that a scriptwriter has to address.

In what follows I focus on a particular prompt, describe the scripts based on this prompt and the follow up instructional activities.

TABLE OF VALUES SCRIPTING TASK

The following task was given to 20 prospective teachers enrolled in a course “Investigations in Mathematics.” Among the goals of the course was to strengthen teachers’ mathematical knowledge.

PART 1

You are given the beginning of an interaction between a teacher and students, and your task is to extend this imaginary interaction in the form of a dialogue.

Teacher: Consider the following table of values. What function can this describe?	
Alex: $y = 3x$	
Teacher: And why do you say so?	
Alex: Because you see numbers on the right are 3 times numbers on the left.	
Jamie: I agree with Alex, but is this the only way?	
Teacher: ...	

x	y
1	3
2	6
3	9
4	12
5	
6	

PART 2

You are asked to explain your choice of approach, that is, why did you choose a particular example, what student difficulties do you foresee, why do you find a particular explanation appropriate, etc.

PART 3 (OPTIONAL)

The way you understand the idea yourself could be different from the way you explain it to a student. If this is the case, please indicate how you could clarify the issue for yourself or for a ‘mathematically mature’ colleague.

SYNOPSIS OF RESULTS

A detailed analysis of the results is found in Zazkis and Marmur (2018). In general, the functions mentioned in the scripts in most cases featured an unbound domain, usually \mathbb{R} , and continuity. Below I exemplify excerpts from several scripts.

DIFFERENT WAYS OF DESCRIBING $Y = 3X$

Most scripts focused on different ways to represent $y = 3x$. Examples from several scripts exemplify this tendency:

Teacher: Do you have another way Jamie?
Jamie: When I did it I came up with an equation $y = x^2 + (3 - x)x$.
Teacher: Well this is the beauty of mathematics; things can be represented different ways. [...]
Alex: I think I see the teacher's point. Although it may be the same equation, we can represent it differently.
 Watch: $y = 3x ; x = y/3$
Jamie: Then I guess we could also just say... $0 = 3x - y$
Alex: Ohhh right I didn't even consider that!
Teacher: This is excellent work [...]

CONNECTING THE POINTS AND 'COVERING' THE REAL NUMBERS

Several script writers found different ways to connect the four colinear points.

Teacher: Excellent question Jamie [...] Why don't we start by plotting these points. And by we, I mean you.
 [Student plots the points]
Teacher: Good, so how would it look if we used Alex's function?
Jamie: It would have a straight line through all the points.
Teacher: Yes, but how else can we connect these points?
Jamie: I suppose we could do a zig zag line.
Teacher: Sure, that would work. But we want this to be a function, so what rule do we need to follow?

ACKNOWLEDGING A POLYNOMIAL

Two scriptwriters acknowledged the possibility of a polynomial that contains the 4 points.

Teacher: The points could be modeling anything! There is nothing there that says it has to be a line.
Jamie: Can we find an equation for that though?
Teacher: Certainly, but I need to talk about degrees of freedom. In our table of values we could make up 6 values of y and therefore we have 6 degrees of freedom. Simple enough?
Jamie: Mhmm.
Teacher: So we need to find a polynomial with at least 6 degrees of freedom to describe it, that is a polynomial with at least 6 terms.
Alex: So a 5th order polynomial?
Teacher: Exactly Alex, we could find a polynomial of the form $y = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ that fits the table of values.

While in this excerpt the existence of a polynomial is noted, a particular polynomial was not provided.

EXEMPLIFYING A POLYNOMIAL

Only one script writer provided a polynomial that includes four colinear points.

Alex: I guess it's making us think outside the box a little, but yeah, our other answers are kind of lousy.
 [...]
Teacher: Then let me give you an extension.
 Check out this function $y = x^4 - 10x^3 + 35x^2 - 47x + 24$.
Alex: Where did you get that from?
 [...]
Alex: But it's not a line!
Jamie: Who cares? It's a function. And I guess it takes going to the power of four to hit all four points.
Teacher: I'll leave you to it. Figure out how to derive that equation! I didn't just pull it out of thin air.

Regardless of the claim of the teacher-character, that he "didn't just pull it out of thin air," the script writer acknowledged that the polynomial function was generated by a computer program. While the correctness of this example can be easily verified both algebraically and graphically, finding it became one of the follow up activities.

FOLLOW UP LESSONS

The scripts generated by teachers provide valuable information for a teacher educator for orchestrating a follow up lesson. In particular, scripts provide a lens for studying participants' personal example spaces, as well as the perceived limited example spaces of their imagined students.

CONSIDERING ADDITIONAL EXAMPLES

One direction of the follow up activity was to acknowledge that a function does not need to be defined by a single 'rule' or equation. The following prompt initiated the focus of functions defined piecewise.

Teacher: Consider the following table of values. What function can this describe?	
Alex: It doesn't fit a function.	
Teacher: And why do you say so?	
Alex: For 1, 2 and 3 you clearly see that this is x-squared. But for 4 it isn't.	
Teacher: And what about other values?	
Jamie: I see that there is another pattern. $11 = 4 \times 3 - 1$; $14 = 5 \times 3 - 1$, so $3x - 1$, same for 6.	
Teacher: Interesting...	

x	y
1	1
2	4
3	9
4	11
5	14
6	17

Further, the students were presented with a set-based definition of a function used in contemporary disciplinary mathematics (based on Usiskin et al., 2003, p. 70)

The Cartesian product of two sets A and B , denoted $A \times B$, is the set of all ordered pairs (a,b) such that $a \in A$ and $b \in B$.

For any sets A and B , a function f from A to B , $f: A \rightarrow B$, is a subset f of the Cartesian product $A \times B$ such that every $a \in A$ appears once and only once as a first element of an ordered pair (a,b) in f .

This definition opens the gate for infinitely many possibilities of defining a function that contains the four given points, for example:

$$f : \{(1,3), (2,6), (3,9), (4,12), (5,77), (6,88)\}$$

CREATING A POLYNOMIAL

Following the suggestion that there exists a polynomial function that contains the four colinear points, the task for students was to create such a function. While initially the task presented some confusion, a suggestion to consider a function with four "zeros" provided a stepping stone.

Figure 1 presents the construction: The polynomial function $h(x)$ is created as the sum of the function $f(x) = (x - 1)(x - 2)(x - 3)(x - 4)$ and the linear function $g(x) = 3x$.

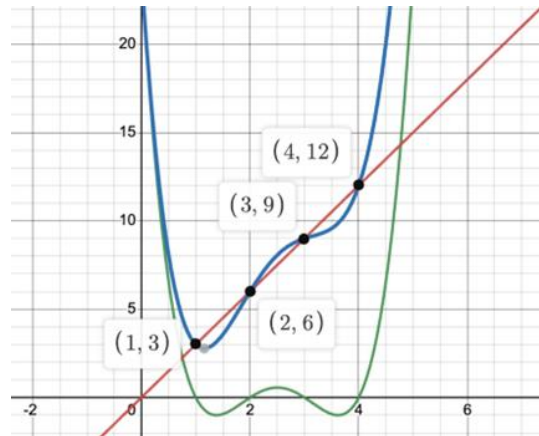


Figure 1. Polynomial with colinear points.

In fact, many of such polynomials can be found, by adding to $g(x)$ a multiple of $f(x)$, as exemplified in Figure 2.

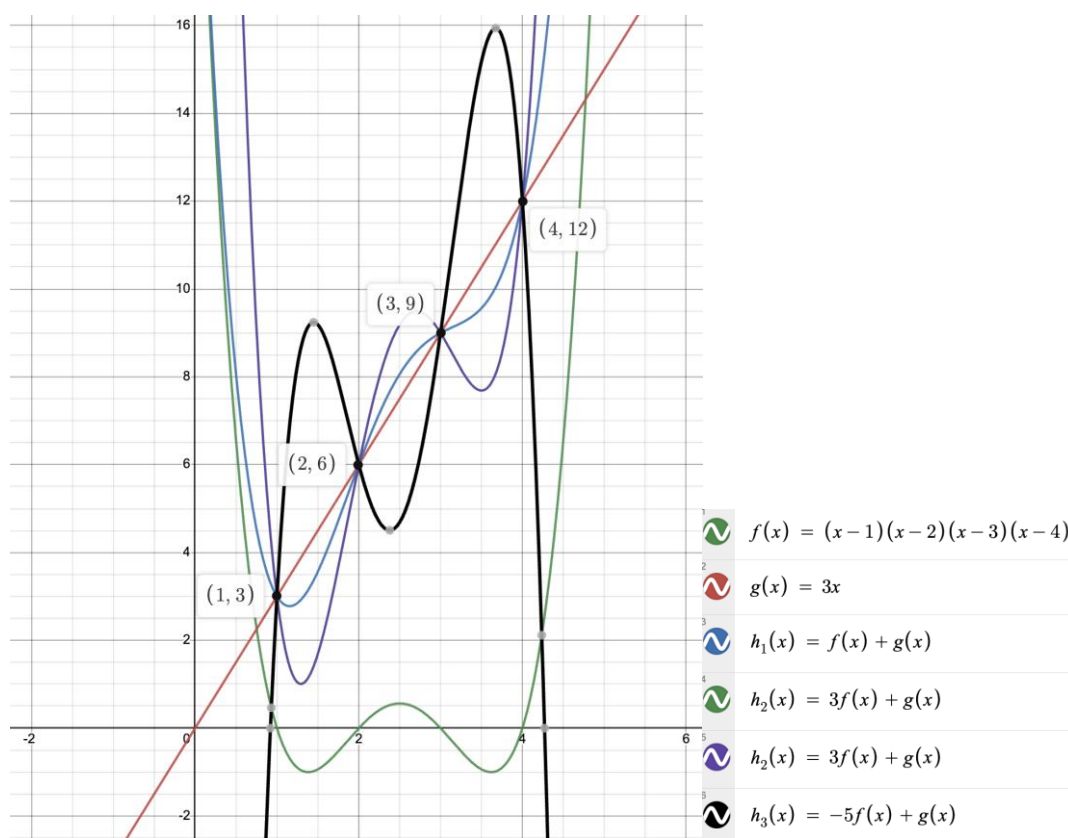


Figure 2. A family of polynomials with four colinear points.

While this approach can be extended to five, six, or more colinear points, a natural question is: How can we create a polynomial with infinitely many colinear points? I invite the reader to explore several solutions.

REFERENCES

- Lajoie, C. (2018). Learning to act in-the-moment: Prospective elementary teachers' role-playing on numbers. In G. J. Stylianides & K. Hino (Eds.), *Research Advances in the Mathematical Education of Pre-service Elementary Teachers: An International Perspective* (pp. 231–243). Springer Cham.
- Lajoie, C., & Maheux, J. F. (2013). Richness and complexity of teaching division: Prospective elementary teachers' roleplaying on a division with remainder. In B. Ubuz, Ç. Haser, & M. A. Mariotti (Eds.), *Proceedings of the Eight Congress of European Research in Mathematics Education* (pp. 3155–3164). Middle East Technical University.
- Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge University Press.
- Lowenstein, A. J. (2007). Role play. In M. J. Bradshaw & A.J. Lowenstein (Eds.), *Innovative teaching strategies in nursing* (4th ed., pp. 173–182). Jones & Bartlett.
- Mason, J., & Watson, A. (2009). The Menousa. *for the learning of mathematics*, 29(2), 32–37.
- Watson, A., & Mason, J. (2005). *Mathematics as a constructive activity: Learners generating examples*. Erlbaum.
- Usiskin, Z., Peressini, A. L., Marchisotto, E., & Stanley, D. (2003). *Mathematics for high school teachers: An advanced perspective*. Pearson Education Inc.

- Zazkis, R., Liljedahl, P., & Sinclair, N. (2009). Lesson Plays: Planning teaching vs. teaching planning. *for the learning of mathematics*, 29(1), 40–47.
- Zazkis, R., Sinclair, N., & Liljedahl, P. (2013). *Lesson play in mathematics education: A tool for research and professional development*. Springer.
- Zazkis, R., & Herbst, P. (Eds.) (2018). *Scripting approaches in mathematics education: Mathematical dialogues in research and practice*. Springer.
- Zazkis, R., & Marmur, O. (2018). Scripting tasks as a springboard for extending prospective teachers' example spaces: A case of generating functions. *Canadian Journal of Science, Mathematics and Technology Education*, 18(4), 291–312.

BETTER LIVING THROUGH COMBINATORICS

Edward Doolittle
First Nations University of Canada

My first encounter with a combinatorial design problem was the struggle to construct a round robin tournament for my grade 8 math class. Since then, I have found some knowledge of combinatorial designs has aided me in finding balanced tournament designs for the University of Regina curling league, designs for efficient mixing of groups for my classes, and breakout sessions for conferences. I teach some combinatorial design in my MATH 101 class as a simple application of modular arithmetic, and I show how the game Spot It is designed, and how prime numbers are important in its design. As a further application, I will discuss the solution of the following International Mathematical Olympiad problem:

An $n \times n$ matrix whose entries come from the set $S = \{1, 2, \dots, 2n - 1\}$ is called a silver matrix if, for each $i = 1, 2, \dots, n$, the i th row and the i th column together contain all elements of S . Show that

- (a) there is no silver matrix for $n = 1997$;*
- (b) silver matrices exist for infinitely many values of n*

Paper not available at time of publication.

New PhD Reports

Présentations de thèses de doctorat

ÉTUDE DE L'ÉVOLUTION DE TÂCHES MATHÉMATIQUES ROUTINIÈRES À TRAVERS LEUR EXPLOITATION COLLECTIVE : LE CAS DE LA TÂCHE DES NOTES DE MUSIQUE EN CLASSE DE 6E ANNÉE

STUDY OF THE EVOLUTION OF ROUTINE MATHEMATICAL TASKS THROUGH THEIR COLLECTIVE INVESTIGATION: THE CASE OF THE MUSIC NOTES TASK IN A 6TH GRADE CLASSROOM

Geneviève Barabé
Université de Montréal

INTRODUCTION

La résolution de problèmes est, pour plusieurs, au cœur de ce qu'est de *faire* des mathématiques (e.g., Arsac et al., 1988; Halmos, 1980). Depuis toujours, elle a occupé une place centrale dans les travaux de recherche en didactique des mathématiques et en *mathematics education* (Liljedahl & Cai, 2021). Plusieurs chercheurs se sont en ce sens penchés sur la construction, la recherche et l'utilisation de *bonnes* tâches mathématiques pour la classe. Ces bonnes tâches mathématiques, par leurs caractéristiques intrinsèques, permettent d'engager les élèves dans une activité mathématique à la fois riche et authentique (voir e.g., Hoshino et al., 2017). Suivant la distinction bien connue de Polya (1945), il est possible de constater que les travaux de recherche ont mis une emphase sur les tâches non routinières, c'est-à-dire des tâches qui nécessitent de surmonter un défi ou une incertitude pour pouvoir les résoudre. À l'opposé, les tâches routinières, couramment appelées « exercices », sont généralement associées à la mise en œuvre d'une activité mathématique pauvre chez les élèves, puisque celles-ci ne nécessitent qu'une application de concepts et de procédures déjà connus pour pouvoir les résoudre (Gavaz et al., 2021). Pour plusieurs, la résolution de ce type de tâches ne se qualifie d'ailleurs pas de résolution de problèmes (Ibid.; Liljedahl, 2020). Alors que le potentiel des tâches non routinières pour la classe de mathématiques a largement été montré par les travaux de recherche (voir à cet effet la récente synthèse de Liljedahl & Cai, 2021), les tâches routinières ne semblent pas avoir reçu autant d'attention de la part de notre domaine de recherche ; sans doute à cause de leur apparence peu porteuse au regard de l'activité mathématique des élèves.

Several researchers, however, have highlighted that the classroom environment could play a significant role in students' mathematical activity, such that the nature of the task may matter little in relation to how it is exploited in the classroom. In this vein, the report from the *Problem Solving: Definition, Role and Pedagogy* working group of the 2016 Annual Meeting of the Canadian Mathematics Education Study Group suggests that a small "twist" can be introduced to push students' activity further: "We said that a 'conventional' problem can become a great learning opportunity by pushing it: by putting a small twist on it" (Hoshino et al., 2017, p. 158). Similarly, Mason (2019) argues that he is increasingly convince that it is not the task itself that is rich but rather how it is used in the classroom. Moreover, in a presentation titled "It's not the task, it's the follow-up!" delivered to the general public, Small (2022) asserts that "many of us would argue that it's less about the task itself and more about the questions teacher asks as and after students work on it." Also, in a more radical manner, Beghetto (2017) suggests that teachers should only use routine tasks, which can be found in abundance in their textbooks, to introduce uncertainty into the mathematics classroom.

Ce potentiel que peut avoir l'exploitation de tâches routinières pour la classe de mathématiques est toutefois peu documenté par les travaux de recherche, et demeure davantage au niveau d'hypothèses. Voulant éclairer ce

phénomène, cette recherche s'est intéressée à comprendre ce qui peut se produire lorsque des tâches routinières sont exploitées de sorte à stimuler l'activité de résolution de problèmes en classe.

SNIPPET OF THE THEORETICAL BACKGROUND

This research is grounded in the biological cognitive theory of enaction (e.g. Maturana & Varela, 1992) and in research conducted under such a theoretical lens (e.g., Kieren, 1995; Martin et al., 2006; Proulx & Maheux, 2017; Towers & Martin, 2015). This theory offers a way to understand individuals' actions through their inter-actions with/in the environment. The emphasis on “inter” and “action” with the hyphen highlights that, from an enactivist perspective, knowledge is observed in one's actions and point to the fundamental role of the environment in those actions (Kieren, 1995). Taking enaction as a grounding perspective invites us to study the evolution, which, in this study, concerns the evolution of routine tasks and their transformations through inter-actions with/in the classroom. Moreover, the focus on the role of inter-actions in one's actions invites us to consider the role of one another in the actions being made and thus to look at the whole class as an entity, as a collectivity that brings forth a mathematical activity together.

Cette recherche prend également appui sur les travaux de recherche sur l'enseignement par résolution de problèmes axé sur l'investigation mathématique (e.g., Borasi, 1996; Cobb et al., 1992; Lampert, 1990; Papert, 2000). À partir d'un travail sur des tâches mathématiques, ces travaux proposent d'investiguer les idées mathématiques proposées par les élèves en classe pour stimuler, amener plus loin et maintenir en vie leur activité mathématique. Dans ces travaux, les tâches mathématiques proposées à résoudre sont vues comme un point de départ pour stimuler l'activité mathématique en classe. Les contenus mathématiques travaillés émergent alors de l'activité mathématique qui prend place en classe. C'est en ce sens que, dans ces travaux, être en activité mathématique prime sur l'apprentissage de contenus mathématiques prédéterminés par l'enseignant. Une analyse de travaux de recherche ancrés dans une approche investigative des mathématiques à l'école a permis de dégager différentes pratiques de mathématisation (Bauersfeld, 1995) pouvant jouer un rôle dans l'exploitation de tâches mathématiques en classe. Ces pratiques de mathématisation sont l'explication et la justification, la validation, l'argumentation, la formulation de conjecture, l'exemplification, l'utilisation de symboles et de représentations, le surpassement des erreurs et incertitudes, la formulation de questions mathématiques ainsi que le recours à un corpus de connaissances mathématiques établies. Ces pratiques offrent un cadre qui permet d'interpréter les actions mathématiques mises en avant en classe en contexte d'exploitation de tâches routinières.

These two levels of theoretical groundings enable an examination of the evolution of routine mathematical tasks through their collective exploitation in the classroom, as well as the role that mathematizing practices, put forth by the collectivity, could play in this evolution of routine tasks. More details regarding these theoretical foundations can be found in French in Barabé (2022) and, to some extent, in English in Barabé (2023).

LE CAS DE LA TÂCHE DES NOTES DE MUSIQUE

This research is part of a larger research project aimed at studying students' mathematical activity in a problem-solving context (e.g., Proulx, 2018). During this project, sessions were conducted, and videotaped, with grade five (10-11 years old), grade six (11-12 years old), and grade eight (13-14 years old) classrooms, totaling 56 sessions spread out over the school year. The tasks used in these sessions were not specifically designed to be effective mathematical tasks nor to further students' mathematical activity. Instead, teachers selected the tasks, mainly from their textbooks, in line with their planned curriculum. Mathematical concepts, procedures, or strategies to solve them had already been taught by the classroom teacher. Consequently, the tasks used in these sessions are considered routine tasks.

Pour étudier l'évolution de tâches routinières à travers leur exploitation collective en classe, plusieurs des 56 séances ont semblé tout particulièrement intéressantes au regard de l'activité mathématique qui a été réalisée. C'est notamment le cas de la tâche des notes de musique :



Figure 1. Énoncé de la tâche des notes de musiques.

Cette tâche routinière a été proposée à résoudre à une classe de 6e année et a mené à une investigation mathématique d'une cinquantaine de minutes à travers lesquelles la tâche a évolué et s'est transformée au fil des inter-actions avec la collectivité. La suite de cet article prend appui sur cette séance pour illustrer les analyses conduites et certains résultats de la recherche.

DESCRIPTION OF THE SESSION

Once the task is stated and displayed on the board, approximately ten seconds are given for reflection on the solution. Initially, an answer of $\frac{3}{10}$ is proposed and explained: "There are 10 music notes, and 3 of them are circled." While several claim to have obtained the same answer, the Principal Investigator (PI) asks if there could be other answers. The answer $\frac{1}{3}$ is then suggested but is immediately invalidated by an explanation of miscounting the total number of music notes. At the board, the PI crosses out one of the music notes to facilitate the explanation of the strategy leading to this answer. The board looked like this:



Figure 2. The music notes task with one music note crossed out.

Des explications et justifications mathématiques appuyant le fait que le dessin représente maintenant trois groupes de 3 notes de musique sont alors mises en avant afin d'expliciter la réponse $\frac{1}{3}$ obtenue :

Rose : *Avec neuf notes, exemple, on peut faire trois paquets de trois, et il y en a un d'encerclé, donc ça fait un tiers.*

CE : *Ok, attends un petit peu, je vais essayer de le faire en même temps. Ok, donc là, on a enlevé cette note-là, hein. Et là, tu nous dis qu'on va faire trois paquets...*

Rose : *Si exemple, on peut faire trois paquets de 3, déjà là, ça fait sur trois.*

Au tableau, les paquets sont encerclés, comme ceci :



Figure 3. Traces de la stratégie du $\frac{1}{3}$.

This strategy then leads the collectivity to explain and justify the equivalence between the answer $\frac{1}{3}$ and the equivalent fraction $\frac{3}{9}$ based on the drawing. These discussions set the tone for the rest of the session as the investigation of this incorrect answer opens doors to new avenues. Indeed, from this point forward, the collectivity begins to seek and propose equivalent fractions to generate new possible answers to the initial routine task.

La réponse $\frac{6}{20}$ est alors proposée et des explications sur le fait qu'il faut doubler le numérateur et le dénominateur sont données ($\frac{3}{10} = \frac{6}{20}$). La collectivité se demande alors comment la réponse $\frac{6}{20}$ peut être expliquée à partir du dessin. L'idée de fraction équivalente est proposée, mais l'explication est incomplète et mène la collectivité à tenter d'expliquer ce que sont des fractions équivalentes. Une explication est à ce moment proposée, à partir d'un dessin d'une barre de chocolat illustrant $\frac{2}{10}$ et $\frac{1}{5}$. Le tableau ressemble à ceci :

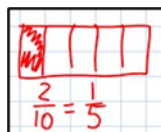


Figure 4. Traces de l'explication de ce que sont des fractions équivalentes.

L'explication suivante est donnée :

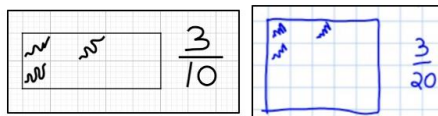
- Nadia : *Quand j'ai cinq parties, c'est comme si un cinquième, c'était égal à deux dixièmes.*
 CE : *Ok, donc ce que Nadia nous dit, c'est que si on regarde les petits carrés, si j'en prends deux sur les dix carrés, ça fait deux dixièmes tu disais. Mais si on regroupe les petits carrés par deux, et on regarde une bande complète, ça fait un cinquième.*
 Nadia : *C'est équivalent.*
 CE : *Donc deux dixièmes, ça serait équivalent à un cinquième.*

Following this discussion, a new explanation is provided using the example of $\frac{50}{100}$, which is asserted to be equal to $\frac{100}{100}$ with the explanation that double of 50 is 100. The collectivity then corrects the error made to obtain the equivalent fraction $\frac{100}{200}$. The proposition that $\frac{50}{100} = \frac{1}{2}$ is then made and justified by the fact that 50 is half of 100 and that 1 out of 2 means half. The example of $\frac{100}{200}$ is justified by the explanation that $100 + 100 = 200$. This leads the collectivity to seek different fractions equivalent to $\frac{1}{2}$. A method to find a multitude of equivalent fractions is then suggested:

- PI: *So, I have 8, what do I do?*
 Timothée: *You multiply it by two.*
 PI: *Okay, I multiply it by two. 8 times 2, that gives me...*
Some students: 16!
 Timothée: *So, since it's 16, the number you multiplied, well, the 8, it's going to be half of 16. After that, we put it in fraction form.*

The collectivity then seeks to determine if the method always works. Various examples aimed at testing it are suggested, adding to the list of different fractions equivalent to $\frac{1}{2}$ already found. The proposed method is considered valid by the collectivity.

Une nouvelle réponse $\frac{9}{30}$ est alors proposée en tant que réponse à la tâche initiale. La réponse est justifiée par le fait que $3 \times 3 = 9$ et que $10 \times 3 = 30$, ce qui donne que $\frac{9}{30} = \frac{3}{10}$. La stratégie de multiplier par le même nombre au numérateur et au dénominateur est ainsi reprise, mais une incertitude quant à sa validité est soulevée. Une explication explicitant que de multiplier par 2 ou par 3 n'a pas d'importance tant que la multiplication soit réalisée au numérateur et au dénominateur est donnée, mais davantage d'explications sont demandées. Le contre-exemple de $\frac{3}{10} = \frac{3}{20}$ est proposé et discuté. Les traces suivantes sont laissées au tableau :

Figure 5. Traces des discussions autour de la non-équivalence entre $\frac{3}{10}$ et $\frac{3}{20}$.

Ces discussions mènent la collectivité à se demander s'il est possible de représenter $\frac{3}{20}$ dans le dessin du $\frac{3}{10}$. Différentes propositions sont discutées, exemplifiées et justifiées, et mènent la collectivité à se demander ce qu'il faut faire à partir du dessin du $\frac{3}{10}$ pour donner des morceaux deux fois plus gros, tout en donnant quand même $\frac{3}{10}$. Une proposition de couper en deux les morceaux est mise en avant, et la question de comment l'écrire avec l'écriture fractionnaire est posée. Cette question occupe la collectivité pendant un certain temps dans un mode de travail individuel ou en petites équipes. De retour en plénière, différentes réponses sont proposées et prises en note au tableau : $\frac{10}{5}$, $\frac{3}{10}$, $\frac{1,5}{5}$, et $\frac{2,5}{5}$. Un

désaccord est exprimé envers certaines réponses puis la collectivité discute simultanément des réponses $\frac{3}{10}$ et $\frac{1,5}{5}$. Celles-ci sont expliquées et représentées par le dessin suivant :

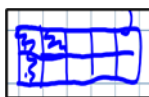


Figure 6. Traces de l'explication du $\frac{1,5}{5}$ et du $\frac{3}{10}$.

L'idée que les morceaux sont deux fois plus gros lorsque divisé en cinquième plutôt qu'en dixième est discutée. La cloche sonne toutefois sur cette discussion ce qui met fin à l'exploitation de la tâche routinière des notes de musique.

ANALYSE SYNTHÉTIQUE DE L'ÉVOLUTION DE LA TÂCHE DES NOTES DE MUSIQUE

Through the collective investigation of the music notes task, various strategies were proposed and investigated. This investigation led the collectivity not only to attempt to solve the routine task, but also to address mathematical sub-tasks that emerged from inter-actions in the classroom. These sub-tasks held the collectivity's attention for a certain amount of time. Through this collective problem-solving activity, the routine task underwent transformations; it evolved. The following figure provides an overview aiming to illustrate the evolution of the routine task in this 6th-grade class.

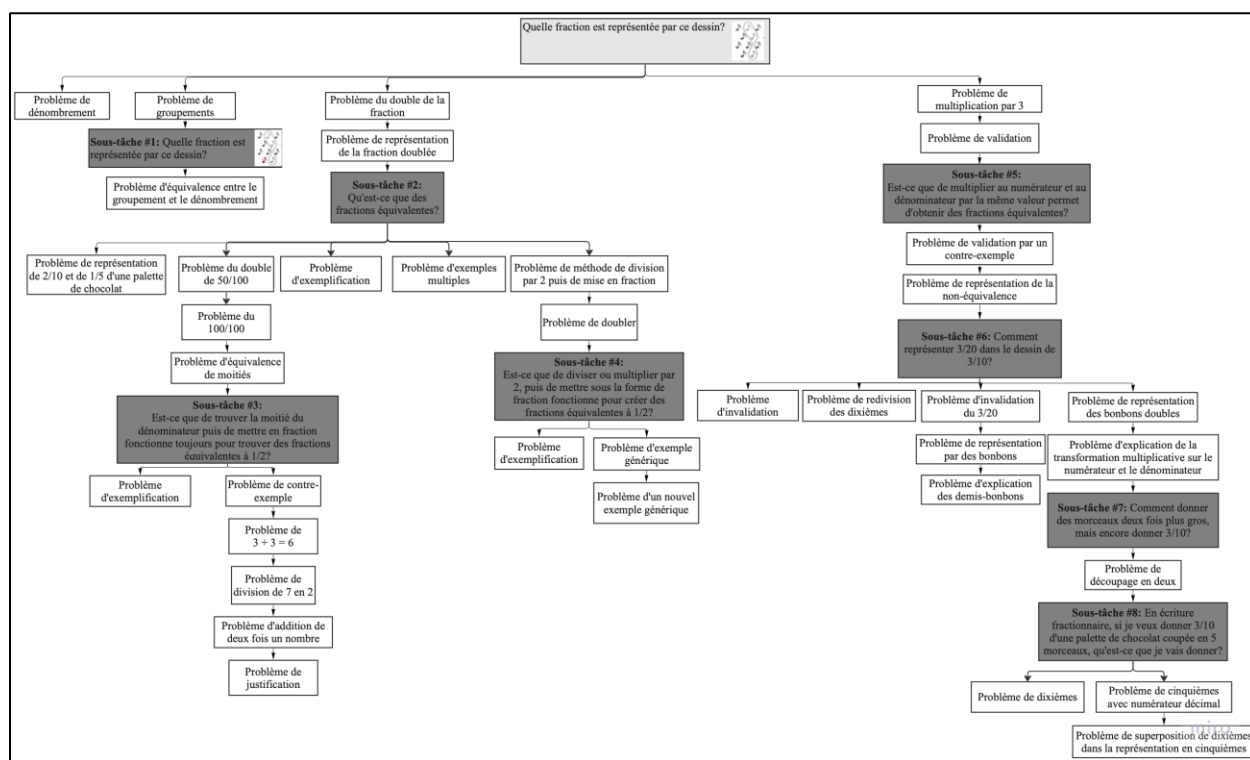


Figure 7. Évolution de la tâche des notes de musique en classe de 6^e année.

As shown in the previous figure (Figure 7), eight mathematical sub-tasks, represented in the grey boxes, emerged from the inter-actions in the classroom through the collective investigation of the initial routine task. These sub-tasks are

1. What fraction is represented by this drawing?
2. What are equivalent fractions?
3. Does finding half of the denominator and putting it into fraction form work to find fractions equivalent to $\frac{1}{2}$?

4. Does dividing or multiplying by 2, and then putting it into fraction form, work to find fraction equivalent to $\frac{1}{2}$?
5. Does multiplying the numerator and denominator by the same value result in equivalent fractions?
6. How do you represent $\frac{3}{20}$ in the drawing of $\frac{3}{10}$?
7. How do you give pieces that are twice as big but still represent $\frac{3}{10}$?
8. In fractional notation, if I want to give $\frac{3}{10}$ of a chocolate bar cut into 5 pieces, what am I going to give?

These eight sub-tasks emerged contingently from the activity that took place in the classroom that day. Even though a routine task was given to solve, uncertainties emerged from the collective activity of solving it; arising from the collective validation of a proposed strategy, from an imprecise explanation, from the overcoming of an error, and so on. Mathematizing practices put forth by the collectivity triggered the evolution of the task. For example, *overcoming the mathematical error* of the answer $\frac{1}{3}$ and its *explanation* led the collectivity to engage in a significant work on equivalent fractions. Also, *validating* the method of finding half of the denominator and putting it in fraction form to find an equivalent fraction to $\frac{1}{2}$ led the collectivity to *exemplify*, *validate*, and *generalize* to the multiplication or division by two in order to find equivalent fraction to $\frac{1}{2}$. Moreover, the *use of symbolism and representation* helped the collectivity to make sense of the discussion about having chocolate pieces or candies twice as big or twice as small. All these mathematizing practices triggered the evolution of the initial routine task, leading to sub-tasks to emerge, while also contributing to their resolution.

DISCUSSION

Cette recherche met en lumière que l'exploitation collective de tâches routinières peut générer une activité mathématique authentique en classe. La recherche soulève, notamment, le rôle fondamental de la collectivité sur l'évolution des tâches routinières proposées à résoudre en classe. Dans l'ensemble des séances analysées, la résolution collective de tâches routinières a permis de faire émerger des sous-tâches mathématiques sur lesquelles la collectivité s'est penchée à résoudre. Ces sous-tâches proviennent des différentes incertitudes vécues par la collectivité à travers l'exploitation collective des tâches routinières. L'engagement de la collectivité envers ces incertitudes semble favoriser l'émergence de nouvelles possibilités qui offrent alors d'autres avenues à investiguer ; ce que Beghetto (2020) appelle des *actionable uncertainties*. La recherche montre également le rôle des pratiques de mathématisation mises en avant par la collectivité dans l'émergence de sous-tâches mathématiques en classe.

At the level of the collectivity, it then becomes evident that the task, initially perceived as routine, is not approached in a routine manner but becomes more complex. This leads us to believe that the collective activity of solving routine tasks implies more than solving it individually. It involves examining each other's strategies and ideas, assessing their mathematical validity, in order to make sense of them and move forward the collective activity. It entails *collective* problem-solving, where productive uncertainties can emerge, guiding the collectivity to explore new mathematical problems arising from this collective enterprise. In this regard, Agre in 1982, already asserted: "If a time-tested procedure exists for bringing about desired state of affairs, there may not exist a problem, because carrying out the procedure may be easy. But if extra effort is required because the income is in doubt, the situation may qualify as a problem." (p. 131). While a dichotomy between routine and non-routine tasks is often emphasized, this research suggests paying closer attention to the lived mathematical activity, especially when the proposed task could evolve through its inter-action with the collectivity. In that sense, this research proposes a reconciliation between routine and non-routine tasks, viewing both as authentic problem-solving situations.

RÉFÉRENCES

- Agre, G.P. (1982). The concept of problem. *Educational Studies*, 13(2), 121–142.
- Arsac, G., Germain, G., & Mante, M. (1988). *Problème ouvert et situation-problème*. IREM de Lyon.
- Barabé, G. (2022). *Étude de l'évolution de tâches mathématiques routinières à travers leur exploitation collective en classe* [thèse de doctorat]. Université du Québec à Montréal. <https://archipel.uqam.ca/16441/>

- Barabé, G. (2023). Emergence of uncertainties and mathematical problems through collective investigation on routine task. *Canadian Journal of Science, Mathematics and Technology Education*, 23, 818–831. <http://dx.doi.org/10.1007/s42330-024-00308-2>
- Bauersfeld, H. (1995). The structuring of the structures: development and function of mathematizing as a social practice. In L. Steffe & J. Gale (Eds.), *Constructivism in education* (p. 137–158). Routledge.
- Beghetto, R. A. (2017). Lesson unplanning: toward transforming routine tasks into non-routine problems. *ZDM: Mathematics Education*, 49, 987–993.
- Beghetto, R. A. (2020) Uncertainty. In V. Glăveanu (Ed.), *The Palgrave encyclopedia of the possible*. Palgrave Macmillan, Cham.
- Borasi, R. (1996). *Reconceiving mathematics instruction: A focus on errors*. Ablex Publishing.
- Cobb, P., Wood, T., Yackel, E., & McNeal, B. (1992). Characteristics of classroom mathematics traditions: An interactional analysis. *American Educational Research Journal*, 29(3), 573–604.
- Gavaz, H. O., Yazgan, Y., & Arslan, Ç. (2021). Non-routine problem solving strategy flexibility: A quasi-experimental study. *Journal of Pedagogical Research*, 5(3), 40–54.
- Halmos, P. R. (1980). The art of mathematics. *The American Mathematical Monthly*, 87(7), 519–524.
- Hoshino, R., Polotskaia, E., & Reid, D. (2017). Problem solving: definitions, role and pedagogy. In S. Oosterle, D. Allan, & J. Holm (Eds.), *Proceedings / actes 2016 annual meeting / rencontre annuelle of the Canadian Mathematics Education Study Group / Groupe Canadien d'étude en didactique des mathématiques* (p.151–162). CMESG/GCEDM.
- Kieren, T. (1995). Mathematics teaching (In-the-middle): Enactivist view on learning and teaching mathematics. *Proceedings of Canadian National Mathematics Leadership conference*. Kingston, Ontario.
- Lampert, M. (1990). When the problem is not the question and the solution is not the answer: mathematical knowing and teaching. *American Educational Research Journal*, 27(1), 29–63.
- Liljedahl, P. (2020). *Building thinking classrooms in mathematics*. Corwin Press Inc.
- Liljedahl, P., & Cai, J. (2021). Empirical research on problem solving and problem posing: a look of the state of the art. *ZDM: Mathematics Education*, 53, 723–735.
- Martin, L., Towers, J., & Pirie, S. (2006). Collective mathematical understanding as improvisation. *Mathematical Thinking and Learning*, 8(2), 149–183.
- Mason, J. (2019). Evolution of a task domain. *Digital Experience in Mathematics Education*, 5, 145–165.
- Maturana, H., & Varela, F. J. (1992). *The tree of knowledge*. Shambhala.
- Papert, S. (2000). What's the big idea? Toward a pedagogy of idea power. *IBM System Journal*, 39(3-4), 720–729.
- Polya, G. (1945). *How to solve it*. Doubleday.
- Proulx, J. (2018). On teaching actions in mathematical problem-solving contexts. In T. E. Hodges, G. J. Roy, & A. M. Tyminski (Eds.), *Proceedings of the 40th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 1060–1067). University of South Carolina & Clemson University.
- Proulx, J., & Maheux, J. -F. (2017). From problem solving to problem posing and from strategies to laying down a path in solving: Taking Varela's ideas to mathematics education research. *Constructivist Foundation*, 13(1), 160–190.
- Small, M. (2022, February 22nd). *It's not the task it's the follow-up* [presentation]. <https://www.onetwoinfinity.ca/its-not-the-task-its-the-follow-up/>
- Towers, J., & Martin, L. (2015). Enactivism and the study of collectivity. *ZDM: Mathematics Education*, 47, 247–256.

A TRI-PLE ANALYSIS OF THINKING MULTIPLICATIVELY AROUND/WITH *TOUCHTIMES*

Canan Güneş
Simon Fraser University

A picture is worth a thousand words.

So, I invite you, dear reader, to have a look at the following image, and think about what you see.



Figure 1. Image from Muhamad Nayem, MN 2019.

I found this image on social media platform X which came with the following text:

If you can see a beach, ocean sky, rocks and stars then you are an artist.

But it's not a painting, it is the lower part of the car [door] which needs to be repaired. (nayem, 2019)

At first, I engaged with this image like an artist. After reading the accompanying text, the focus of my gaze shifted from the top of the image to its bottom, and there emerged the car, for me. This experience made me realize, once again, how we make meaning depends on how we engage with the images which is also true for mathematical models like *TouchTimes* (Jackiw & Sinclair, 2018)

TouchTimes (hereafter TT) is an iPad application designed to animate users' bodies in specific ways to help them develop a robust understanding of multiplication, which is not limited to quick recall of multiplication facts. TT consists of two worlds. In the Zaplify world, learners can create three types of objects: yellow horizontal and vertical lines and orange dots. While the lines emerge at the fingertip of the user, the dots emerge when the lines intersect (Figure 2a). The Grasplify world allows users to create a collection of individual circular discs (pips) with one hand and multiple copies of this collection (pods) with the other hand (Figure 2b).



Figure 2. a) Zaplify and b) Grasplify worlds in TT.

Even though the graphics of these two worlds are different, each world is embedded with multiplicative relationships which are aligned with various researchers' conceptualization of multiplication. In my dissertation, I explored how learners think multiplicatively while engaging with these TT worlds that respond to one's actions based on these multiplicative relationships.

MULTIPLICATIVE VERSUS ADDITIVE RELATIONSHIPS

Multiplication is crucial to gaining an understanding of other mathematical concepts such as proportion (Hino & Kato, 2019). However, studies show that students have certain difficulties not only in recalling multiplication facts and computing multiplicative expressions, but also in solving multiplicative word problems (Degrande et al., 2018). Researchers attributed these difficulties to how multiplication is introduced to students. Many school curricula introduce multiplication to students in relation to various concepts such as skip counting, arrays, division and repeated addition. Among these concepts, the last one seems to be the fundamental concept on which multiplication is based. However, an over-emphasis on the repeated addition model is thought to have a negative effect on students' multiplicative thinking (Vergnaud, 1988).

When students are given problems that can be solved both by additive and multiplicative approaches, they tend to approach the quantities additively. Upon facing quantities in the problems, students are more prone to focus on the difference and the sum between the quantities, rather than the multiplicative relationships among them (Degrande et al., 2018). In the former case, one structures the quantity based on a single unit count. In the latter, one structures a quantity considering the intricate relationships between multiple unit counts. Let me elaborate on this contrast over the quantification of apples in Figure 3.

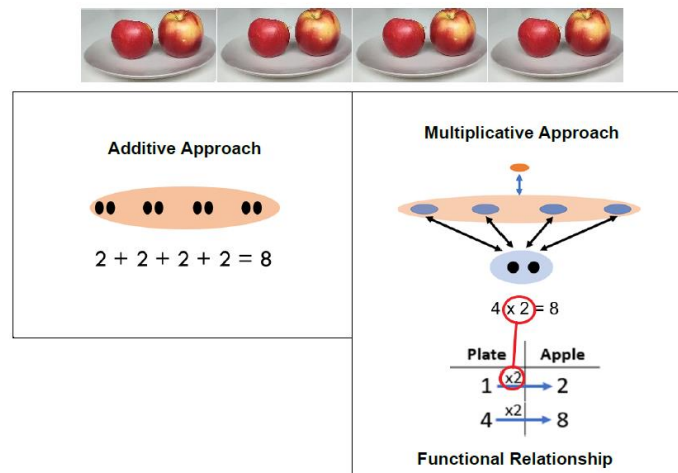


Figure 3. Additive and multiplicative approaches to relate quantities.

We can quantify apples in two different ways: either we add two repeatedly four times or we multiply four by two. In the first way, the only quantity we operate with is the number of apples which is represented by two. We add them four times as there are four plates, but the number of plates is not highlighted in this equation, unlike the number of apples. The only unit we are counting with is *one apple* which is represented in the model with the black dots.

In the second way of quantification, two different quantities are emphasized at once. Unlike in the additive case, “2” does not represent the number of apples, but the number of apples per plate, a relationship between the plates and the apples. Vergnaud (1988) called this a functional relationship. According to this relationship, one plate corresponds to two apples, and this correspondence holds for each plate. Thanks to this relationship, we can calculate the total number of apples not by counting the apples one by one but by counting the plates. Davydov (1992) described this situation as the indirect measurement of quantity through the transfer of unit count, in other words unitizing. In brief, when we think multiplicatively, we simultaneously play with two distinct unit counts that have a functional relationship.

My dissertation (Güneş, 2022) is a collection of three published articles in which I examined emergent multiplicative thinking from three perspectives: (1) the affordances of TT and pencil-and-paper to develop multiplicative thinking; (2) how a young learner makes sense of TT when the TT tasks are solved with corresponding pencil-and-paper activities; (3) how students' multiplicative thinking emerges as they interact with/around TT to collaboratively solve a unitizing task. The following sections presents the summary of each article. Before engaging with these summaries, I invite you, dear reader, to watch this [short video](#) and imagine as if you are using the TT to have a sense of its two worlds, and to familiarize yourself with the TT terminology that I use in this paper.

ARTICLE 1: THE ANALYSIS OF A MODEL–TASK DYAD IN TWO SETTINGS: ZAPLIFY AND PENCIL-AND-PAPER

This article followed from a question I was asked multiple times when I introduced Zaplify to different audiences (researchers, teachers, and teacher candidates): “What can Zaplify do that we can’t do with pencil-and-paper?” To answer this question, I drew on the theory of semiotic mediation (TSM) and examined the semiotic potentials of Zaplify and of pencil-and-paper with respect to the same task that can be solved in these two different settings.

According to TSM (Bartolini Bussi & Mariotti, 2008), whenever individuals use an artefact to achieve a mathematical task in a social context, they use the artefact in a certain way and create certain signs both to achieve the given task and to generate shared meanings. TSM categorizes these signs according to their relationship to the artefact and to the mathematical culture. The artefact sign plays a role in expressing the relationship between the task and the artefact. It is associated with the operations conducted to achieve the task. The mathematical sign expresses the relationship between the artefact use and mathematical knowledge, and it is aligned with the existing mathematical culture. The pivot sign “may refer both to the activity with the artefact; in particular [it] may refer to specific instrumented actions, but also to natural language, and to the mathematical domain” (Bartolini Bussi & Mariotti, 2008, p. 757). It plays an important role in the evolution of artefact signs into mathematical signs. If an artefact can mediate artefact signs that can evolve into mathematical signs, it is said that the artefact has a semiotic potential.

The evolution of artefact signs into mathematical signs is the aim of mathematics education and this is achieved by the semiotic mediation of the artefacts and the cultural mediation of the teacher (Bartolini Bussi & Mariotti, 2008). At this point, Mariotti (2012) considered the analysis of an artefact’s semiotic potential as an a priori phase in designing a successful teaching sequence. The focus of such analysis is on the possible signs (e.g., words, gestures, images) created by the users which can be predicted from examining the tasks in relation to the artefact. Within the literature, I have not found a robust description of a method to conduct such analysis. So, I created two different types of data sets: a video recording of my engagement with Zaplify and still images that captured my engagement with pencil-and-paper to solve the same mathematical task. The analysis of these different data sets revealed some important considerations while identifying the semiotic potentials of artefacts. I elaborated on this methodological contribution in my thesis (Güneş, 2022). In this paper, by comparing the semiotic potentials of Zaplify and pencil-and-paper with respect to the same task, I aim to point to the added value of Zaplify in evoking various meanings of multiplication, not to argue that one is better than the other.

I analyzed the semiotic potential of pencil-and-paper and Zaplify based on the task “Make the product 198 by counting up with four-ples?” which is impossible to make in Zaplify but can be created on paper if one ignores the term “four-ple” (Figure 4). This is a specific Zaplify term that refers to exactly four dots on a line. I created this term to help students experience four as a composite unit.

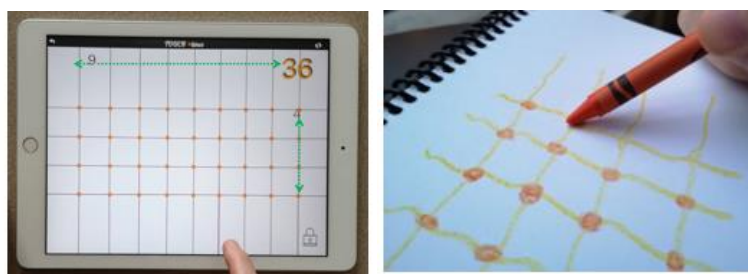


Figure 4. Counting up with four-ples by using a) Zaplify; b) pencil and paper.

The analysis shows that the task solutions undergo significant changes depending on the technological settings. Even though the end product of the model–task dyads might look the same in both settings the multiplicative product emerges from different actions that would mediate quite different meanings for multiplication. For example, the end product may prompt learners to create the same signs in both settings such as four-ple that can be related to the notion of composite unit. However, while a four-ple is created with a single touch all at once in TT, it emerges sequentially when drawn on paper (Figure 4). While the former action can be related to Davydov’s multiplicative notion of unitizing, the latter is more aligned with additive structures as it is created through the sequential operation on a single unit count. Through this study I learned that while designing tasks that involve mathematical models, rather than focusing only on the end product, considering the whole process that leads to the end product would reveal the extensive potential meanings the model–task dyad can mediate.

ARTICLE 2: RECIPROCAL INFLUENCES IN A DUO OF ARTEFACTS: IDENTIFICATION OF RELATIONSHIPS THAT SERVE TO MULTIPLICATIVE THINKING

The importance of multimodality in teaching and learning mathematics has been acknowledged for decades. Aligned with this emphasis, I studied how a child makes sense of multiplicative relationships when he creates and engages with Zaplify objects by using both pencil-and-paper and an iPad. This combined use of a physical pedagogical artefact and its digital counterpart is described as a duo of artefacts (Maschietto & Soury-Lavergne, 2013). In most of the research studies, the duo of artefacts was provided to the students in a specific order in which the digital artefact follows the non-digital one (e.g., Maschietto, 2018). This order might be related to the assumption that the digital artefact expands the affordances of the non-digital one. However, I suspected that this order might hinder the potential of the physical artefact to enrich the affordances of the digital counterpart, as each artefact has unique affordances as I elaborated on in Article 1. Therefore, drawing on TSM, I explored how using a duo of artefacts back-and-forth might influence a student’s understanding of Zaplify world.

In this research, I focused on the signs a six-year-old child Zach (pseudonym) created while using the duo of artefacts and I analyzed the signs considering Arzarello et al.’s (2009) concept of semiotic bundle. There are two ways to analyze a semiotic bundle: synchronic and diachronic analysis. The former focuses on a specific moment where the subject produces different signs spontaneously. The latter focuses on the evolution of the signs produced by the subject in successive moments. I also analyzed different signs created by different artefacts at different time points in a synchronic manner in order to examine the relationship between the artefact signs. This constitutes a methodological contribution of this article.

I created the data from the videorecording of two interviews with Zach who knew number names until 100 and demonstrated a sense of quantity by matching number words with the appropriate number of objects. Each interview began with number-making tasks and was followed by drawing tasks. This cycle was repeated multiple times during the interviews. During the number-making tasks, first Zach and I created products collaboratively in Zaplify, then I asked him what happens if I press my finger on one side of the screen and he predicted the new product. After number-making tasks, Zach created the signs in Figure 5 when I asked him how the dot was.

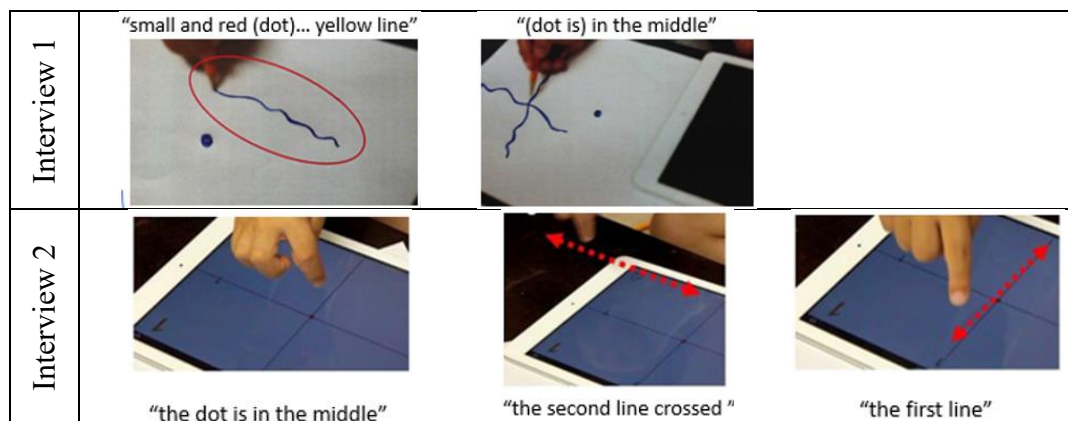


Figure 5. The signs Zach created after number-making tasks in each interview (the red dashed arrows are added to the image to represent back-and-forth tracing gesture of Zach’s index finger).

Initially, Zach used the words “small” and “red” to describe the dot. These artefact signs refer to the physical features of the dot, not its position with respect to the lines. When I drew Zach’s attention to the other Zaplify objects by asking him “Were there anything else other than the dot?” Zach uttered the word “yellow line.” Again, Zach created signs related to the physical features of the objects rather than their orientation (e.g., horizontal/vertical), which is important in terms of distinguishing the two unit counts involved in multiplication. When I hinted at the orientation by asking where the yellow line was, instead of describing its position verbally, Zach created a visual sign by drawing on the paper. This sign illustrates the line in horizontal orientation as in the Zaplify, yet separate from the dot. So, it seems that Zach did not relate the dot with the horizontal line except for their quantities. For one dot, he created one line, which is aligned with the idea of one-to-one correspondence which weighs more in additive thinking.

Zach pointed to the spatial relationship between the signs in our second trial. After Zach and I together made a dot the second time on Zaplify, I asked him to draw a dot on the paper. In this episode, Zach drew one dot and then two lines next to the dot, which intersected each other. So, this drawing suggests that Zach perceived the lines in relation to each other, yet not necessarily related to the dot. After this drawing, Zach started to create intersecting lines on the screen by using his second drawing as a reference to make one dot with the iPad.

Zach started to relate the intersection of lines and the dot in a consistent manner in Interview 2. He referred to the intersection point via the sign “the middle,” which he created during a drawing task during Interview 1. Zach created the verbal signs “the middle” and “these lines” together with tracing gestures. These signs all together suggest that Zach related the intersection point of the lines to the location of the dot. His pointing gesture and the word “middle” refer to the intersection point, and the tracing of the lines refers to the perpendicular lines. According to Zach’s verbal accounts, the intersection seems to be necessary for the dot to appear. He stated that “the second line crossed the first line. The dot is with the second line.”

Some signs which were created in one setting were re-created in another setting later such as the word “middle”. The dynamic gestures which are aligned with the act of drawing lines on paper were re-enacted over the static lines of Zaplify world (Figure 5). Through these re-enactions the static Zaplify lines obtained dynamic characteristics such as ‘cross’ing each other which mediated the relationship between dots and the lines. In light of these findings, I understand that repeating back-and-forth use of a duo of artefacts might enrich learners’ meaning-making experience and mediate the transmission of signs between the settings which may facilitate the identification of mathematical relationships embedded in dynamic mathematical models.

ARTICLE 3: A QUANTITATIVE SHIFT TOWARDS MULTIPLICATIVE THINKING

Compared to Articles 1 and 2, this piece focuses on a different dimension of learning mathematics by using digital technology. In the first one, I focused on the semiotic potentials of Zaplify and pencil-and-paper. In the second one, I studied how these potentials unfolded when a young child used both artefacts back-and-forth. In this article, I explored how a child structured a quantity in Grasplify while collaborating with a peer and a researcher. Rather than a student who uses a duo of artefacts, this study focuses on a duo of students who use a single artefact.

This study was framed by enactivism (Maturana & Varela, 1987/1992) which describes cognition as bringing forth one’s own world instead of creating a mental representation of an objective reality. According to this theory, organisms cognize through a history of recurrent interaction with their environment. During these interactions, changes in the environment trigger some actions in organisms’ structures, and these changes trigger some actions in the environment. Cognitive structures emerge from these interactions. If the organism maintains its unity, then it is said that the organism and the environment are structurally coupled. If the source of perturbations is another organism, then it is said that the organisms are coupled socially. Drawing on this conceptualization of cognition, I examined how children collaboratively structure quantities in order to solve a unitizing task in TT and how children couple with their environment, as well as with other individuals also engaged in this same environment, in order to solve a unitizing task in TT.

To address these questions, I analyzed a video recording of two third graders who interacted around/with Grasplify. The students had never formally been taught multiplication at the moment the video recorded. At first, they explored Grasplify through free play and then one of the students (Jacy-pseudonym) was asked to make seven with one finger. In order to solve this task, the students must first create seven pips on one side and then create a pod on the other side.

Therefore the solution to this task requires that students first use multiple fingers to create seven pips, distinguish pips from pods and press a pod-making finger on the opposite side of the screen, and enact the many-to-one correspondence between pips and pods by pressing one pod-making finger to make seven. The students discovered all of these gestures during free play at different times and synthesized them, yet in a nonlinear fashion, to solve the task successfully. Due to space constraints, in this paper, I only share a few instances from free-play episode that illustrate how students structured quantity in different ways.

At the beginning of the free play session, Jacy was pressing her single finger, and later on, she started pressing multiple pip-making fingers sequentially as she counted up by one (Figure 6 a&b). These gestures are more aligned with additive thinking as the quantity was created by using pips, the only unit count. As the session progressed, Jacy generated a new gesture that is more aligned with multiplicative relationships. She started dragging pips to change the shape of pods indirectly which I associate with many-to-one correspondence between two multiplicative unit counts.



Figure 6. The gestures Jacy discovered during free play.

This shift in the structure of the gestures occurred gradually, through a history of recurrent interactions between Jacy and her environment. Within this history, I identified significant events that happened right before the gestural transitions. One of these events involves some accidental touches on the screen. For example, before Jacy started to drag pips to change the shape of the pods, there were thirteen pods and Jacy was dragging the pods one by one as she held five pips. While trying to drag an existing pod, she accidentally created a pip and dragged it across the screen for four seconds, creating circular paths. During this time, Jacy addressed the covarying relationship between the shape of pods and the movement of pips by pointing to one of the pods, dragging her two pip-making fingers towards each other and further away repeatedly, and saying “look here, I am doing the exact same thing” (Figure 6c). This suggests that the covarying gesture emerged through structural coupling between Jacy and Grasplify world.

Another significant event happened when Jacy was holding a pip on the screen and Kyra tapped her index finger near Jacy’s pip and created another pip. After this event, Jacy started pressing multiple fingers. This suggests that Jacy’s multitouch gesture emerged through a ‘social coupling’ between the students. However, I argue that it is not only Kyra’s tapping gesture that triggered this gestural transition but the relationship between Kyra’s tapping gesture and the emergence of another pip. Therefore, as a theoretical contribution of this article, I propose this incident as an example of a socio-structural coupling between Jacy, Kyra and pips.

CONCLUSION

As evident in the perspective image I shared at the beginning of this summary, I hope to show in my thesis (Güneş, 2022) that the way we engage with images, even visually, changes our meaning-making. Dynamic models like TT that embed specific mathematical relationships through the unique affordances of multitouch technology enrich learner’s engagement with these relationships not only visually but also haptically by prompting specific ways of using one’s own and also others’ bodies including non-human bodies such as paper-and-pencil and TT objects.

REFERENCES

- Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics*, 70(2), 97–109. <https://doi.org/10.1007/s10649-008-9163-z>

- Bartolini Bussi, M. G., & Mariotti, M. (2008). Semiotic mediation in the mathematics classroom: Artefacts and signs after a Vygotskian perspective. In L. D. English (Ed.), *Handbook of international research in mathematics education* (2nd ed., pp. 746–783). Routledge.
- Davydov, V. V. (1992). The psychological analysis of multiplication procedures. *On Learning Problems in Mathematics*, 14(1), 3–67.
- Degrande, T., Hoof, J. Van, Verschaffel, L., & Dooren, W. Van. (2018). Open word problems: Taking the additive or the multiplicative road? *ZDM: Mathematics Education*, 50(1), 91–102. <https://doi.org/10.1007/s11858-017-0900-6>
- Güneş, C. (2022). *A tri-ple analysis of thinking multiplicatively around/with TouchTimes* [doctoral dissertation, Simon Fraser University]. Summit Research Repository. <https://summit.sfu.ca/item/35255>
- Hino, K., & Kato, H. (2019). Teaching whole-number multiplication to promote children's proportional reasoning: A practice-based perspective from Japan. *ZDM: Mathematics Education*, 51(1), 125–137. <https://doi.org/10.1007/s11858-018-0993-6>
- Jackiw, N. & Sinclair, N. (2019). *TouchTimes* [iPad application software]. Tangible Mathematics Group. <https://apps.apple.com/ca/app/TT/id1469862750>
- Mariotti, M. A. (2012). Proof and proving in the classroom : Dynamic Geometry Systems as tools of semiotic mediation. *Research in Mathematics Education*, 14(2), 163–185. <https://doi.org/10.1080/14794802.2012.694282>
- Maschietto, M. (2018). Classical and digital technologies for the Pythagorean Theorem. In L. Ball, P., Drijvers, S., Ladel, H.-S., Siller, M., Tabach, & C. Vale (Eds.), *Uses of technology in primary and secondary mathematics education* (pp. 203–225). Springer. <https://doi.org/10.1007/978-3-319-76575-4>
- Maschietto, M., & Soury-Lavergne, S. (2013). Designing a duo of material and digital artefacts: The pascaline and Cabri Elem e-books in primary school mathematics. *ZDM: Mathematics Education*, 45(7), 959–971. <https://doi.org/10.1007/s11858-013-0533-3>
- Maturana, H. R., & Varela, F. J. (1987/1992). *The tree of knowledge: the biological roots of human understanding* (revised; R. Paolucci, ed.). Shambhala Publications, Inc.
- Muhamad Nayem, MN [@Muhamad]. (2019, July 2). *if you can see a beach, ocean sky, rocks and stars then you are an artist, But its not a painting its lower part of the car gate which needs to be repaired* [post]. X. <https://x.com/nxyxm/status/1146104108006420481?lang=en>
- Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 141–161). Lawrence Erlbaum Associates.

EXPLORATORY STUDY OF MATHEMATICAL ACTIVITY DURING GAME SESSIONS IN THE ELEMENTARY CLASSROOM

ÉTUDE EXPLORATOIRE DE L'ACTIVITÉ MATHÉMATIQUE LORS DE SÉANCES DE JEUX EN CLASSE DU PRIMAIRE

Sabrina Héroux
Université du Québec à Montréal

There is a long tradition of teaching through games, and research has shown that students can improve their mathematical knowledge through the use of games in the elementary classroom. This thesis (Héroux, 2023) asks what happens mathematically when a game is played in the elementary classroom. This exploratory research highlights the mathematical activity (e.g., concepts, processes, and reasoning) when playing a mathematical game in the classroom that appears to be related to different formal components of the game (e.g., purpose, mechanics) and also appears to be related to the classroom (e.g., known/new concept, anticipated/emergent mathematical process). What emerges from the analysis is the richness of what it can mean to do mathematics in the primary classroom. There are also specific avenues and questions for research on the mathematical activity that students engage in when immersed in a game in the primary classroom.

Il existe une longue tradition d'enseignement par le jeu et des recherches ont montré que les élèves peuvent améliorer leurs connaissances mathématiques par l'utilisation de jeux en classe du primaire. Dans cette thèse (Héroux, 2023), on s'est demandé ce qui se passe mathématiquement lorsque l'on joue à un jeu en classe du primaire. Cette étude exploratoire a mis en lumière que l'activité mathématique (p. ex. concepts, processus et raisonnements) lorsque l'on joue à un jeu mathématique en classe semble liée à différentes composantes formelles du jeu (p. ex. finalité, mécanique) et apparaît aussi liée à la classe (p. ex. concept connu/nouveau, processus mathématique anticipé/émergent). Il se dégage des analyses la richesse de ce que peut signifier faire des mathématiques en classe du primaire. On retrouve aussi de manière précise des pistes et des questions de recherche sur l'activité mathématique déployée par les élèves lorsqu'ils sont immergés dans un jeu en classe du primaire.

RESEARCH PROBLEM / PROBLÉMATIQUE

Games have been used in teaching for centuries (Rabecq-Maillard, 1969). However, a certain mystery remains regarding mathematical games for didactic purposes. Studies have shown how mathematical games can have positive effects on students in mathematics (e.g., Ernst, 1986) and can even contribute to specific mathematical learning (e.g., Brousseau, 1998).

Les jeux sont utilisés en enseignement depuis des siècles (Rabecq-Maillard, 1969). Pourtant, un certain mystère demeure en ce qui concerne les jeux mathématiques à des fins didactiques. Des études ont démontré comment des jeux mathématiques peuvent avoir des effets positifs sur les élèves en mathématique (p. ex. Ernst, 1986) et qu'ils peuvent même contribuer à des apprentissages mathématiques précis (p. ex. Brousseau, 1998).

As convincing as these works are, they say little about the mathematical activity experienced by elementary students while they play. This is why this thesis addresses in a broader way the possible contribution of the game on the student's mathematical activity. This seems important to document the extent to which the game offers students a rich environment in which they are led to do mathematics.

Si convaincants que soient ces travaux, ils en disent peu sur l'activité mathématique telle que vécue par les élèves du primaire pendant qu'ils jouent. C'est pourquoi cette thèse aborde de manière plus large la contribution possible du jeu sur le travail mathématique des élèves. Cela semble important pour documenter dans quelle mesure le jeu propose aux élèves un environnement riche dans lequel ils sont amenés à faire des mathématiques.

CONCEPTUAL FRAMEWORK / CADRE CONCEPTUEL

In order to clarify the research question, different authors work has been assembled to clarify what is meant by 'game', and what can be expected regarding 'what happens mathematically' during a 'classroom' game session (for more details, see Héroux, 2023). As can be seen in Figure 1, this provided a kind of *analytical key*.

Afin de préciser la question de recherche, les propos de différents auteurs ont été assemblés pour préciser ce que l'on entend par « jeu », et ce à quoi on peut s'attendre en ce qui concerne « ce qui se passe mathématiquement » au cours d'un jeu « en classe » (pour plus de détails, voir Héroux, 2023). Comme on peut le voir à la figure 1, cela a permis de constituer une sorte de *clé d'analyse*.

- | | |
|--|--|
| <ul style="list-style-type: none"> ● Formal characteristics of a game <ul style="list-style-type: none"> ○ Fictional side ○ Rules ○ Mechanics ○ Purpose ○ Players ● Elements of a classroom game session <ul style="list-style-type: none"> ○ Location (e.g., classroom, other space) ○ Moments (e.g., introduction, game, return) ○ Teacher's postures (e.g. player, game master, pedagogical, educational) ○ Student's postures (e.g., player, actor, learner, student). ● Mathematical aspects <ul style="list-style-type: none"> ○ Concepts (e.g., prime numbers, commutativity of multiplication, neutral element of addition, base ten) ○ Processes (e.g., counting a collection, representing a number, ordering numbers, comparing fractions) ○ Reasoning (e.g. deduction, combinatorics, conjecture) | <ul style="list-style-type: none"> ● Caractéristiques formelles d'un jeu <ul style="list-style-type: none"> ○ Côté fictif ○ Règles ○ Mécanique ○ Finalité ○ Joueurs ● Éléments d'une séance de jeu en classe <ul style="list-style-type: none"> ○ Lieux (p. ex. : classe, autre local) ○ Moments (p. ex. : présentation, partie, retour) ○ Postures de l'enseignante (p. ex. : joueur, maître de jeu, pédagogique, éducatif) ○ Postures de l'élève (p. ex. : joueur, actant, apprenant, élève). ● Aspects mathématiques <ul style="list-style-type: none"> ○ Concepts (p. ex. : nombres premiers, commutativité de la multiplication, élément neutre de l'addition, base dix) ○ Processus (p. ex. : dénombrer une collection, représenter un nombre, ordonner des nombres, comparer des fractions) ○ Raisonnements (p. ex. : déduction, combinatoire, conjecture) |
|--|--|

Figure 1. Analysis key / clé d'analyse.

This study proposes to observe students during a series of game sessions to document the mathematical activity involved. This led to the formulation of the following two research objectives: taking into account the formal characteristics of the game and the elements of a classroom game session, observe and explain the concepts, the mathematical processes and reasoning that arise as students play mathematical games in primary school class and secondly, Identify specific research

Dans le cadre de cette étude, on propose de procéder à l'observation des élèves pendant une série de séances de jeux afin de documenter l'activité mathématique qui s'y mobilise. Ceci a amené à formuler les deux objectifs de recherche suivants : premièrement, en tenant compte des caractéristiques formelles du jeu et des éléments d'une séance de jeu en classe, observer et expliquer les concepts, les processus et les raisonnements mathématiques qui apparaissent pendant que les élèves

leads and questions on mathematical activity related to the formal characteristics of the game and elements of a classroom game session.

jouent à des jeux mathématiques en classe du primaire et deuxièmement, dégager de manière précise des pistes et des questions de recherches sur l'activité mathématique en lien avec les caractéristiques formelles du jeu et les éléments d'une séance de jeu en classe.

METHODOLOGY / DEVIS MÉTHODOLOGIQUE

The methodological approach chosen for this study is exploratory research (Stebbins, 2001). Compared to the agenda of a meeting, exploratory research is based on conceptual aspects while having a point called *varia* that gives a place to ideas that would not have been presented during an initial conceptualization. The rigor of exploratory research is based on the criterion of scientific generativity (Proulx, 2015). The paths and questions of research generated must be linked to the conceptual elements but also a reflection on the nature what is produced during the research.

Data collection for this study took place in a Grade 5 class (24 students). The game sessions (about 60 minutes) were videotaped. The choice of games was made with the teacher according to the level of the students and what she wanted to address. As can be seen in Figure 2, several methodological choices have been made regarding the games so that the sessions are deliberately different from each other to generate the most ideas.

L'approche méthodologique retenue pour cette étude est la recherche exploratoire (Stebbins, 2001). Comparée à l'ordre du jour d'une réunion, la recherche exploratoire s'appuie sur des aspects conceptuels tout en ayant un point appelé *varia* qui fait place à des idées qui n'auraient pas été présentées lors d'une première conceptualisation. La rigueur de la recherche exploratoire repose sur le critère de scientificité de la générativité (Proulx, 2015). Les pistes et les questions de recherches générées doivent être liées aux éléments conceptuels, mais aussi une réflexion sur la nature ce qui est produit durant la recherche.

La collecte de données pour cette étude a eu lieu dans une classe de 5^e année (24 élèves). Les séances de jeux (environ 60 minutes) ont fait l'objet d'un enregistrement vidéo. Le choix des jeux s'est fait avec l'enseignante en fonction du niveau des élèves et de ce qu'elle désirait aborder. Comme on peut le voir à la figure 2, plusieurs choix méthodologiques ont été faits relativement aux jeux afin que les séances soient volontairement différentes les unes des autres pour générer le plus d'idées.

- | | |
|--|--|
| <ul style="list-style-type: none"> • Mathematical concepts <ul style="list-style-type: none"> ◦ Properties of numbers ◦ Whole of a fraction ◦ Proportionality ◦ Grouping and exchanges ◦ Chains of operations ◦ Properties of operation • Mathematical processes <ul style="list-style-type: none"> ◦ Calculate the product of a multiplication ◦ Determine the properties of a number ◦ Adding fractions (including the fraction equivalence process if necessary) ◦ Calculate an operation • Mathematical reasoning <ul style="list-style-type: none"> ◦ Choose the place to put a token • Presentation of the game <ul style="list-style-type: none"> ◦ Video and short discussion ◦ Still image and short demonstration ◦ Rule sheet sent to the teacher ◦ Very short presentation ◦ Simulation • Ways to play <ul style="list-style-type: none"> ◦ Collectively ◦ Team of 4 ◦ Individually • Feedback on the game <ul style="list-style-type: none"> ◦ Collectively ◦ Individually | <ul style="list-style-type: none"> • Concepts mathématiques <ul style="list-style-type: none"> ◦ Propriétés des nombres ◦ Entier/Tout d'une fraction ◦ Proportionnalité ◦ Groupements et échanges ◦ Chaîne d'opération ◦ Propriétés des opérations • Processus mathématiques <ul style="list-style-type: none"> ◦ Calculer le produit d'une multiplication ◦ Déterminer les propriétés d'un nombre ◦ Additionner des fractions (incluant le processus d'équivalence de fraction si nécessaire) ◦ Calculer une opération • Raisonnements mathématiques <ul style="list-style-type: none"> ◦ Choisir la case où placer son jeton • Présentation du jeu <ul style="list-style-type: none"> ◦ Vidéo et courte discussion ◦ Image fixe et courte démonstration ◦ Feuille de règle envoyée à l'enseignante ◦ Très courte présentation ◦ Simulation • Façon de jouer <ul style="list-style-type: none"> ◦ Collectivement ◦ Équipe de 4 ◦ Individuelle • Retour sur le jeu <ul style="list-style-type: none"> ◦ Collectivement ◦ Individuelle |
|--|--|

Figure 2. Methodological choices / choix méthodologiques.

During the processing and analysis of the results of this study, many times when mathematical aspects are clearly recognizable were identified and labeled according to the analysis *key* and emerging categories.

Sixteen mathematical moments were retained and are presented in detail in the thesis (see Héroux, 2023) since there was a distinction between the mathematical aspects, the elements of the class and the formal characteristics of the games. This choice was also made in order to find the level of detail needed to generate new leads and research questions.

Lors du traitement et de l'analyse des résultats de cette étude, plusieurs moments où des aspects mathématiques étaient clairement reconnaissables ont été identifiés et étiquetés en fonction de la *clé* d'analyse et de catégories émergentes.

Seize moments mathématiques ont été retenus et sont présentés en détail dans la thèse (voir Héroux, 2023) puisqu'il y avait une distinction entre les aspects mathématiques, les éléments de la classe et les caractéristiques formelles des jeux. Ce choix a aussi été fait dans le but de trouver le niveau de détail nécessaire afin de générer de nouvelles pistes et des questions de recherche.

RESULTS / RÉSULTATS

In the five game sessions, mathematical concepts (e.g., chance, property of numbers) could be addressed. We also encountered mathematical processes (e.g., calculating the product of a multiplication, determining the properties of a number). The abundance of mathematical concepts and processes observed during game sessions illustrates the richness of mathematics when playing in class. There was also a marked observation of mathematical reasoning (e.g., choosing the square where to place your token, choosing the properties of a number). Although not all games involved mathematical reasoning, their presence brought a significant advantage since it will be possible to play the same game several times in a class to develop strategies. There is therefore mathematical activity that appears while students play math games in primary school class. In light of these results, we can wonder whether it would be possible to cover an entire curriculum with mathematical games.

Dans les cinq séances de jeux, on a vu que des **concepts mathématiques** (p. ex. hasard, propriété des nombres) pouvaient être abordés. On a aussi rencontré des **processus mathématiques** (p. ex. calculer le produit d'une multiplication, déterminer les propriétés d'un nombre). L'abondance de concepts et processus mathématiques relevés durant les séances de jeux illustre la richesse des mathématiques lorsque l'on joue en classe. On a également fait l'observation de façon marquée des **raisonnements mathématiques** (p. ex. choisir la case où placer son jeton, choisir les propriétés d'un nombre). Bien que ce ne soit pas tous les jeux qui comportaient des raisonnements mathématiques, leurs présences ont apporté un plus non négligeable puisqu'il sera possible de jouer plusieurs fois au même jeu dans une classe afin de développer des stratégies. Il y a donc de l'activité mathématique qui apparaît pendant que les élèves jouent à des jeux mathématiques en classe du primaire. À la lumière de ses résultats, on peut se demander s'il serait même possible de couvrir tout un programme de formation avec des jeux mathématiques.

DISCUSSION

The observed mathematical activity is closely related to playing a game in class and a variety of ways to involve it have been noted. For example, classroom game session can be an opportunity for students to encounter concepts, processes and reasoning that may be already known or quite new. The known elements could be worked on in the sense of implementing them, strengthening their mastery or revising their understanding. As for the new ones, they may have appeared as a presentation or been discovered. Finally, we saw that certain concepts and processes appeared

L'activité mathématique observée est intimement liée au fait de jouer à un jeu en classe où on a noté une variété de manières de le faire intervenir. Par exemple, le jeu en classe peut constituer l'occasion pour les élèves de rencontrer des concepts, de processus et des raisonnements pouvant être déjà connus ou tout à fait nouveaux. Les éléments connus ont pu être travaillés au sens de les mettre en œuvre, d'en renforcer la maîtrise ou d'en revisiter la compréhension. Quant aux éléments nouveaux, ils ont pu apparaître sous forme de présentation ou de découverte. Enfin, on a vu que

during games without having been anticipated, providing the teacher with opportunities to engage (or not) the class on this field. Questions and avenues of research have been raised in this study regarding the preparation of the teacher's interventions and also the expression of students regarding mathematical activity.

The cross-sectional analysis helped to show how the mathematical activity when playing in class can be present both at the time of the presentation and during the games and on the return. It would be interesting to have a deeper reflection on how to include mathematics without revealing too much while leaving room for some emergence.

As for the postures of the teacher and the students they were all observed, sometimes changing during the game. It could also be the subject of more in-depth research at a later date since it has been possible to observe a tension between the playful dimension and the didactic dimension which could affect mathematical activity.

Mathematical concepts were useful in games because their adequate mobilization allowed to win (or finish) the game. They can be inside the game, but their involvement in the playful dimension is not always of the same importance. In games, mathematical processes and reasoning are used to make us progress towards victory. Applying a mathematical process is therefore related to the mechanics of the game since the game is not only based on the answer obtained. The predominant place of mathematical processes in games shows that mathematics is not simply added for school purposes but that they are part of the game. A possible avenue of research would be to search deeper into the mechanics of the game and how winning contributes to the expression of mathematical activity.

certain concepts and processes are appearing during games without having been anticipated, providing the teacher with opportunities to engage (or not) the class on this field. Questions and avenues of research have been raised in this study regarding the preparation of the teacher's interventions and also the expression of students regarding mathematical activity.

L'analyse transversale a aidé à montrer comment l'activité mathématique lorsque l'on joue en classe peut être présente tant au moment de la présentation que durant les parties et lors du retour. Il serait intéressant d'avoir une réflexion plus poussée sur la manière d'inclure des mathématiques sans en révéler trop tout en laissant place à une certaine émergence.

Quant aux postures de l'enseignante et des élèves, elles ont toutes été observées, changeant parfois en cours de partie. Elles pourraient également faire l'objet de recherches ultérieures plus approfondies puisque l'on a pu constater une tension entre la dimension ludique et la dimension didactique qui pourrait affecter l'activité mathématique.

Les concepts mathématiques ont été utiles dans les jeux, car leur mobilisation adéquate permettait de gagner (ou terminer) le jeu. Ils peuvent donc se trouver à l'intérieur du jeu, mais leur implication dans la dimension ludique n'est pas toujours de la même importance. Dans les jeux, les processus mathématiques et les raisonnements servent quant à eux à faire progresser vers la victoire. Le fait d'appliquer un processus mathématique est donc lié à la mécanique du jeu puisque le jeu ne repose pas seulement sur la réponse obtenue. La place prépondérante qu'occupent les processus mathématiques dans les jeux démontre que les mathématiques ne sont pas simplement ajoutées pour des fins scolaires, mais qu'elles sont parties prenantes du jeu. Une éventuelle piste de recherche serait de se pencher plus à fond à la mécanique du jeu et à la façon de gagner contribuant à l'expression de l'activité mathématique

CONCLUSION

This thesis (Héroux, 2023) constitutes a contribution on the methodological level to the development of exploratory research since it contains details and adaptations for research in educational science as well as in mathematics didactics. There was also construction of conceptual clarifications surrounding the characteristics of a game, the elements of a classroom game session and the mathematical activity during a mathematical game. Other benefits include the

Cette thèse constitue une contribution sur le plan méthodologique au développement de la recherche exploratoire puisque l'on y retrouve des précisions et des adaptations pour la recherche en science de l'éducation et en didactique des mathématiques. Cette thèse a également permis des clarifications conceptuelles entourant les caractéristiques d'un jeu, les éléments d'une séance de jeu en classe et l'activité mathématique durant un jeu mathématique. Dans les retombées, il faut

development of three games that have been offered to the teacher involved in the research.

At the end of this research, we must highlight some limits. We could obviously go much further and explore in detail the design and modalities of each game. Then, some elements of a classroom game session could be better prepared including feedback. Finally, on the analysis side, mathematical reasoning could not be analyzed as attentively as concepts and processes and mathematical communication was not addressed during the games.

The end of this doctoral project is a springboard to study other games or other aspects of the mathematical activity of students and the teacher in primary school.

aussi compter la conception de trois jeux qui ont été offerts à l'enseignante participante.

Au terme de cette recherche, on doit souligner quelques limites. On pourrait évidemment aller beaucoup plus loin et explorer en détail le design et les modalités de chaque jeu. Ensuite, certains éléments des séances de jeu en classe pourraient être mieux préparés (p. ex. les retours). Finalement, du côté des analyses, les raisonnements mathématiques n'ont pas pu faire l'objet d'une analyse aussi attentionnée que les concepts et les processus et on ne s'est pas penchée sur la communication mathématique au cours des jeux.

La fin de ce projet doctoral constitue un tremplin pour étudier d'autres jeux ou d'autres aspects de l'activité mathématique des élèves et de l'enseignante en classe du primaire.

REFERENCES / RÉFÉRENCES

- Brousseau, G. (1998). *Théories des situations didactiques*. La Pensée sauvage.
- Ernest, P. (1986). Games: A rationale for their use in the teaching of mathematics in school. *Mathematics in School*, 15(1), 2–5.
- Héroux, S. (2023). *Étude exploratoire de l'activité mathématique durant des jeux en classe du primaire* [Thèse de doctorat]. Université du Québec à Montréal.
- Proulx, J. (2015). Going beyond validity criteria in mathematics education research: towards the generativity of a research study. *Chroniques : Fondements et épistémologie de l'activité mathématique*. <http://chroniques.uqam.ca/index.php/2015/07/02/beyondvalidity/>
- Rabecq-Maillard, M.-M. (1969). *Histoire des jeux éducatifs*. F. Nathan.
- Stebbins, R. A. (2001). *Exploratory research in the social sciences*. Sage.

STUDENTS' EMBODIED EXPERIENCES OF SPATIAL CAPABILITY

Josh Markle
University of Alberta

Oh, how wise the body is. (Tokarczuk, 2018, p. 187)

ABSTRACT

Spatial reasoning has been identified as integral to general mathematical capability and the potential for individuals to flourish in life beyond formal mathematics education. Specifically, the ability to visualize shape and space is an aspect of spatial reasoning that is consistently associated with achievement in mathematics. There is also a considerable literature supporting the idea that the body plays a constitutive role in developing mathematical understanding. In my dissertation (Markle, 2022a), I developed and applied an enactive hermeneutic framework to describe and interpret students' embodied experiences of spatial visualization. Enactive hermeneutics draws on a theoretical framework of enactivism, in which cognition is viewed as a complex phenomenon emerging out of the interactions between an organism and the environment, and carnal hermeneutics, which posits that the body is both interpretive and interpretable. Data were generated through my work with students in grade 12 pre-calculus and calculus courses as they took part in a series of lessons designed to foreground spatial visualization in their problem posing and solving, and included video recordings of lessons, written work (e.g., problem solving and reflection), and classroom observation. Results from this study were presented in a manuscript-based format. They included the development of a theoretical framework for interpreting students' experiences in the mathematics classroom, with a particular focus on visualization, a phenomenon that is often difficult to observe and interpret in a classroom context; a conceptualization and analysis of tentativeness as both a strategy and affordance in problem solving in spatial contexts; and the development and application of a novel tool for visualizing how students make sense of mathematics through the body, in particular through sensation (e.g., touch) and orientation, in the context of a spatial reasoning exercise.

INTRODUCTION

I was honoured to present in a New PhD session at this year's meeting (especially so because my supervisor, Dr. Jo Towers, chaired our session). Many thanks to all who attended. In this paper, I provide some philosophical and theoretical context for my study, introduce the participants I worked with, and summarize the foci of the papers that constituted the bulk of my manuscript-based dissertation¹.

The purpose of my study was to offer a deeper understanding of students' embodied experiences of working spatially in the mathematics classroom. This requires some unpacking: First, I was (and still am) interested in the role the body played in learning and doing mathematics. The term *body*, in the sense I employed it, entailed physical movement as well as the potential for movement. This last part makes more sense when one is aware of my specific interest in visualization, which is a critical aspect of spatial reasoning. That is, one can imagine the body's movement through

¹ The current paper uses material from the first and final chapters of my dissertation (Markle, 2022a).

visualization—its potential for movement. To further complicate things, I conceived of the body as not necessarily confined to the boundary of the skin—more on this below.

Back to the idea of working spatially: in my work, I used the term *spatial capability* instead of the more familiar *spatial reasoning* in order to capture the dynamic, embodied, and spatial ways we engage the world. Drawing on Nussbaum's (2011) capabilities approach, I wanted to a) attempt to consolidate all of the different terminology used to refer to spatial acts (e.g., spatial reasoning, spatial ability, spatiality, and so on) and b) speak to the inherent potentiality of spatial actions and how the means by which we experience space “arise, blend, and self-transform” (Davis and the Spatial Reasoning Study Group, 2015, p. 142).

RESEARCH QUESTION

Drawing on carnal hermeneutics (Kearney & Treanor, 2015) and Merleau-Ponty (1968), I understood our bodies to serve “as a precondition for the existence of all other sensations—visual, acoustic, olfactory, gustatory—which participate in it” (Kearney, 2015, p. 28). Our understanding and experience of the world is not only mediated by the body but sustained by it and the bodies of others. In terms of my own classroom teaching practice, I was interested in the way I elicited the spatial capabilities of my students and how I often failed to acknowledge the body's role in perceptual and conceptual understanding. Through my research question—*How can we understand students' embodied experiences of spatial capability in mathematics?*—I probed the ways in which our bodies brought forth mathematical experiences and how we might interpret spatial capability through the body.

SOME KEY IDEAS FROM PHILOSOPHICAL HERMENEUTICS

One of my earliest influences in pursuing this work was Gadamer's philosophical hermeneutics. In Gadamer's (1989) hermeneutics, tradition is an elemental feature of our understanding, and this had implications for how I conducted myself in my research. Working with students in the mathematics classroom, as I did, involves foregrounding one's own prejudices and pre-understandings when a *text* speaks. In much of Gadamer's work, the written word is the paragon text. But because my interest was in the role of the body, I was drawn to an approach to hermeneutics that took a more expansive view of understanding, namely Kearney and Treanor's (2015) carnal hermeneutics. I discuss these ideas in more detail below, but before I do, I want to talk about two sections in Gadamer's foundational text, *Truth and Method* (1989), that indirectly informed my work.

SPACE AND BODY IN HERMENEUTIC PHENOMENOLOGY

Two important features of the hermeneutic experience that Gadamer (1989) takes up in his discussion of the ontology of the work of art are *play* and *festival*. Although the role of the body is not explicitly mentioned in either discussion, each points to a broader, more embodied conception of understanding, one not wholly dependent on the uttered or written word. In fact, both *play* and *festival* evoke the in-betweenness of understanding. Gadamer (1989) began by characterizing play as a clue to ontological explanation and wrote that this sort of play has “its own, even sacred, seriousness” (p. 102). But this is not with respect to the player being serious about a game or taking a game to seriously. Rather, the player

...does not know this [seriousness] in such a way that, as a player, he actually *intends* this relation to seriousness. Play fulfills its purpose only if the player loses himself in play. Seriousness is not merely something that calls us away from play; rather, seriousness in playing is necessary to make the play wholly play. (Gadamer, 1989, p. 103, italics in original)

In this sense, one is not playing a game seriously but rather is serious in being played. As Gadamer (1989) wrote, the “player knows very well what play is, and that what he is doing is ‘only a game’; but he does not know what exactly he ‘knows’ in knowing that” (p. 103).

The idea of play is taken up frequently in education, but Gadamer means something different. Play is not simply an object to be acted on by a subject, the player, but rather is something the player undergoes. It is a transformative experience that Gadamer (1989) connected to the experience of art: much like play, the “work of art has its true being in the fact that it becomes an experience that changes the person who experiences it” (p. 103). Although it was not intended at the outset, I think one of my manuscripts (Markle, 2022b) speaks to this idea.

Having introduced this notion of play, Gadamer then pursued the topic in a curious way, one that I think highlights the tension underlying the primacy he attributed to the written word. First, Gadamer (1989) explored the metaphorical use and etymology of the word *play*. He noted we frequently use play as a metaphor—"the play of light, the play of the waves...the interplay of limbs"—and that these metaphors all point to movement (Gadamer, 1989, p. 104). "In each case," Gadamer (1989) wrote, "what is intended is to-and-fro movement that is not tied to any goal that would bring it to an end" (p. 104). Etymologically, Gadamer (1989) linked play with the German word *spiel* (English: game), which originally meant *dance* (p. 104). Taken together, these premises lead Gadamer to conclude that play is not something that an individual does, but rather is something done between people. In fact, Gadamer (1989) suggested that play consists in this movement between bodies: "it makes no difference who or what performs this movement. The movement of play, as such, has no substrate" (p. 104). Even more succinctly, "play [is] a process that takes place 'in between'" (Gadamer, 1989, p. 109).

This highlights the tension underlying Gadamer's insistence that understanding is most fully realized in written language. Etymologically linking play to the original meaning of *spiel* (dance) is an example of this tension. Does one best understand dance through language, through conversation? On the one hand, we see that Gadamer believed understanding fully realized itself in the written word; on the other, Gadamer's notion of movement in play seems to allow for a more expansive view of the interpretive event. For example, Gadamer (1989) wrote that "understanding must be conceived of as a part of the event in which meaning occurs," (p. 157), but what does this mean for events that involve the body, whether it be dancing, playing an instrument, or as I describe in one manuscript (Markle, 2021), tracing the outline of a canoe's hull through the air with one's hand? Vilhauer (2016) noted that this is indeed problematic for Gadamer's notion of understanding: what then can we say about the "artist's interpretation of a song she has heard, or a dance she has seen, in her own performance of it" (p. 175)?

Has she not really understood the meaning of the music or dance if she has not yet discussed it (with herself or others), or written about it? And what about the interpretation that occurs when an audience watches a ballet, views a painting, experiences a jazz festival, or takes part in an ancient tea ritual? (Vilhauer, 2016, p. 175)

In the final lines of *Truth and Method* (1989), Gadamer evoked Heidegger in quoting the poet Stefan George: "Where the word breaks off, no thing may be" (p. 483), suggesting that understanding cannot be fully realized in absence of the written or uttered word. But what of the spectator of the ballet, in which one's own body feels each pull and thrust of the collective of dancers in one's midst? In viewing a painting, can one reconcile what one knows of the work—its author, its style, and so on—with the tingling sensation it elicits on the back of one's neck or the drop it effects in one's stomach?

Returning to the notion of spatial capability and spatial acts in the classroom, I argue they involve the ways we experience the physical spaces and places around us in complex ways. Gadamer's discussion of the *festival* is worth considering here, in particular as a means by which we can take a broader perspective of what Gadamer means by language. Gadamer invoked the notion of *festival* to explain the temporality of understanding. He began by reasserting the work of art as play: the work's "actual being cannot be detached from its presentation and that in this presentation the unity and identity of a structure emerge" (Gadamer, 1989, p. 120). Gadamer again refers to flux, or movement, between an artwork and an observer, for example. The play itself is repeated over and over again, between the artwork and those who observe it, in such a way that each repeated presentation "is as original as the work itself" (Gadamer, 1989, p. 120).

It is here that Gadamer referred to the festival, which repeats year after year but never in the same way: the "festival that comes round again is neither another festival nor a mere remembrance of the one that was originally celebrated" (Gadamer, 1989, p. 121). This is because the festival, by its nature, does not consist in a particular time or place, but in being celebrated. Like the play of a work of art, it "is not to say that [the festival] is of a subjective character and has its being only in the subjectivity of those celebrating it" (Gadamer, 1989, p. 121). Instead, Gadamer (1989) suggested that the individual celebrating the festival (or being celebrated by the festival) is present in the sense that "one is able to forget one's own purposes" (Gadamer, 1989, p. 122). To be present, Gadamer argued, is to *participate*. While not explicit, I think Gadamer's notion of the festival suggests a broader view of the nature of language in understanding. In asking a forgetfulness, the festival seems to require a participation from the entire body, as Gadamer characterizes true participation in the festival as "being totally involved in and carried away by what one sees" (Gadamer, 1989, p. 122).

CARNAL HERMENEUTICS: TURNING BACK TO THE BODY

This is the final heavy, philosophical part. As I noted above, I was interested in how understanding was grounded in experience in the context of spatial acts in the mathematics classroom. Philosophical hermeneutics offered an inroad in this respect, but I also wanted to speak to our embodied experiences, and so I was drawn to the notion of carnal hermeneutics. Kearney and Treanor (2015) wrote that carnal hermeneutics addresses the “surplus of meaning arising from our carnal embodiment, its role in our experience and understanding, and its engagement with the wider world” (p. 1). It recognizes that while we are indeed languaged beings, the linguistic turn in hermeneutics of Gadamer and Ricoeur emphasized the “temporality of understanding” at the expense of the “spatiality of the flesh” (Kearney & Treanor, 2015, p. 17). A carnal approach to hermeneutics posits that sensation is interpretation, which is an idea I try to take up in one of the manuscripts, which was co-authored with Dr. Towers (Markle & Towers, 2023). In that piece, we drew on Kearney’s (2015) three connotations of sense: sensation, as when we smell the spring air or touch the bark of a tree; meaning, as when we *make sense* of a math problem; and orientation, which refers to “how we orient ourselves in space and time, how we move towards or away from, fore and aft, hither and thither” (p. 16). This is not restricted to how we orient ourselves in three-dimensional, bodily space, but also to the mathematical spaces that are derived from that bodily space. Carnal hermeneutics pointed to that liminal gap in the very distinction between bodily and mathematical spaces—and “where there is a gap there is a surplus: something more to be understood” (Casey, 2015, p. 73).

DATA GENERATION

Data generation occurred in two Grade 12 classes with a total of 36 participants at a large, western Canadian high school. One of the courses was a calculus class in which 16 participants took part in two sessions. Subsequently, five of those participants were invited and agreed to participate in a third and final session. All of the sessions with this first group took place virtually due to COVID 19. The second class was a pre-calculus class in which 20 participants took part in three in-person sessions. Both classes were part of the school’s International Baccalaureate (IB) program.

I planned and taught all of the lessons, which were designed to elicit spatial approaches to problem posing in solving in a variety of ways. Activities included paper folding to explore the properties of parabolas and several visualization exercises (e.g., visualizing and mentally operating on functions in response to verbal, symbolic, and graphical prompts). Data generation was confined to the classroom interactions of students and the researcher, and captured through video recording and participants’ written work, which included reflections on their experiences of visualizing.

Sessions for both classes were designed to elicit spatial approaches to problem posing and solving. Activities common to both classes included visualizing conic sections based on J.L Nicolet’s stop animation videos [see Markle (2021); for the interested reader, these beautiful videos were also taken up by de Freitas and Sinclair (2014) and in a previous CMESG meeting—see Chorney et al. (2019)], folding paper to investigate the properties of parabolas (adapted from Hull, 2013), and several exercises in which students performed mental operations on functions given a verbal, symbolic, or visual prompt. Participants used several means of investigating these activities, including individual and collective sketching, discussion, and digital geometry environments such as Desmos (2024) and GeoGebra (2023).

Each class also used material from the RabbitMath (*The RabbitMath curriculum*, n.d.) curriculum. In the pre-calculus class, participants worked with “Parabola & Line,” which focuses on finding the equations for lines tangent to a parabola. In the calculus class, participants worked with “Walking on the Hill,” in which they found the maximum height of a path traced over a circular paraboloid. These projects were chosen because they drew together compelling visual inscriptions of the problems with rigorous algebraic approaches. They also invited students to pose and solve problems in a variety of ways. For example, although the “Walking on the Hill” problem was used in a pre-calculus class, some participants offered solutions using derivatives, in addition to visual and algebraic solutions.

SUMMARY OF THE MANUSCRIPTS

My dissertation (Markle, 2022a) consisted mainly of three manuscripts (Markle, 2021, 2022b; Markle & Towers, 2023) and they all sit at the intersection of spatial capability in mathematics education, embodied cognition, and hermeneutic phenomenology. My objective was to describe and interpret how students experienced the complex role the body plays in visualizing mathematics in the classroom. However, though I do want my work to inform classroom practice, I do not intend to suggest it was a definitive account. I was not interested in drawing a functional relationship

between the gesturing body and mathematical content knowledge, for example, but rather in opening up the topic—namely, the role our bodies play in spatial reasoning—to question. This included foregrounding my own roles as a researcher, teacher, and body amongst bodies in the classroom. Moules et al. (2015) noted that hermeneutic interpretation is “divergent rather than convergent: it involves carefully opening up associations that strengthen understanding of the topic rather than focusing in on a single governing theme” (p. 117). Throughout each work, I drew on ideas from enactivism (Varela et al., 1991), which provided the theoretical framework through which I could begin to understand the way embodied mathematical understanding arose as a relationship between *things*. It was hermeneutic interpretation that allowed me to see those *things* at a particular moment for particular individuals in particular places. Hermeneutic interpretation was thus a “movement through the landscape of the topic, such that perspectives change with...varied points of view...and are informed by reference to disciplinary and other pertinent literature” (Moules et al., 2015, p. 119). Below, I provide a brief overview of each manuscript.

In Markle (2021), I drew together insights from the theory of enactivism and the philosophy of carnal hermeneutics to develop an enactive hermeneutic approach, which I employed to better understand how students experienced a spatial visualization exercise. Enactive hermeneutics, as I conceive it, foregrounds the principles of enactivism, such as the notion that cognition is a complex phenomenon emerging out of interactions between an organism and its environment, and the sensibilities of carnal hermeneutics, in which the body is seen as the medium through which we bring forth a world of meaning. In Markle (2021), I characterize visualization as a genuinely sensorimotor act; that is, mental imagery is not simply a static representation in the mind’s eye, but rather a dynamic phenomenon that enlists the capacities of the entire body and its potential for movement. Against that theoretical frame, an enactive hermeneutic account put students’ experiences of visualization—the way a visualization *strayed*, for example—into high relief.

In Markle (2022b), I was interested in the strategies students employed when working in spatial ways on a math problem, how they interacted with each other and the environment, and what that could tell me about spatial capability, in general, and visualization, in particular. There are two salient points from this work. One is a more refined characterization of visualization, which I described as a “sensuous phenomenon that emerges out of the interaction between bodies and the environment, one in which ‘imagined perceptual experiences [are] influenced by the way our senses and bodies enfold and involve us in the world’” (p. 21). As I describe in the paper, visualization is still a sensorimotor act that enlists the physical body, but this characterization locates it in the interaction between organisms and the environment. A second point follows from my interpretation of how students experienced and engaged in visualization, which I characterized as *tentativeness*. Tentativeness exemplifies the tension I alluded to above: it is both an emergent, participatory quality of problem solving and posing (i.e., it is a world unto itself) and an affordance that broadens the possibility for adaptive action in an environment (i.e., it is an extension into the world).

In the final manuscript, which became Markle and Towers (2023), I further developed the carnal hermeneutic foundations of my approach to understanding students’ embodied experiences of spatial reasoning. Specifically, I elaborated on Kearney’s (2015) and Merleau-Ponty’s (1968) notion of flesh as the medium through which the body’s movement brings forth a world of meaning in the mathematics classroom. Within an enactive hermeneutic framework, I interpreted video recordings of lessons in part through the process of *Bodymarking* (Towers et al., 2023), which is one way of characterizing the cadence and orientation of the body’s movement in the classroom. My analysis foregrounded the crucial role the body, through its capacities for sensation and orientation, played in developing mathematical understanding. Specifically, it highlighted the complex and varied ways students engage with each other and the affordances available to them in the mathematics classroom through the senses, such as touch and sight.

CONCLUDING REMARKS AND ACKNOWLEDGEMENTS

I appreciated the opportunity to share a bit about this work with you, both in person at CMESG in Regina and in these proceedings. I would like to end by thanking my supervisor, Dr. Jo Towers for her generosity, guidance and support. I was fortunate to have a supervisory committee made up of experts in both mathematics education and hermeneutic philosophy: my gratitude to Dr. Brent Davis and Dr. James C. Field. I am grateful to Dr. Nancy Moules, not only for serving as an external examiner, but also for her teaching and for supporting my work over the past several years through the Canadian Hermeneutic Institute. Many thanks to both Dr. Nathalie Sinclair and Dr. Miwa Takeuchi, who also served as external examiners. As is true now in my new role at the University of Alberta, I credit any success I have to being surrounded by such great people. And speaking of great people, I would be remiss not to thank the students I worked with while engaged in this study.

REFERENCES

- Casey, E. S. (2015). Skin deep: Bodies edging into place. In R. Kearney & B. Treanor (Eds.), *Carnal hermeneutics* (pp. 159–172). Fordham University Press.
- Chorney, S., Sinclair, N., & Coles, A. (2019). Relation, ritual and romance: Rethinking interest in mathematics learning. In J. Holm & S. Mathieu-Soucy (Eds.), *Proceedings / actes 2018 annual meeting / rencontre annuelle of the Canadian Mathematics Education Study Group / Groupe Canadien d'étude en didactique des mathématiques* (pp. 77–84). CMESG/GCEDM.
- Davis, B. & the Spatial Reasoning Study Group. (2015). *Spatial reasoning in the early years: Principles, assertions, and speculations*. Routledge.
- de Freitas, E., & Sinclair, N. (2014). *Mathematics and the body: Material entanglements in the classroom*. Cambridge University Press.
- Desmos. (2024). *Desmos Graphing Calculator* (Version 6.33.0). Desmos.
<https://www.desmos.com/calculator>
- Gadamer, H. G. (1989). *Truth and method* (J. Weinsheimer, Trans.). Continuum.
- GeoGebra. (2023). *GeoGebra* (Version 6.0.841.0) [Computer software]. GeoGebra.
- Hull, T. (2013). *Project origami: Activities for exploring mathematics*. CRC Press.
- Kearney, R. (2015). The wager of carnal hermeneutics. In R. Kearney & B. Treanor (Eds.), *Carnal hermeneutics* (pp. 15–56). Fordham University Press.
- Kearney, R., & Treanor, B. (Eds.). (2015). *Carnal hermeneutics*. Fordham University Press.
- Markle, J. (2021). Enactive hermeneutics as an interpretive framework in the mathematics classroom. *Philosophy of Mathematics Education Journal*, 37(August), 1–27.
- Markle, J. (2022a). *Spatial capability as embodied understanding* [Unpublished dissertation]. University of Calgary.
- Markle, J. (2022b). Tentativeness as strategy and affordance in visualization. *for the learning of mathematics*, 42(1), 20–24.
- Markle, J., & Towers, J. (2023). Bodymarking: Interpreting embodied experiences of spatial reasoning. In X. Robichaud, O. Fellus, D. Martinovic, & V. Freiman (Eds.), *MACAS2022: Creating space for mathematics to emerge between, within, and across contexts and disciplines* (pp. 130–139). Université de Moncton and the Authors.
https://www.umoncton.ca/umcs-macas2022/sites/umcs-macas2022.prod.umoncton.ca/files/wf/macass_2022_proceedings.pdf
- Merleau-Ponty, M. (1968). *The visible and the invisible* (C. Lefort, Ed.; A. Lingis, Trans.). Northwestern University Press.
- Moules, N., McCaffrey, G., Field, J. C., & Laing, C. M. (2015). *Conducting hermeneutic research*. Peter Lang.
- Nussbaum, M. (2011). *Creating capabilities: The human development approach*. Harvard University Press.
- The RabbitMath curriculum*. (n.d.). RabbitMath. <https://www.rabbitmath.ca/curriculum>
- Tokarczuk, O. (2018). *Drive your plow over the bones of the dead* (A. Lloyd-jones, Trans.). Fitzcarraldo Editions. (Original work published 2009).
- Towers, J., Markle J., & Jacinto, E. L. (2023). Bodymarking: An interpretive framework for analyzing embodied action in classrooms. *Canadian Journal of Science, Mathematics and Technology Education*, 23(1), 66-79.
- Varela, F. J., Thompson, E., & Rosch, E. (1991). *The embodied mind: Cognitive science and human experience*. MIT Press.

Vilhauer, M. (2016). Verbal and nonverbal forms of play: Words and bodies in the process of understanding.
In G. Warnke (Ed.), *Inheriting Gadamer* (pp. 161–180). Edinburgh University Press.

EMBODIED CURIOSITY IN THE MATHEMATICS CLASSROOM THROUGH THE AFFORDANCE OF THE GEOMETER'S SKETCHPAD

Sheree Rodney
Ontario Tech University

This research uses data collected from grade nine students (14-15 years old) at two secondary schools in Jamaica to examine how curiosity, embodiment, and digital technology relate to the construction of mathematical meanings. In doing so, I designed a theoretical framework that I named Embodied Curiosity to illustrate how these relationships become possible when students interact with circle geometry theorem in a dynamic geometry environment (DGE). The framework is grounded in theories of embodied cognition and draws on Pickering's (1995) account of agency, where human, material and disciplinary agents work together in the learning process. In this research, curiosity is reconceptualized as something observable and a relationship that emerges when learners engage with digital technology tools. This study highlights the significance of considering curiosity and DGEs as essential aspects of embodied learning.

INTRODUCTION

Mathematics education research was dominated by the Cartesian mind–body divide for an extended period, but more recent studies have shifted focus to embodiment (Lakoff & Núñez, 1997/2000), and mathematics learning (de Freitas & Sinclair, 2014; Hall & Nemirovsky, 2012; Radford et al., 2009; Sinclair & Heyd-Metzuyanim, 2014). While there is a growing interest in researching embodiment, I find it unclear how theories of embodiment consider the affective domain of learning. Furthermore, there is high research interest about how to encourage students meaning-making using technology (Schneider & Gu 2022) but little is said about the role emotions play in the meaning-making process in a technology-enhanced mathematical environment. My research is concerned with the intersection of embodiment (which is relatively new in mathematics education), the affective domain of learning (specifically curiosity), and the influence of digital technology (such as *The Geometer's Sketchpad*) in facilitating embodied learning. In order to better understand these relationships, I designed a theoretical framework which I called 'Embodied Curiosity' with the aim of gaining insights into learners' meaning-making processes when the focus is on embodiment, curiosity, and digital technology. This paper aims to present the framework and demonstrate its application as a research tool for exploring mathematical meaning-making. In doing so, I use empirical data collected from grade nine students in Jamaica who engaged the dynamic geometry environment (DGE) to explore the circle geometry theorem. In this report, I used data from a specific episode, drawn from a broader pool, to reconceptualize curiosity as a construct that is linked to digital technology through the body. The research questions that guide this study are *What are the physical markers of curiosity in the secondary mathematics classroom?* and *To what extent does Embodied Curiosity foster the construction of mathematical meanings?*

RELATED LITERATURE

Curiosity, as we understand it, cannot be seen with the naked eye; as a result, it cannot be touched or measured. For this reason, formulating a precise definition or developing theories about curiosity presented challenges. In much research, curiosity is seen as an internal motive with a biological function that influences human behaviour and fosters active learning (Oudeyer et al., 2016; Kidd & Hayden, 2015). However, Loewenstein (1994) argues that previous definitions did not capture certain significant characteristics of it. One such example is its intensity (the pain of not having information) and another its transience (how short-lived curiosity is). As a result, Loewenstein

reconceptualized curiosity as an information-gap construct. That is, he suggests that curiosity is “a form of cognitively induced deprivation that arises from a gap in knowledge or understanding” (p. 75). Implying that curiosity is as a result of the imbalance between what we know and what we want to know. Taking this into consideration, Loewenstein’s information-gap definition forms the basis of this research, where uncertainties, doubts or wonderings are seen as signals for missing information.

Similar to curiosity, embodiment, is an elusive term to define. However, Lakoff and Núñez (1997/2000) argued that the human body must be given a fundamental role in shaping the mind and that mathematical ideas are shaped by our everyday experiences. They further suggest that the sensorimotor system of the brain and body plays an important role in how knowledge is constructed. In making this claim, they further argued that mathematical meanings emerge in the teaching-learning process through multimodal means. That is, elements such as the body, languages, and material artefacts (including digital technology) are considered as central components of how students and teachers think mathematically. In this context, it becomes evident that the processes of meaning-making and cognition are intricately connected to physical experiences which encompass actions, emotions and perceptions.

Mathematical meaning-making processes are consistently changing, creating a space for the development of innovative and critical involvement ways in which children learn (Radford et al., 2011). This shift moves the focus away from traditionally recognizing the teacher as the sole dispenser of knowledge, to embracing learners as capable of constructing their own meanings. Based on this, there is a constant need to better understand how meanings are constructed in mathematics education. While Radford et al. (2011) argue that the mathematical meaning-making process involves the way learners think and communicate about mathematical concepts when they interact in a social context, Jaworski and Didis (2014) argues that it is the connections that learners make both within the context of mathematics and the physical world, as well as the processes of socialization within a culture. Radford et al. (2011) also proposed that mathematical meanings can be achieved through written, oral, bodily and other communication cues. It is within these contexts that my research explores mathematical meaning.

As technological advancements continue to rise, the demand for seamlessly incorporating digital technologies in mathematics classrooms is growing rapidly. Dynamic geometry software (DGS) such as Cabri (Cabrilog S.A.S, 2004), GeoGebra (2023), and *The Geometer’s Sketchpad* (or *Sketchpad*, KCP Technologies, n.d.), is being widely promoted to enhance learners’ experience with hands-on manipulation of mathematical concepts, as well as the potential for more collaborative learning. One reason for this is the dynamic capabilities of DGS. Nicholas Jackiw (2006), argues that besides the aesthetic, visual and symbolic capabilities of the software, it provides opportunity to explore the dynamic actions of a construction easily. According to Jackiw (2006) this can be achieved by moving and manipulating elements through the intuitive dragging feature of the software. Expanding on Arzarello et al.’s (2002) dragging modalities, this research investigates how learners interact with the *Sketchpad*’s (KCP Technologies, n.d.) dragging feature to create and convey meanings. According to Arzarello et al. learners’ dragging practices follow certain modalities when they perform geometric tasks in a dynamic environment. These include *wandering*, *bound*, *guided*, *dummy locus*, *line*, *linked dragging*, and the *dragging test*. Considering these dragging practices, I have expanded Arzarello et al.’s (2002) modalities to include a form of dragging that I have termed **curious dragging**. From an Embodied Curiosity standpoint, curious dragging involves incorporating any of Arzarello et al.’s (2002) modalities along with other elements such as temporality, speed and emotion. For example, bound dragging occurs when the learner moves a constructed point that is linked to an object. However, if this is done in a slow meticulous manner or quickly with enthusiasm this is interpreted as **curious dragging**.

THEORETICAL CONSIDERATION

Grounded in theories of embodied cognition, which considers knowledge construction in relation to the way human bodies operate in the environment (Lakoff & Núñez, 1997/2000), Embodied Curiosity, illustrated as a stratigraphic structure in Figure 1, encompasses the interplay among four core elements—curiosity, body movement, digital technology and mathematical meanings.

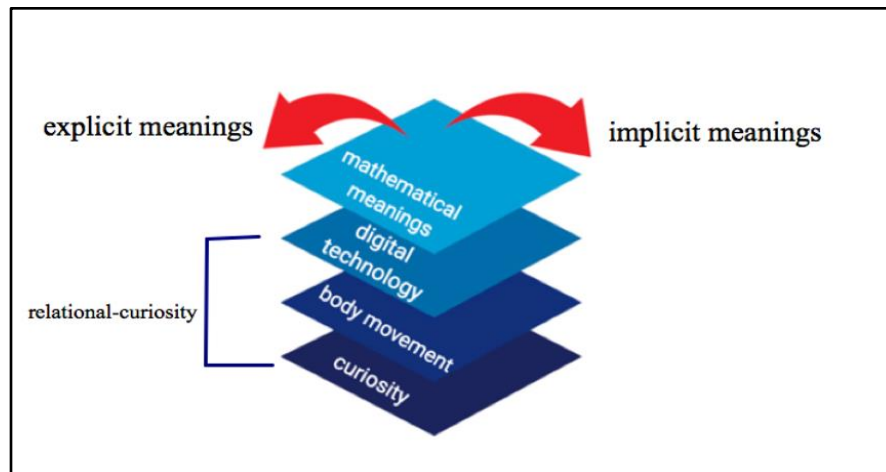


Figure 1. The Embodied Curiosity model.

The four layers metaphorically assume semi-permeable ability like the membrane of a biological cell to accommodate movement of ‘instances’ from one layer to another. Instances, in this sense mean factors such as students’ wondering questions (e.g., ‘what ifs?’, ‘suppose...’) and learners’ body movements that indicate uncertainty or doubt. These body movements could be visual fixation, eye gazes, raised or furrowed eyebrows, facial expressions, leaning forward and backwards, or any visual signals that indicate the presence of surprise. The framework adheres to three main principles. The initial principle proposes that bodily actions serve as a bridge that connects trait-curiosity (our inherent ability to contemplate the world) with digital technology. This relationship is referred to as relational-curiosity. The second principle positions curiosity as temporal and emergent, meaning that it is unplanned and emerges in real time. In other words, it is unpredictable how curiosity will manifest. The third principle involves learners engaging in curious dragging when they interact within a dynamic learning environment.

Drawing on Pickering’s (1995) concept of agency and the underlying assumption that knowledge is generated through interactions with human, material and disciplinary agencies, the concept of ‘The Mangle’ is used to express the tripartite relationship among trait-curiosity, body movement and digital technology. In the mangle of practice theory, Pickering (1995) places more emphasis on performance or ‘doing’ rather than on cognition. In this way, his focus is on what humans can do, what the material (*Sketchpad*, KCP Technologies, n.d.) does and how these performances intertwine or ‘mangle’ with each other. He argues that, in people’s desire to understand the world around them or construct knowledge, they are led to do certain things, thus encountering resistance from various sources, including material objects. Resistance usually hinders the smooth running or data collection of a process and that knowledge is generated through this ‘dance of agency’ of resistance and accommodation between people and things. For Pickering (1995), agency has to do with the influence of one thing onto the other. Fundamentally, in this research context, agency is the influence of trait-curiosity onto the digital technology with the body acting as an agent between the two.

METHODOLOGY

This study incorporates data derived from video recordings of classroom observations and the students’ work at two high schools located in the Caribbean. To safeguard the schools’ identity, I am referring to them as Schools X and Y. They were purposely selected because of their technology-rich learning environment. They operate with fully equipped computer labs with at least 30 functional computers dedicated to students use. The participants were grade nine students (ages 14 and 15 years old), who were randomly selected from a pool of grade nine classes. The two respective classroom teachers, Sammy and Andrew (pseudonyms given), were also a part of the study. At the time of the research, the students were beginning the geometry section of the mathematics curriculum, and the topic of circle geometry theorem was chosen for investigation. Moreover, the students had encountered geometry in earlier grades and were already acquainted with the fundamental properties of the circle in a static context. Nevertheless, their introduction to circle geometry theorems was a first-time experience, and notably, both teachers and students were also using *Sketchpad* (KCP Technologies, n.d.) for the first time.

The classroom observations were done over a three-week period where students engaged in tasks using only *Sketchpad* (KCP Technologies, n.d.). They were allowed to collaborate with each other either working on individual computers or by sharing one computer in pairs or triads. The duration of the classroom sessions was forty-five minutes, and the classroom interactions were video recorded. Students' work was saved and retrieved from the schools' server after each session. The data was analyzed over a six-month period in a continuous, iterative manner to ascertain the attributes of students learning that could be interpreted as observable curiosity. These instances were extracted from the data pool, and a second-phase analysis carried out to identify episodes where relational-curiosity (Curiosity-body movement-technology) was evident. Ultimately, the chosen episodes underwent analysis to identify instances of the emergence of mathematical meanings. Six specific episodes were selected and incorporated into my dissertation (Rodney, 2021). However, this paper utilizes a single episode to illustrate the potential applicability of Embodied Curiosity for mathematics learning. The participants in this episode are Ali and Joni (pseudonyms given), and they were working on a task that involves investigating the relationship between the angles in a cyclic quadrilateral.

OPPOSITE ANGLES IN A CYCLIC QUADRILATERAL

In this session, Ali and Joni (students from School X), were working together in a shared-computer arrangement. They were investigating the connection between the opposite angles in a cyclic quadrilateral and were tasked with describing their observations when one, two or three of the angles were fixed at ninety degrees. The task was communicated verbally, and students encouraged to listen attentively before starting, as it would not be presented in the familiar written format that they were accustomed to.

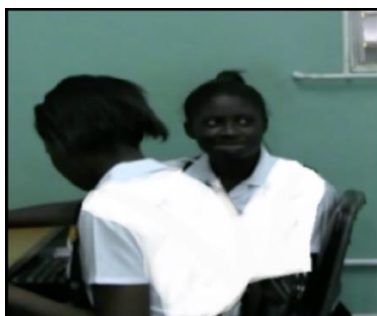


Figure 2. Ali with a bewildered look on her face.

As soon as the task was given, Ali, with a bewildered look on her face (Figure 2), asked, “Should they have the same measurement?” This was interpreted as her first sign of uncertainty. Her immediate question, coupled with the bewildered facial expression, indicated the onset of curiosity. These combined factors seemed to be physical markers of curiosity and suggested that the Embodied Curiosity process was initiated. In this scenario, Ali’s wondering triggered her body to act upon her uncertainty, suggesting a possible movement from the layer of curiosity to the layer of body movement. The use of the word “they” in her question implied the creation of multiple shapes, and the words “same measurement” implied that within these shapes, there was a need for angular measurements. Ali sought solutions using the technology, initially constructing a circle and incorporating a quadrilateral—specifically a rectangle—within it (Figure 3a). This action highlighted the curiosity-body-technology relationship and provided evidence of **relational-curiosity**. It also revealed an information-gap in Ali’s understanding of the cyclic quadrilateral.

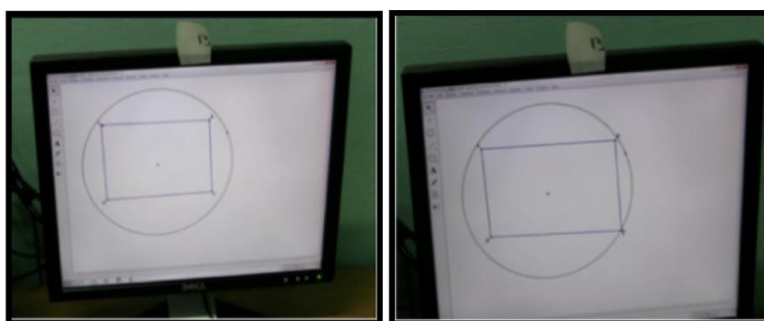


Figure 3. (a) Quadrilateral detached from the circle. (b) Ali engaged in curious dragging.

Furthermore, the initial diagram by the girls in Figure 3(a), wherein all four vertices are detached from the circumference, revealed the ‘missing’ knowledge that this unique quadrilateral generally requires all four vertices to be touching the circle to maintain its concyclic property. Ali briefly glanced at the other computer screens in the room and then started to adjust her diagram. She began by dragging each vertex of the quadrilateral onto the circumference glancing back and forth as she moved each point individually. In a sense, Ali’s brief glances (wandering eyes), as well as the decision to drag each point of the quadrilateral onto the circumference (Figure 3b), showed that the eyes and the computer were engaged in a ‘dance of agency’. That is, the body (in this case, the eyes) performed the action of glancing, which further triggered the hand to react to the technology (dragging the vertices onto the circumference). A process that was repeated for all four vertices. This back-and-forth exchange effectively addressed Ali’s uncertainty of whether or not the rectangle should be connected to the circle.

In this interplay, Ali engaged in **curious dragging**, that is, Arzarello et al.’s (2002) linked dragging (linking a point to an object and moving it onto that object), but in a slow and cautious manner. Perhaps this was done to avoid disrupting any previously discovered properties. In performing curious dragging in such a deliberate manner, Ali’s engagement with *Sketchpad* (KCP Technologies, n.d.) presented an opportunity for movement to take place between the layer of body movement and the digital technology. After a few minutes had passed, the girls were able to use the ‘measure’ tool (one of the affordances of the software) to obtain the angular measurement of each angle in the quadrilateral. They could explicitly state that “the sum of the opposite angles adds up to ninety degrees” but were unable to state that these angles were supplementary. This was interpreted as the emergence of two types of mathematical meanings, one that is clearly stated (explicit meanings) and one that is implied (implicit meanings).

The students were asked to use their bodies to perform a re-enactment of the computer-based task (Figure 4). Ali and Joni were among the students who offered to participate. During the re-enactment, Ali suggested that, to depict the on-screen task, some individuals should form a circle, while others were needed for the quadrilateral. She said, “We need four people in the middle for the rectangle.”



Figure 4. Students using their bodies to re-enact the cyclic quadrilateral.

The students took a few steps back to position themselves as points on the circumference. This action was interpreted as the emergence of the circle definition as the locus of a point equidistance from the centre. Perhaps this action was to make the circle visible to their peers or to ensure that they had adequate space for the rectangle (four students). Ali instructed the four students on the inside to “hold hands” and then further said, “Hold on to my hand” to the student nearest to her, indicating that there was a need for the rectangle to be touching on the circumference. After completing the re-enactment, they proceeded to call out the names of the students who were positioned opposite to each other. However, Ali introduced a ‘new’ piece of knowledge to the experience. She exclaimed, “Wait! They add up to three hundred and sixty degrees.” It was clear that Ali was aware of a second relationship. However, using the pronoun “they” implies that she was uncertain about which attribute of the shapes had this relationship. This mathematical connection did not become apparent when the task was performed using *Sketchpad* (KCP Technologies, n.d.). It seemed that the dynamic capability of *Sketchpad* (KCP Technologies, n.d.), combined with students’ re-enactment with their bodies, acted as a trigger for the construction of ‘new’ mathematical meanings.

CONCLUSION

The episode discussed in this paper presents a novel perspective on utilizing curiosity for the construction of mathematical meanings. It accentuates the mediating role of digital technology in the meaning-making process and

highlights the significance of considering curiosity and digital technology for embodied learning. The result from this episode shows that students' wondering and uncertainties manifest through questions and corresponding body movements, serving as potential indicators of observable curiosity. The Embodied Curiosity framework underscores the significant role of the body in influencing the mind and the digital technology as an embodied tool collaboratively shaping learners' mathematical meaning-making processes. In keeping with the information-gap definition of curiosity and the relationship among the core elements of the Embodied Curiosity framework, this work emphasizes a new way of thinking about curiosity. It connects to the material tool through the body, especially when there is uncertainty about the mathematical ideas. How students learn mathematics with technology is not only beneficial to the pedagogical goals of mathematics education but also provides a pathway that connects mathematics to the body and the material world. A viable approach going forward is to recognize the importance of DGEs as crucial tools for nurturing curiosity in learners.

REFERENCE

- Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practices in cabri environment. *ZDM: Mathematics Education*, 34(3), 66–72.
- Cabrillog S.A.S. (2004). *Cabri 3D* (Version 1.0.3) [Computer software]. cabri.com
- de Freitas, E., & Sinclair, N. (2014). *Mathematics and the body: Material entanglements in the classroom*. Cambridge University Press.
- GeoGebra. (2023). *GeoGebra* (Version 6.0.841.0) [Computer software]. GeoGebra.
- Hall, R., & Nemirovsky, R. (2012). Modalities of the body engagement in mathematical activity and learning. *The Journal of the Learning Sciences*, 21(2), 207–215.
- Jackiw, N. (2006). Mechanism and magic in the psychology of dynamic geometry. In N. Sinclair, D. Pimm, & W. Higginson (Eds.), *Mathematics and the aesthetic: New approaches to an ancient affinity* (pp. 145–159). Springer.
- Jaworski, B., & Didis, G. M (2014). Relating student meaning-making in mathematics to the aims for and design of teaching in small group tutorials at university level. In S. Oesterle, P. Liljedahl, C. Nicol, & D. Allan (Eds.), *Proceedings of the Joint Meeting of PME 38 and PME-NA 36* (Vol. 3, pp. 377–384). PME.
- KCP Technologies. (n.d.). *The Geometer's Sketchpad* (Version 5) [Computer software]. Key Curriculum Press.
- Kidd, C., & Hayden, B. (2015). The psychology and neuroscience of curiosity. *Neuron*, 88(3), 449–460.
- Lakoff, G. & Núñez, R. (1997/2000). *Where mathematics comes from: How the embodied mind brings mathematics into being*. Basic Books.
- Loewenstein, G. (1994). The psychology of curiosity: A review and reinterpretation. *Psychological Bulletin*, 116(1), 75–98.
- Oudeyer, P.-Y., Gottlieb, J., & Lopes, M. (2016). Intrinsic motivation, curiosity and learning: Theory and applications in education technologies. *Progress in Brain Research*, 299, 257–283.
- Pickering, A. (1995). *The mangle of practice: Time, agency and science*. University of Chicago Press.
- Radford, L., Edwards, L., & Arzarello, F. (2009). Beyond words. *Educational Studies in Mathematics*, 70(2), 91–95.
- Radford, L., Schubring, G., & Seeger, F. (2011). Signifying and meaning-making in mathematical thinking, teaching, and learning. *Educational Studies in Mathematics*, 77(2), 149–156.
- Rodney, S. G. (2021). *Embodied curiosity in the mathematics classroom through the affordance of The Geometer's Sketchpad* [Doctoral dissertation]. Simon Fraser University. <https://summit.sfu.ca/item/34824>
- Schnaider, K., & Gu, L. (2022). Potentials and challenges in students' meaning-making via sign systems. *Multimodal Technologies and Interaction*, 6(2), 9.

Sinclair, N., & Heyd-Metzuyanim, E. (2014). Learning number with TouchCounts: The role of emotions and the body in mathematical communication. *Technology, Knowledge and Learning*, 19(1–2), 81–99.

REFLECTIVE PRACTICES IN A TA OBSERVATION PROGRAM

Zack Wolske

ABSTRACT

We describe a program for peer observation of tutorials focused on encouraging reflective practice and goal-setting for new Teaching Assistants (TAs) working in large, introductory, university math courses. The program is suitable for observing up to 50 TAs during a 12-week term, with costs of five TA hours per observation.

INTRODUCTION

Teaching assistants (TAs) in large university courses (enrollment over 250) may be the only members of the instructional team who regularly meet students in a small-group setting (under 40 students), putting them in a position to have a large influence on student learning. However, they are also the least experienced members and have limited institutional support to improve their teaching, in part due to their precarity.

This report describes a program of structured *peer observations* between TAs with the goal of encouraging *reflective practice* and *goal-setting*. We selected formative frameworks for the observation protocols and took steps to ensure that individual teaching performance was not reported so that new TAs would feel supported and not evaluated.

The program ran from 2019-2022, organized by two postdocs within a math department, along with 19 observation TAs and 172 participating new TAs. Each term involved four distinct phases:

- Planning phase (4 weeks before the term starts) involving only the organizers and faculty, where we set our goals, write meeting plans and information pages, select experienced TAs who applied for leadership roles, and contact new TAs.
- Training phase (4-6 weeks), involving organizers and observers, where we introduce the program and frameworks, they practice observing each other (or us), we reflect on their experiences, and then match them with new TAs.
- Observation phase (6-8 weeks), involving mostly observers and new TAs, where they arrange the meetings and observations. First time observers check-in with organizers after two observations to bring up any issues.
- Reflection phase (exam week of term), involving organizers and observers, as well as anonymous feedback forms from new TAs, where we reflect on the program and plan changes for the next iteration.

Fall terms had significantly more new TAs apply than winter terms (48-55 compared to 3-15); as a result, we hired seven or eight observers in the fall, and only two or three in the winter.

Each term, we hired and trained experienced TAs to work as observers for 6 or 7 new TAs each, observing approximately one each week. We chose to focus on new TAs—graduate or undergraduate students on their first contract—for three reasons:

- our budget did not allow us to observe every TA;
- as the least experienced group, we believe they need and want the most support; and

- we believe that learning reflective practices and goal-setting will improve their teaching in future courses when they are hired again.

We used an observation model centred on goals the new TA had for their session (almost always tutorials with 10-30 students), with a 20-25 minute pre-observation meeting to introduce the program, discuss areas of interest and set goals, and a 30-35 minute post-observation meeting to review what happened and how they might change. The total number of TA hours spent was approximately five per observation—one additional hour for the new TA, two and a half to three hours for the observer to complete the meetings, observation, and planning between them, and some additional time for observation training. Each of the organizers spent 50 hours in the first year to plan and run the program, then approximately 25 in subsequent years.

ACKNOWLEDGEMENTS

Thanks to Michael Breeling for co-organizing the program; Cindy Blois for support in planning; and to Mihai Alboiu, Assaf Bar-Natan, Heng Yi Cheng, Julian Cheng, Carrie Clark, Hannah Constantin, Keegan Dasilva Barbosa, Kathlyn Dykes, Jesse Frohlich, Tresa LeBlanc-Doucet, Vasiliki Lontou, Tarek Mohammad, Ramtin Mojtahedi, Mat Olechnowicz, Eva Politou, Nancy Qiu, Luke Volk, Andrew Wilson, and Jake Zimmermann for their work as observers and mentors.

THEORETICAL FRAMEWORKS

Classroom observational practice is complex and involves many considerations for observer, observee, and the general scheme or process (Schoenfeld, 2013). Documented examples of peer observation generally focus on teachers or other experienced educators (e.g., Allen & LeBlanc, 2005), and many observation methods (Boston et al., 2015) are intended for performance evaluation of educators in permanent positions. For our program, we prefer to use observation as a method of transformation rather than evaluation (Peel, 2005). This allows new TAs to set challenging goals for themselves and focus on improving their teaching through formative feedback, rather than performing to meet a predefined metric, and without fear that future job opportunities may be at risk.

Reflective practice is a powerful method for making positive changes through iteration (Osterman & Kottkamp, 2004). We introduce this concept to observers, then directly and explicitly model it for them across our training sessions. The observation TAs introduce and model the concept for new TAs, with a focus on making small, incremental changes to their tutorial teaching. We expect that observers will also take part in a reflective practice cycle, and that the observation process may build collegiality among TAs (Bell, 2001). Observers guide new TAs using aspects from the DEAL method—Describe, Examine, Articulate Learning—(Ash & Clayton, 2004, 2009) and the “What? So What? Now What?” framework (Rolfe et al., 2001) to encourage nonjudgmental discussion. Each of these are three-staged discussion formats where participants first describe a shared understanding of what events occurred, then examine them to decide what impact they had, and finally consider how they can use what they learned from the discussion. We believe this will focus the post-observation discussion on their original goals and areas of interest; allow them both to describe what they observed without judgment; and demonstrate a concrete method for making iterative improvements through reflective practice. We also appreciate that it leaves the new TA with ideas to try next time and not an evaluation of their previous work. This may lead to more discussions between the observer and new TA when they meet in the department and increase interest in teaching and learning.

PLANNING PHASE

We initially met with all teaching stream faculty four weeks before the term began to hear their ideas for a program to improve tutorials. They varied from additional training in active learning to evaluative and non-evaluative observations. Due to budget constraints, we could only offer training to some of the department’s 250 TAs and opted for new TAs in order to have the largest long-term impact within the department. Focusing on reflective practice also aligned with this long-term goal and was universally supported as a valuable skill that would benefit new TAs. One faculty member had experience with a peer observation program and volunteered to work with us, so we proceeded to the literature to put the theoretical pieces together.

Because the department hires approximately 250 TAs each year for 500-600 positions, and individual courses may have 20-40 TAs as part of the instructional team, there are a number of leadership TA roles (such as head TA, lead grading TA or lead tutorial TA) who act as liaisons between coordinators and TAs. Applicants for TA positions, which may include undergraduate and graduate students from any department, indicate if they would like to be considered for one of these leadership positions. We selected TAs from this pool who had at least two years experience as TAs and offered them 25-30 hour contracts (they would also have additional TA contracts or instructor positions and generally were at the stage of working on PhD theses).

We met with Cindy Blois weekly before the term started to discuss classroom observational practice, the frameworks for discussion, and general logistics. She introduced the importance of allowing one to three days between each meeting (pre and post) and the observation to give time for the observer to come up with observables and consolidate their observation notes and to allow the new TA time to reflect on their goals and what happened in the tutorial.

Before meeting with the observers, but after the term had started, the organizers observed each other in classroom teaching using the protocols. We then wrote summary guidelines and meeting notes for the training phase.

The second terms each year used most of the same material, with small adjustments based on the first term outcomes, and so had a much shorter planning phase. In the next two years, we omitted the full teaching stream consultation, reduced the number of meetings with Dr. Blois, and removed our observations in favour of starting the training earlier.

TRAINING PHASE

Prior to our first meeting with the observation TAs, we arranged activities to introduce the theoretical concepts, share their own experiences with observation, brainstorm initial meeting questions or potential areas of interest new TAs might have, and practice taking observation notes and using the discussion frameworks. We selected one of these activities to debrief during the meeting in order to demonstrate how we would facilitate a post-observation meeting. The organizers chose roles as activity facilitator or observer, then set goals and areas of interest for the activity. During the activity, the observer took notes but did not participate. Immediately after the activity, the observer used the discussion frameworks to talk with the facilitator about the activity, focusing on the selected goals and areas of interest. Observation TAs got to see how note-taking could work in practice and observe discussion methods that focus on areas of interest instead of unexpected events that came up.

At the end of the meeting, observers were paired up (with each other or one of the organizers) to run a pair of observations on their own classes over the next two weeks. Each of them were also course TAs or instructors separate from this contract—there was only one instance in three years where they had no in-class teaching, and in that case, they only observed one of the organizers. In the first year, they were also given a task to report on their role at the end of the term by answering specific questions so that we could make structural or contract changes before the next iteration.

After they had completed their paired observations, we met again to discuss what it was like to be observed; what happened in their meetings; how they took notes and organized their observations; if the observation caused any changes in their teaching; and what they would do differently with new TAs who were not familiar with the processes.

We reached out to all new TAs during this time to invite them to participate in the program, advertised it during their mandatory new TA training day, and found out about their course roles and duties. They were paid for the additional time spent in meetings with the observer, but the program was not mandatory.

OBSERVATION PHASE

Observers were assigned six or seven new TAs, generally selecting those teaching in courses where the observer had experience and who had tutorials at times they were available. We provided a short summary of the program they could share. The observer was responsible for contacting the new TA and scheduling the observations and meetings. After they had completed two observations, they met with one of the organizers to discuss their progress and any additional support they needed. Because of the long-term goals of the program, we did not prioritize observing the new TAs early in the term.

Some new TAs had scheduling conflicts with all observers—this occurred three times in three years, about once every sixty observations. In these cases, one of the organizers met with and observed the new TA.

REFLECTION PHASE

At the end of the term, we met with the observers to hear about their experiences and ways to support future observations. In the first year, we also had them write a report, asking for suggestions or advice for future observers; observation or discussion methods they found effective; and total amount of time spent working. From these meetings in the first and second years, we learned the following:

- It was difficult to observe more than once per week. There are scheduling challenges, especially with new TAs not responding or missing meetings, but the bigger issue was mental exhaustion from frequently taking on roles as observer or discussion guide, then needing to quickly return to their role as researcher or student or TA or lecturer in their own courses.
- New TAs did not have goals or had vague goals. Some expected explicit direction of what to do during the tutorial and numerical evaluations at the end.
- Note-taking during meetings or observations was more difficult than they expected. When handwriting notes, they had to stop observing or engaging in conversation with the new TA.

These were improved in later iterations by suggesting one observation per week, with pre- and post-observation meetings on the same day of the week, one week apart; including “ACTION REQUIRED BY” prompts in their emails; giving observers additional suggestions for refining goals and for steering conversations toward reflective practice; and using note-taking templates.

We had returning observers in every term after the first iteration, which helped during the training and reflection phases by giving new observers (and us) their perspectives as practitioners. In the third year, we hired an observer who had been a new TA in the first iteration, which gave us a better understanding of the entire process and how it was generally received.

New TAs were sent a survey with Likert-scale questions about their experiences, and space for free responses. They universally rated their experience good or great each year, with nearly all respondents saying they would recommend it to other TAs, and most saying it changed how they run their tutorials. There were many requests for additional observations, either later in the term, or in future courses.

YEAR-TO-YEAR CHANGES

Somewhere between the introduction and here, I hope you have wondered “Hey—doesn’t a pandemic start at some point?” The first iteration of the program was entirely in person in the fall, and we completed all but two observations in winter 2020 before moving to emergency remote teaching. All of the observations (as well as planning, training and reflection phases) in fall 2020 and winter 2021 were done online. About half of the observations (including training observations) were in classrooms in fall 2021 and winter 2022, while the rest and the other phases were online. This presented three major problems:

- Some courses did not schedule traditional tutorials, so new TAs had a wider variety of roles;
- Online tutorials tended to be more teacher-centred, in particular, it was difficult to observe learners and learner-learner interactions; and
- People were stressed and overworked (including us).

To the first point, we expanded the observer role to mentoring a new TA and observing any duties they had been assigned. We were able to keep the goal of encouraging reflective practice but explicit goal-setting was more difficult for one-on-one student interactions. For example, in an online help centre, office hour, or when responding to students’ written questions, it was difficult to predict what students would bring up and how they would react to suggestions. As a result, goals for these sessions were vague, and observers had trouble helping to refine them. In at least one case, it was impossible to observe what a new TA would do, because no one came to their office hour. Fortunately, the observation was rescheduled (to a date before a major assignment), and the new TA was able to get feedback from the observer.

To the second point, new TAs and observers tended towards goals, areas of interest, and observables that could be seen entirely through TA actions without relying on student actions. This is not ideal, and invalidates the assumptions behind the theory of classroom observation, in particular, that there are things the observer can see that are impossible for the teacher to see. Observers noted that there were no viable alternatives to this—the students were simply not observable.

To the third point, we did less planning, we moved the reflection phases to the start of the following term, and we did not have observers write end of term reports. Although we had a comparable number of new TAs in all three fall terms, we had fewer in the winters of 2021 and 2022 than in winter 2020. With fewer new TAs, we were able to add some repeat observations of new TAs from the fall term.

Observers noted that scheduling and hosting meetings was much easier online. Some said they would prefer to always have the initial meeting online, or both meetings online, with an in-class observation.

FURTHER REFLECTIONS

The program was very well received, both by observers and new TAs. The program was optional for new TAs, and we had 70-75% of all new TAs opt in (which includes those who were not assigned tutorials). Previous observers enthusiastically returned each term.

I believe the time between meetings and observations were crucial for the observers to process what they would look for and what they had seen, as well as for the new TAs to think about their goals and cool off after the tutorial—especially those who had not thought about setting goals before their pre-observation meeting.

The second meeting, and the paired observations, during the training phase were eye-opening for observers. In particular, those who only had experience with evaluative observations (generally by instructors in their courses) noted that it was hard to avoid telling someone what to do and that they learned a lot by seeing what others did in the classroom. During the later iterations, these were also important venues for returning observers to share their tips and suggestions.

In order to increase the number of observations, I believe it is necessary to hire more observers so that they can focus on at most one observation per week, and arrange parallel training sessions so that they remain in a small group to allow everyone to share their advice and experiences.

REFERENCES

- Allen, D. W., & LeBlanc, A. C. (2005). *Collaborative peer coaching that improves instruction: The 2 + 2 performance appraisal model*. Corwin Press.
- Ash, S. L., & Clayton, P. H. (2004). The articulated learning: An approach to guided reflection and assessment. *Innovative Higher Education*, 29(2), 137–154.
- Ash, S. L., & Clayton, P. H. (2009). Generating, deepening, and documenting learning: The power of critical reflection in applied learning. *Journal of Applied Learning in Higher Education*, 1(1), 25–48.
- Bell, M. (2001). Supported reflective practice: A programme of peer observation and feedback for academic teaching development. *The International Journal for Academic Development*, 6, 29–39.
- Boston, M., Bostic, J., Lesseig, K., & Sherman, M. (2015). A comparison of mathematics classroom observation protocols. *Mathematics Teacher Educator*, 3(2), 154–175. <https://doi.org/10.5951/mathteaceduc.3.2.0154>
- Osterman, K. F., & Kottkamp, R. B. (2004). *Reflective practice for educators* (2nd ed.). Corwin Press.
- Peel, D. (2005) Peer observation as a transformatory tool? *Teaching in Higher Education*, 10(4), 489–504. DOI:10.1080/13562510500239125
- Rolfe, G., Freshwater, D., & Jasper, M. (2001). *Critical reflection in nursing and the helping professions: A user's guide*. Palgrave Macmillan.

Schoenfeld, A.H. (2013) Classroom observations in theory and practice. *ZDM: Mathematics Education*, 45, 607–621. <https://doi.org/10.1007/s11858-012-0483-1>

Ad Hoc Sessions

Séances ad hoc

THE SLOW DEATH OF MATH TEACHER JOURNALS IN CANADA: RIP, JOURNAL OF THE SMTS

Egan J. Chernoff
University of Saskatchewan
Simon Fraser University

The worry that stems from proposing an *ad hoc* session at CMESG/GCEDM, in my opinion, is that you do not know who or how many people will attend, which begins immediately after you muster up the courage to write down your name and working title on whatever whiteboard, wherever located, days before your session. Making matters worse for me this year in Regina, I was using terms like “death,” “rest in peace,” or “RIP,” and the nebulous term “math teacher journals.” These words, for the record, were not part of a ‘if it bleeds it leads’ campaign. As such, I decided to take a very measured approach when people, while playing pool at the dinners or while swimming in the hot pools or wandering the tunnels of Moose Jaw, asked me what my *ad hoc* was all about. I would just mention, casually, that Canadian math teacher journals, here in Canada, are a bit of a passion project of mine (Chernoff et al., 2016; Chernoff et al., 2019; Chernoff & Sterenberg, 2014). In what follows, I recount what took place over a 20-minute *ad hoc* session when, to my delight, five tables full of colleagues, from all over Canada, took time out of there conference schedule to attend my *ad hoc*.

Me at the front, with a few slides thrown together the nights before, and five tables strewn about the room, littered with lots of very large chart paper and many marker pens, we were ready to begin. The first task, a simple one, for each of the tables to identify Canadian provinces (10) and territories (3). In no time, each group finished and eagerly awaited the second task, which was to list the corresponding mathematics teacher association or society or organization or insert-relevant-group-descriptor-here for each province and territory. Interrupting the groups, I gave them a third task, that is, to identify the corresponding math teacher journals and, if they could, the current editor of the journal.

With a room full of colleagues from across the country, we were (almost) able to complete all tasks, and the room was able to spot a fake journal, *Circumpolar: Journal of the Northwest Territories Mathematics Teachers’ Association (NTMTA)*, that I had added to the list. The tasks, however, were mostly a primer for a general discussion of the state of mathematics teacher journals across the country. Colleagues that had published their work in such journals shared their stories, which included tips and pitfalls to avoid. Adding to the stories, I took the opportunity to share my experience with such journals and how my university has discounted such publications for years and years. Near the end, we discussed the importance of CMESG/GCEDM attendees publishing their work in math teacher journals in Canada. The conclusion was clear, do it...while you still can.

REFERENCES

- Chernoff, E. J., Russell, G. L., & Sriraman, B. (Eds.). (2019). *Selected Writings from the Journal of the Saskatchewan Mathematics Teachers’ Society: Celebrating 50 years (1961-2011) of vinculum*. Information Age Publishing.
- Chernoff, E. J., Liljedahl, P., & Chorney, S. (Eds.). (2016). *Selected writings from the Journal of the British Columbia Association of Mathematics Teachers: Celebrating 50 (1962-2012) years of Vector*. Information Age Publishing.
- Chernoff, E. J., & Sterenberg, G. (2014). *Selected writings from the Journal of the Mathematics Council of the Alberta Teachers’ Association: Celebrating 50 years (1962-2012) of delta-K*. Information Age Publishing.

PARA-IMAGING: RE-PRESENTING STUDENTS' MENTAL IMAGES

Tina Rapke¹, Marc Husband², & Cristina De Simone¹
York University¹, St. Francis Xavier University²

Para-imaging operates similarly to revoicing, except teachers utilize materials instead of verbal methods to dynamically re-present what students say they see in their minds, including movement, while solving mental math problems. The need for more attention to this topic is surprising given the recognized importance of images (e.g., Pirie & Kieren, 1994) in comprehending mathematics. We shared videos of para-imaging in the context of Number Talks. Existing research on Number Talks (Matney et al., 2020) has highlighted the crucial role of teachers' actions during these discussions. While significant attention has been given to verbal instructional methods like revoicing during whole group discussion, there has been comparatively less focus on the practice of visually re-presenting the mental images.

We discussed the crucial element of movement and the teacher's role that we observed in the videos. For instance, in the case of $11+11$, it is difficult to re-present statically decomposing 11 into 10 and 1. Similarly, how can one draw a static re-presentation for the 10s and the 1s going together? See the video of para-imaging for the movement using small circular plastic dots to re-present 11 (Dr. Tina & Dr. Marc, 2023). In terms of the role of the teacher, we delved into how the teacher may assert their own concepts while ensuring that students' perspectives remain visible. There was a discussion on the legitimacy of the teacher's ideas and whether it was justifiable to enforce them. In the video (Dr. Tina & Dr. Marc, 2023), one can see that the teacher was imposing ideas by changing the spatial arrangement of the 11 circular objects and replacing 10 of the circular objects with a 10-rod.

Members of the audience suggested that para-imaging allows teachers to concentrate on crucial elements in solution strategies. While revoicing, the teacher might subtly steer the conversation in a different direction to promote their own discussion points (O'Connor & Michaels, 1996, p.74). A fundamental aspect of re-voicing and para-imaging is allowing the teacher to attribute intellectual content to a specific student. It has been recommended that re-voicing is an effective strategy that should be used by all educators (Herbel-Eisenmann et al., 2009), para-imaging may hold the same potential.

REFERENCES

- Dr. Tina & Dr. Marc [@DrTina_DrMarc]. (2023, October 11). *Mental math: 2×11* [Video]. YouTube. <https://www.youtube.com/watch?v=UEW8VKCHaq4&t=2s>
- Herbel-Eisenmann, B., Drake, C., & Cirillo, M. (2009). "Muddying the clear waters": Teachers' take-up of the linguistic idea of revoicing. *Teaching and Teacher Education*, 25(2), 268–277. <https://doi.org/10.1016/j.tate.2008.07.004>
- Matney, G., Lustgarten, A., & Nicholson, T. (2020). Black holes of research on instructional practice: The case of number talks. *Investigations in Mathematics Learning*, 12(4), 246–260. <https://doi.org/10.1080/19477503.2020.1804273>
- O'Connor, M. C., & Michaels, S. (1996). Shifting participant frameworks: Orchestrating thinking practices in group discussion. In D. Hicks (Ed.), *Discourse, learning, and schooling* (pp. 63–103). Cambridge University Press. <https://doi.org/10.1017/CBO9780511720390.003>
- Pirie, S. E. B., & Kieren, T. E. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics*, 26, 165–190. <https://doi.org/10.1007/BF01273662>

COD AND BAO; NBC CAT; NOT TBD

Zack Wolske

The title of this *ad hoc* session has an unusual property: each pair of words share one letter. This was inspired by Ed Doolittle’s session on using block designs for randomized small groups with nice features, and the small-group discussion protocol at CMESG 2023 using the sets of numbers on our name tags.

The goal is to construct sets of ‘words’ (or with younger students, collections of symbols or images) so that any two have one common letter. I wanted to see how mathematicians and math educators would approach this, and what issues they would predict when using an activity like this in a classroom. We arranged groups of four to five at round tables with paper, markers and scissors.

Some participants started with particular words they liked (for example, their own names) and built additional ‘words’ around them. Others in their group used an abstract representation, starting with ABC, ADE, BDF, then adding more words and letters that satisfied the constraint.

Participants with background knowledge of the problem—from attending Ed’s session as participant or speaker, or from their own work with combinatorial designs—knew methods, or diagrams, or counting principles that helped construct larger collections with four letters in each set; or collections where either two words shared one letter, or there was a third word that was disjoint from both of them. These spurred additional questions about generalizations: from Asmita Sodhi, “what about words with two letters in common?”, and from Ed Doolittle, “can we programmatically search through actual dictionaries to find collections?” Asmita later shared results about “biplanes” with a group of math circle leaders, while Ed has written such a program to be shared.

Issues that came up:

- When using symbols, or characters from other alphabets, we had to point to indicate the common element. This may make it accessible to very young students and generally slowed down the game.
- The sets are hard to construct by trial and error! Students may get frustrated or get stuck with small collections of words. Systematic approaches, like using a grid with letters as columns and words are rows, was helpful. It may also help to give a model, like the title, and have them construct collections using their own images, like stickers.
- There are trivial examples that make a boring game: {ABC, ADE, AFG, AHI, AJK, ...}. Some additional constraint, like “the same letter isn’t in every word” may be necessary to mention.
- When playing around a table, it is easier to see a match when the letters are in their usual orientation toward the player. This can be used to create collections that advantage one player, by writing every letter upright and turning the cards to face them or creating cards in a way that distributes the orientations in various directions. In this case, we should take care to not use letters that are rotations of each other.

Thanks everyone for participating and helping me think about different ways to represent these concepts from finite (projective) geometry!

Mathematics Gallery

Galerie Mathématique

CONNECTING MATH TO OUR LIVES AND COMMUNITIES: MEASURING OUR IMPACT

Ellen Carter
St. Francis Xavier University

Connecting Math to Our Lives and Communities (CMTOLC) is a focused in-community mathematics outreach program run in partnership between local Mi'kmaw and African Nova Scotian communities in Eastern Nova Scotia and a team of St. Francis Xavier University (StFX) professors: Drs. Lisa Lunney Borden, Ellen Carter, Robert van den Hoogen, Tara Taylor, and Marcia English. The program began in 2015 and continues to grow and evolve, now reaching over 250 youth annually. Reciprocal relationships have been and continue to be central to the operation of the program. Building on asset-based, community-first approaches (Zinga et al., 2009), CMTOLC emerged *with* communities over time (Wiseman et al., 2020). Community leaders share priorities for topics to be explored, identify Elders and knowledge holders who may share their knowledge with youth directly or indirectly, and provide ongoing feedback. Meaningful, relevant, hands-on inquiry modules are then designed by the CMTOLC coordinator to encourage the investigation of these topics while underscoring the inherent mathematics.

StFX students with community connections, and mathematical knowledge are hired and trained to *facilitate* programming in each of the seven communities through bi-weekly, in-community sessions during the academic year, and through day camps in the same locations through the summer. Each session begins with discussion intended to center youth voices, to make connections to existing knowledge, and elicit curiosity. A variety of differentiated inquiry-based tasks in each session allows for breadth and depth of exploration and individual youth agency to choose the activities that are most meaningful for them. Facilitators are prepared not only to lead activities but build relationships, connect to what students share in the opening discussion, prompt further discourse, and highlight mathematical concepts. At the end of each session, facilitators and youth come back together to synthesize learning and respond to the questions of “*What did you like?*”, “*What did you learn?*”, and “*What more do you want to know?*”. Facilitators report themes from these discussions to further inform programming goals and module development.

Most recently, the CMTOLC team began to explore the long-term impact of the program on youth, communities, and post-secondary student facilitators. Preliminary findings indicate that youth feel more confident in their mathematics abilities, and community embedded and culturally relevant explorations support and encourage youth to participate in further studies in STEM. Facilitators, too, reported experiencing mathematics differently resulting in deepened understanding of the meaningfulness of mathematics and the role of community knowledges.

REFERENCES

- Wiseman, D., Lunney Borden, L., Beatty, R., Jao, L., & Carter, E. (2020). Whole-some artifacts: (STEM) teaching and learning emerging from and contributing to community. *Canadian Journal of Science, Mathematics and Technology Education*, 20, 264–280. <https://doi.org/10.1007/s42330-020-00079-6>
- Zinga, D., Styres, S., Bennett, S., & Bomberry, M. (2009). Student success research consortium: Two worlds community-first research. *Canadian Journal of Native Education*, 32(1), 19–37. <https://doi.org/10.14288/cjne.v32i1.196483>

Appendices

Annexes

Appendix A / Annexe A

WORKING GROUPS AT EACH ANNUAL MEETING / GROUPE DE TRAVAIL DES RENCONTRES ANNUELLES

1977 *Queen's University, Kingston, Ontario*

- Teacher education programmes
- Undergraduate mathematics programmes and prospective teachers
- Research and mathematics education
- Learning and teaching mathematics

1978 *Queen's University, Kingston, Ontario*

- Mathematics courses for prospective elementary teachers
- Mathematization
- Research in mathematics education

1979 *Queen's University, Kingston, Ontario*

- Ratio and proportion: a study of a mathematical concept
- Minicalculators in the mathematics classroom
- Is there a mathematical method?
- Topics suitable for mathematics courses for elementary teachers

1980 *Université Laval, Québec, Québec*

- The teaching of calculus and analysis
- Applications of mathematics for high school students
- Geometry in the elementary and junior high school curriculum
- The diagnosis and remediation of common mathematical errors

1981 *University of Alberta, Edmonton, Alberta*

- Research and the classroom
- Computer education for teachers
- Issues in the teaching of calculus
- Revitalising mathematics in teacher education courses

- 1982 *Queen's University, Kingston, Ontario*
- The influence of computer science on undergraduate mathematics education
 - Applications of research in mathematics education to teacher training programmes
 - Problem solving in the curriculum
- 1983 *University of British Columbia, Vancouver, British Columbia*
- Developing statistical thinking
 - Training in diagnosis and remediation of teachers
 - Mathematics and language
 - The influence of computer science on the mathematics curriculum
- 1984 *University of Waterloo, Waterloo, Ontario*
- Logo and the mathematics curriculum
 - The impact of research and technology on school algebra
 - Epistemology and mathematics
 - Visual thinking in mathematics
- 1985 *Université Laval, Québec, Québec*
- Lessons from research about students' errors
 - Logo activities for the high school
 - Impact of symbolic manipulation software on the teaching of calculus
- 1986 *Memorial University of Newfoundland, St. John's, Newfoundland*
- The role of feelings in mathematics
 - The problem of rigour in mathematics teaching
 - Microcomputers in teacher education
 - The role of microcomputers in developing statistical thinking
- 1987 *Queen's University, Kingston, Ontario*
- Methods courses for secondary teacher education
 - The problem of formal reasoning in undergraduate programmes
 - Small group work in the mathematics classroom
- 1988 *University of Manitoba, Winnipeg, Manitoba*
- Teacher education: what could it be?
 - Natural learning and mathematics
 - Using software for geometrical investigations
 - A study of the remedial teaching of mathematics
- 1989 *Brock University, St. Catharines, Ontario*
- Using computers to investigate work with teachers
 - Computers in the undergraduate mathematics curriculum
 - Natural language and mathematical language
 - Research strategies for pupils' conceptions in mathematics

Appendix A • Working Groups at each Annual Meeting

- 1990 *Simon Fraser University, Vancouver, British Columbia*
- Reading and writing in the mathematics classroom
 - The NCTM “Standards” and Canadian reality
 - Explanatory models of children’s mathematics
 - Chaos and fractal geometry for high school students
- 1991 *University of New Brunswick, Fredericton, New Brunswick*
- Fractal geometry in the curriculum
 - Socio-cultural aspects of mathematics
 - Technology and understanding mathematics
 - Constructivism: implications for teacher education in mathematics
- 1992 *ICME–7, Université Laval, Québec, Québec*
- 1993 *York University, Toronto, Ontario*
- Research in undergraduate teaching and learning of mathematics
 - New ideas in assessment
 - Computers in the classroom: mathematical and social implications
 - Gender and mathematics
 - Training pre-service teachers for creating mathematical communities in the classroom
- 1994 *University of Regina, Regina, Saskatchewan*
- Theories of mathematics education
 - Pre-service mathematics teachers as purposeful learners: issues of enculturation
 - Popularizing mathematics
- 1995 *University of Western Ontario, London, Ontario*
- Autonomy and authority in the design and conduct of learning activity
 - Expanding the conversation: trying to talk about what our theories don’t talk about
 - Factors affecting the transition from high school to university mathematics
 - Geometric proofs and knowledge without axioms
- 1996 *Mount Saint Vincent University, Halifax, Nova Scotia*
- Teacher education: challenges, opportunities and innovations
 - Formation à l’enseignement des mathématiques au secondaire: nouvelles perspectives et défis
 - What is dynamic algebra?
 - The role of proof in post-secondary education
- 1997 *Lakehead University, Thunder Bay, Ontario*
- Awareness and expression of generality in teaching mathematics
 - Communicating mathematics
 - The crisis in school mathematics content

1998 *University of British Columbia, Vancouver, British Columbia*

- Assessing mathematical thinking
- From theory to observational data (and back again)
- Bringing Ethnomathematics into the classroom in a meaningful way
- Mathematical software for the undergraduate curriculum

1999 *Brock University, St. Catharines, Ontario*

- Information technology and mathematics education: What's out there and how can we use it?
- Applied mathematics in the secondary school curriculum
- Elementary mathematics
- Teaching practices and teacher education

2000 *Université du Québec à Montréal, Montréal, Québec*

- Des cours de mathématiques pour les futurs enseignants et enseignantes du primaire/Mathematics courses for prospective elementary teachers
- Crafting an algebraic mind: Intersections from history and the contemporary mathematics classroom
- Mathematics education et didactique des mathématiques : y a-t-il une raison pour vivre des vies séparées?/Mathematics education et didactique des mathématiques: Is there a reason for living separate lives?
- Teachers, technologies, and productive pedagogy

2001 *University of Alberta, Edmonton, Alberta*

- Considering how linear algebra is taught and learned
- Children's proving
- Inservice mathematics teacher education
- Where is the mathematics?

2002 *Queen's University, Kingston, Ontario*

- Mathematics and the arts
- Philosophy for children on mathematics
- The arithmetic/algebra interface: Implications for primary and secondary mathematics / Articulation arithmétique/algèbre: Implications pour l'enseignement des mathématiques au primaire et au secondaire
- Mathematics, the written and the drawn
- Des cours de mathématiques pour les futurs (et actuels) maîtres au secondaire / Types and characteristics desired of courses in mathematics programs for future (and in-service) teachers

2003 *Acadia University, Wolfville, Nova Scotia*

- L'histoire des mathématiques en tant que levier pédagogique au primaire et au secondaire / The history of mathematics as a pedagogic tool in Grades K–12
- Teacher research: An empowering practice?
- Images of undergraduate mathematics
- A mathematics curriculum manifesto

2004 *Université Laval, Québec, Québec*

- Learner generated examples as space for mathematical learning
- Transition to university mathematics
- Integrating applications and modeling in secondary and post secondary mathematics
- Elementary teacher education – Defining the crucial experiences
- A critical look at the language and practice of mathematics education technology

2005 *University of Ottawa, Ottawa, Ontario*

- Mathematics, education, society, and peace
- Learning mathematics in the early years (pre-K – 3)
- Discrete mathematics in secondary school curriculum
- Socio-cultural dimensions of mathematics learning

2006 *University of Calgary, Calgary, Alberta*

- Secondary mathematics teacher development
- Developing links between statistical and probabilistic thinking in school mathematics education
- Developing trust and respect when working with teachers of mathematics
- The body, the sense, and mathematics learning

2007 *University of New Brunswick, New Brunswick*

- Outreach in mathematics – Activities, engagement, & reflection
- Geometry, space, and technology: challenges for teachers and students
- The design and implementation of learning situations
- The multifaceted role of feedback in the teaching and learning of mathematics

2008 *Université de Sherbrooke, Sherbrooke, Québec*

- Mathematical reasoning of young children
- Mathematics-in-and-for-teaching (MifT): the case of algebra
- Mathematics and human alienation
- Communication and mathematical technology use throughout the post-secondary curriculum / Utilisation de technologies dans l'enseignement mathématique postsecondaire
- Cultures of generality and their associated pedagogies

2009 *York University, Toronto, Ontario*

- Mathematically gifted students / Les élèves doués et talentueux en mathématiques
- Mathematics and the life sciences
- Les méthodologies de recherches actuelles et émergentes en didactique des mathématiques / Contemporary and emergent research methodologies in mathematics education
- Reframing learning (mathematics) as collective action
- Étude des pratiques d'enseignement
- Mathematics as social (in)justice / Mathématiques citoyennes face à l'(in)justice sociale

2010 *Simon Fraser University, Burnaby, British Columbia*

- Teaching mathematics to special needs students: Who is at-risk?
- Attending to data analysis and visualizing data
- Recruitment, attrition, and retention in post-secondary mathematics
Can we be thankful for mathematics? Mathematical thinking and aboriginal peoples
- Beauty in applied mathematics
- Noticing and engaging the mathematicians in our classrooms

2011 *Memorial University of Newfoundland, St. John's, Newfoundland*

- Mathematics teaching and climate change
- Meaningful procedural knowledge in mathematics learning
- Emergent methods for mathematics education research: Using data to develop theory / Méthodes émergentes pour les recherches en didactique des mathématiques: partir des données pour développer des théories
- Using simulation to develop students' mathematical competencies – Post secondary and teacher education
- Making art, doing mathematics / Créer de l'art; faire des maths
- Selecting tasks for future teachers in mathematics education

2012 *Université Laval, Québec City, Québec*

- Numeracy: Goals, affordances, and challenges
- Diversities in mathematics and their relation to equity
- Technology and mathematics teachers (K-16) / La technologie et l'enseignant mathématique (K-16)
- La preuve en mathématiques et en classe / Proof in mathematics and in schools
- The role of text/books in the mathematics classroom / Le rôle des manuels scolaires dans la classe de mathématiques
- Preparing teachers for the development of algebraic thinking at elementary and secondary levels / Préparer les enseignants au développement de la pensée algébrique au primaire et au secondaire

2013 *Brock University, St. Catharines, Ontario*

- MOOCs and online mathematics teaching and learning
- Exploring creativity: From the mathematics classroom to the mathematicians' mind / Explorer la créativité : de la classe de mathématiques à l'esprit des mathématiciens
- Mathematics of Planet Earth 2013: Education and communication / Mathématiques de la planète Terre 2013 : formation et communication (K-16)
- What does it mean to understand multiplicative ideas and processes? Designing strategies for teaching and learning
- Mathematics curriculum re-conceptualisation

2014 *University of Alberta, Edmonton, Alberta*

- Mathematical habits of mind / Modes de pensée mathématiques
- Formative assessment in mathematics: Developing understandings, sharing practice, and confronting dilemmas
- Texter mathématique / Texting mathematics
- Complex dynamical systems
- Role-playing and script-writing in mathematics education practice and research

2015 *Université de Moncton, Moncton, New Brunswick*

- Task design and problem posing
- Indigenous ways of knowing in mathematics
- Theoretical frameworks in mathematics education research / Les cadres théoriques dans la recherche en didactique des mathématiques
- Early years teaching, learning and research: Tensions in adult-child interactions around mathematics
- Innovations in tertiary mathematics teaching, learning and research / Innovations au post-secondaire pour l'enseignement, l'apprentissage et la recherche

2016 *Queen's University, Kingston, Ontario*

- Computational thinking and mathematics curriculum
- Mathematics in teacher education: What, how... and why / Les mathématiques dans la formation des enseignants : quoi, comment... et pourquoi
- Problem solving: Definition, role, and pedagogy / Résolution de problèmes : définition, rôle, et pédagogie associée
- Mathematics education and social justice: Learning to meet the others in the classroom / Éducation mathématique et justice sociale : apprendre à rencontrer les autres dans la classe
- Role of spatial reasoning in mathematics
- The public discourse about mathematics and mathematics education / Le discours public sur les mathématiques et l'enseignement des mathématiques

2017 *McGill University, Montréal, Québec*

- Teaching first year mathematics courses in transition from secondary to tertiary
- L'anxiété mathématique chez les futurs enseignants du primaire : à la recherche de nouvelles réponses à des enjeux qui perdurent / Elementary preservice teachers and mathematics anxiety: Searching for new responses to enduring issues
- Social media and mathematics education
- Quantitative reasoning in the early years / Le raisonnement quantitatif dans les premières années du parcours scolaire
- Social, cultural, historical and philosophical perspectives on tools for mathematics
- Compréhension approfondie des mathématiques scolaires / Deep understanding of school mathematics

2018 *Quest University, Squamish, British Columbia*

- The 21st century secondary school mathematics classroom
- Confronting colonialism / Affronter le Colonialisme
- Playing with mathematics / Learning mathematics through play
- Robotics in mathematics education
- Relation, ritual and romance: Rethinking interest in mathematics learning

2019 St. Francis Xavier University, Antigonish, Nova Scotia

- Problem-based learning in postsecondary mathematics / L'apprentissage par problèmes en mathématiques au niveau postsecondaire
- Teaching primary school mathematics...what mathematics? What avenues for teacher training? / Enseigner les premiers concepts mathématiques à l'école primaire...quelles mathématiques? Quelles avenues pour la formation à l'enseignement?
- Humanizing data / Humaniser les données
- Research and practice: Learning through collaboration / Recherche et pratique : apprendre en collaborant
- Interdisciplinarity with mathematics: Middle school and beyond
Capturing chaos? Ways into the mathematics classroom / Capturer le chaos ? Entrées sur la classe de mathématiques

2021 Online (Virtual)

- Learning Theories / Théories (de l') apprenant
- Pour ou contre les tests : est-ce la bonne question ? / To test or not to test: Is this the question?
- The rewards and challenges of video in the field of mathematics education: Looking back in order to prepare for the future / Les apports et défis de la vidéo pour (la formation à) l'enseignement-apprentissage des mathématiques : regard du passé pour préparer le futur
- How can we be creative with large classes? / Comment composer avec les grands groupes ?
- Returning to our roots: Exploring collaborative possibilities for research and teaching in mathematics and mathematics education

2022 Online (Virtual)

- Contenu et pratiques pour la formation initiale et continue des enseignants : Un regard plus approfondi sur les potentiels, les défis, les pièges et les perspectives / Content and practices for pre-service and in-service teacher education: A deeper look into the potentials, challenges, pitfalls, and prospects
- Critical mathematics working group: Changing mathematics to fit our whole selves / Les mathématiques critiques : On change les mathématiques pour s'adapter à nous-mêmes
- Weaving identity in mathematics education: Fads, fictions, fibers, and freedoms / Le tissage d'une identité dans l'enseignement des mathématiques : les modes, les histoires, les ficelles et les libertés
- Assessment in undergraduate mathematics / Évaluation en mathématiques au postsecondaire
- Matériel de manipulation dans l'apprentissage et l'enseignement des mathématiques au primaire / Manipulatives in elementary mathematics teaching and learning
- Facilitating learning mathematics online / Favoriser l'apprentissage des mathématiques en ligne

2023 University of Regina, Regina, Saskatchewan

- Machi kis kiyih tam uske/nantaw ota: Iteyhtam mas kooch etikwe apekha tam eyiniw misiwe uske kawapahtik ake ihike win ewa kiskino huma kewina (in [Bushland Cree](#)) / Learning from/on/with land/place: Imagining possibilities for braiding Indigenous worldviews, mathematics, and teaching / Apprendre avec la terre/le lieu: imaginer des possibilités pour tresser les visions du monde autochtones, les mathématiques et l'enseignement
- Sawubona. I see you. Je te vois.
- An undergraduate curriculum based on mathematical reasoning skills (wouldn't it be awesome?)
- Games for mathematical learning / Jeux pour l'apprentissage des mathématiques
- Where's the math? Inquiring into early years mathematics curriculum / Où sont les mathématiques? Enquêter sur le programme de mathématiques dans les premières années

Appendix B / Annexe B

PLENARY LECTURES AT EACH ANNUAL MEETING / CONFÉRENCES PLÉNIÈRES DES RENCONTRES ANNUELLES

1977	A.J. COLEMAN C. GAULIN T.E. KIEREN	The objectives of mathematics education Innovations in teacher education programmes The state of research in mathematics education
1978	G.R. RISING A.I. WEINZWEIG	The mathematician's contribution to curriculum development The mathematician's contribution to pedagogy
1979	J. AGASSI J.A. EASLEY	The Lakatosian revolution Formal and informal research methods and the cultural status of school mathematics
1980	C. GATTEGNO D. HAWKINS	Reflections on forty years of thinking about the teaching of mathematics Understanding understanding mathematics
1981	K. IVERSON J. KILPATRICK	Mathematics and computers The reasonable effectiveness of research in mathematics education
1982	P.J. DAVIS G. VERGNAUD	Towards a philosophy of computation Cognitive and developmental psychology and research in mathematics education
1983	S.I. BROWN P.J. HILTON	The nature of problem generation and the mathematics curriculum The nature of mathematics today and implications for mathematics teaching
1984	A.J. BISHOP L. HENKIN	The social construction of meaning: A significant development for mathematics education? Linguistic aspects of mathematics and mathematics instruction
1985	H. BAUERSFELD H.O. POLLAK	Contributions to a fundamental theory of mathematics learning and teaching On the relation between the applications of mathematics and the teaching of mathematics
1986	R. FINNEY A.H. SCHOENFELD	Professional applications of undergraduate mathematics Confessions of an accidental theorist
1987	P. NESHER H.S. WILF	Formulating instructional theory: the role of students' misconceptions The calculator with a college education

1988	C. KEITEL L.A. STEEN	Mathematics education and technology All one system
1989	N. BALACHEFF D. SCHATTNEIDER	Teaching mathematical proof: The relevance and complexity of a social approach Geometry is alive and well
1990	U. D'AMBROSIO A. SIERPINSKA	Values in mathematics education On understanding mathematics
1991	J.J. KAPUT C. LABORDE	Mathematics and technology: Multiple visions of multiple futures Approches théoriques et méthodologiques des recherches françaises en didactique des mathématiques
1992	ICME-7	
1993	G.G. JOSEPH J CONFREY	What is a square root? A study of geometrical representation in different mathematical traditions Forging a revised theory of intellectual development: Piaget, Vygotsky and beyond
1994	A. SFARD K. DEVLIN	Understanding = Doing + Seeing ? Mathematics for the twenty-first century
1995	M. ARTIGUE K. MILLETT	The role of epistemological analysis in a didactic approach to the phenomenon of mathematics learning and teaching Teaching and making certain it counts
1996	C. HOYLES D. HENDERSON	Beyond the classroom: The curriculum as a key factor in students' approaches to proof Alive mathematical reasoning
1997	R. BORASSI P. TAYLOR T. KIEREN	What does it really mean to teach mathematics through inquiry? The high school math curriculum Triple embodiment: Studies of mathematical understanding-in-interaction in my work and in the work of CMESG/GCEDM
1998	J. MASON K. HEINRICH	Structure of attention in teaching mathematics Communicating mathematics or mathematics storytelling
1999	J. BORWEIN W. WHITELEY W. LANGFORD J. ADLER B. BARTON	The impact of technology on the doing of mathematics The decline and rise of geometry in 20 th century North America Industrial mathematics for the 21 st century Learning to understand mathematics teacher development and change: Researching resource availability and use in the context of formalised INSET in South Africa An archaeology of mathematical concepts: Sifting languages for mathematical meanings
2000	G. LABELLE M. B. BUSSI	Manipulating combinatorial structures The theoretical dimension of mathematics: A challenge for didacticians

Appendix B • Plenary Lectures at each Annual Meeting

2001	O. SKOVSMOSE C. ROUSSEAU	Mathematics in action: A challenge for social theorising Mathematics, a living discipline within science and technology
2002	D. BALL & H. BASS J. BORWEIN	Toward a practice-based theory of mathematical knowledge for teaching The experimental mathematician: The pleasure of discovery and the role of proof
2003	T. ARCHIBALD A. SIERPINSKA	Using history of mathematics in the classroom: Prospects and problems Research in mathematics education through a keyhole
2004	C. MARGOLINAS N. BOULEAU	La situation du professeur et les connaissances en jeu au cours de l'activité mathématique en classe La personnalité d'Evariste Galois: le contexte psychologique d'un goût prononcé pour les mathématiques abstraites
2005	S. LERMAN J. TAYLOR	Learning as developing identity in the mathematics classroom Soap bubbles and crystals
2006	B. JAWORSKI E. DOOLITTLE	Developmental research in mathematics teaching and learning: Developing learning communities based on inquiry and design Mathematics as medicine
2007	R. NÚÑEZ T. C. STEVENS	Understanding abstraction in mathematics education: Meaning, language, gesture, and the human brain Mathematics departments, new faculty, and the future of collegiate mathematics
2008	A. DJEBBAR A. WATSON	Art, culture et mathématiques en pays d'Islam (IX ^e -XV ^e s.) Adolescent learning and secondary mathematics
2009	M. BORBA G. de VRIES	Humans-with-media and the production of mathematical knowledge in online environments Mathematical biology: A case study in interdisciplinarity
2010	W. BYERS M. CIVIL B. HODGSON S. DAWSON	Ambiguity and mathematical thinking Learning from and with parents: Resources for equity in mathematics education Collaboration et échanges internationaux en éducation mathématique dans le cadre de la CIEM : regards selon une perspective canadienne / ICMI as a space for international collaboration and exchange in mathematics education: Some views from a Canadian perspective My journey across, through, over, and around academia: "...a path laid while walking..."
2011	C. K. PALMER P. TSAMIR & D. TIROSH	Pattern composition: Beyond the basics The Pair-Dialogue approach in mathematics teacher education
2012	P. GERDES M. WALSHAW W. HIGGINSON	Old and new mathematical ideas from Africa: Challenges for reflection Towards an understanding of ethical practical action in mathematics education: Insights from contemporary inquiries Cooda, wooda, didda, shooda: Time series reflections on CMESG/GCEDM

2013	R. LEIKIN B. RALPH E. MULLER	On the relationships between mathematical creativity, excellence and giftedness Are we teaching Roman numerals in a digital age? Through a CMESG looking glass
2014	D. HEWITT N. NIGAM	The economic use of time and effort in the teaching and learning of mathematics Mathematics in industry, mathematics in the classroom: Analogy and metaphor
2015	É. RODITI D. HUGHES HALLET	Diversité, variabilité et convergence des pratiques enseignantes / Diversity, variability, and commonalities among teaching practices Connections: Mathematical, interdisciplinary, electronic, and personal
2016	B. R. HODGSON C. KIERAN E. MULLER P. TAYLOR	Apport des mathématiciens à la formation des enseignants du primaire : regards sur le « modèle Laval » Task design in mathematics education: Frameworks and exemplars A third pillar of scientific inquiry of complex systems—Some implications for mathematics education in Canada Structure—An allegory
2017	Y. SAINT-AUBIN A. SELDEN	The most unglamorous job of all: Writing exercises 40+ years of teaching and thinking about university mathematics students, proofs, and proving: An abbreviated academic memoir
2018	D. VIOLETTE M. GOOS	Et si on enseignait la passion? Making connections across disciplinary boundaries in preservice mathematics teacher education
2019	J-M. DE KONINCK R. GUTIERREZ	Découvrir les mathématiques ensemble avec les étudiants Mathematics as dispossession: Reclaiming mental sovereignty by living mathematx
2021	S. MAYES-TANG	Teaching on empty: Trauma, achievement, and what's next in our math education community
2022	E. PETITFOUR	Quel enseignement de la géométrie pour les élèves dyspraxiques ?
2023	J. SUH M. NGOM	Cultivating joy, wonder and power through community-based math modeling Building the foundations of mathematical modeling and machine learning

Appendix C / Annexe C

PROCEEDINGS OF ANNUAL MEETINGS / ACTES DES RENCONTRES ANNUELLES

Past proceedings of CMESG/GCEDM annual meetings have been deposited in the ERIC documentation system with call numbers as follows:

<i>Proceedings of the 1980 Annual Meeting</i>	ED 204120
<i>Proceedings of the 1981 Annual Meeting</i>	ED 234988
<i>Proceedings of the 1982 Annual Meeting</i>	ED 234989
<i>Proceedings of the 1983 Annual Meeting</i>	ED 243653
<i>Proceedings of the 1984 Annual Meeting</i>	ED 257640
<i>Proceedings of the 1985 Annual Meeting</i>	ED 277573
<i>Proceedings of the 1986 Annual Meeting</i>	ED 297966
<i>Proceedings of the 1987 Annual Meeting</i>	ED 295842
<i>Proceedings of the 1988 Annual Meeting</i>	ED 306259
<i>Proceedings of the 1989 Annual Meeting</i>	ED 319606
<i>Proceedings of the 1990 Annual Meeting</i>	ED 344746
<i>Proceedings of the 1991 Annual Meeting</i>	ED 350161
<i>Proceedings of the 1993 Annual Meeting</i>	ED 407243
<i>Proceedings of the 1994 Annual Meeting</i>	ED 407242
<i>Proceedings of the 1995 Annual Meeting</i>	ED 407241

<i>Proceedings of the 1996 Annual Meeting</i>	ED 425054
<i>Proceedings of the 1997 Annual Meeting</i>	ED 423116
<i>Proceedings of the 1998 Annual Meeting</i>	ED 431624
<i>Proceedings of the 1999 Annual Meeting</i>	ED 445894
<i>Proceedings of the 2000 Annual Meeting</i>	ED 472094
<i>Proceedings of the 2001 Annual Meeting</i>	ED 472091
<i>Proceedings of the 2002 Annual Meeting</i>	ED 529557
<i>Proceedings of the 2003 Annual Meeting</i>	ED 529558
<i>Proceedings of the 2004 Annual Meeting</i>	ED 529563
<i>Proceedings of the 2005 Annual Meeting</i>	ED 529560
<i>Proceedings of the 2006 Annual Meeting</i>	ED 529562
<i>Proceedings of the 2007 Annual Meeting</i>	ED 529556
<i>Proceedings of the 2008 Annual Meeting</i>	ED 529561
<i>Proceedings of the 2009 Annual Meeting</i>	ED 529559
<i>Proceedings of the 2010 Annual Meeting</i>	ED 529564
<i>Proceedings of the 2011 Annual Meeting</i>	ED 547245
<i>Proceedings of the 2012 Annual Meeting</i>	ED 547246
<i>Proceedings of the 2013 Annual Meeting</i>	ED 547247
<i>Proceedings of the 2014 Annual Meeting</i>	ED 581042
<i>Proceedings of the 2015 Annual Meeting</i>	ED 581044
<i>Proceedings of the 2016 Annual Meeting</i>	ED 581045
<i>Proceedings of the 2017 Annual Meeting</i>	ED 589990
<i>Proceedings of the 2018 Annual Meeting</i>	ED 595075
<i>Proceedings of the 2019 Annual Meeting</i>	ED 610111

Proceedings of the 2021 Annual Meeting ED 620362

Proceedings of the 2022 Annual Meeting ED 660551

NOTES

-There was no Annual Meeting in 1992 because Canada hosted the Seventh International Conference on Mathematical Education that year.

-There was no Annual Meeting in 2020 due to COVID-19.

-Due to financial cuts to services in the United States, ERIC has changed the regulations for depositing documents, and CMESG/GCEDM Proceedings no longer qualify to be given a call number in their system. The 2022 Proceedings is the last proceedings to be listed in ERIC.